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LOWER BOUNDS ON THE DISTANCE DOMINATION NUMBER OF A GRAPH

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ABSTRACT. For an integer $k \geq 1$, a (distance) k-dominating set of a connected graph G is a set S of vertices of G such that every vertex of $V(G) \setminus S$ is at distance at most k from some vertex of S. The k-domination number, $\gamma_k(G)$, of G is the minimum cardinality of a k-dominating set of G. In this paper, we establish lower bounds on the k-domination number of a graph in terms of its diameter, radius, and girth. We prove that for connected graphs G and H, $\gamma_k(G \times H) \geq \gamma_k(G) + \gamma_k(H) - 1$, where $G \times H$ denotes the direct product of G and H.

1. INTRODUCTION

Distance in graphs is a fundamental concept in graph theory. Let G be a connected graph. The *distance* between two vertices u and v in G, denoted $d_G(u, v)$, is the length (i.e., the number of edges) of a shortest (u, v)-path in G. The *eccentricity* $\mathrm{ecc}_G(v)$ of v in G is the distance between v and a vertex farthest from v in G. The minimum eccentricity among all vertices of G is the *radius* of G, denoted by $\mathrm{rad}(G)$, while the maximum eccentricity among all vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. Thus, the diameter of G is the maximum distance among all pairs of vertices of G. A vertex v with $\mathrm{ecc}_G(v) = \mathrm{diam}(G)$ is called a *peripheral vertex* of G. A *diametral path* in G is a shortest path in G whose length is equal to the diameter of the graph. Thus, a diametral path is a path of length $\mathrm{diam}(G)$ joining two peripheral vertices of G. If S is a set of vertices in G, then the *distance*, $d_G(v, S)$, from a vertex v to the set S is the minimum distance from v to a vertex of S; that is, $d_G(v, S) = \min\{d_G(u, v) \mid u \in S\}$. In particular, if $v \in S$, then $d(v, S) = 0$.

The concept of domination in graphs is also very well studied in graph theory. A *dominating set* in a graph G is a set S of vertices of G such

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that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The literature on the subject of domination parameters in graphs, up to the year 1997, has been surveyed and detailed in the two books [8, 7].

In this paper we continue the study of distance domination in graphs, which combines the concepts of both distance and domination in graphs. Let $k \geq 1$ be an integer and let G be a graph. In 1975, Meir and Moon [15] introduced the concept of a distance k-dominating set (called a "k-covering" in [15]) in a graph. A set S is a *k-dominating set* of G if every vertex is within distance k from some vertex of S; that is, for every vertex v of G, we have $d(v, S) \leq k$. The *k-domination number* of G, denoted $\gamma_k(G)$, is the minimum cardinality of a k-dominating set of G. When $k = 1$, the 1-domination number of G is precisely the domination number of G, that is, $\gamma_1(G) = \gamma(G)$. The literature on the subject of distance domination in graphs, up to the year 1997, can be found in the book [9]. Distance domination is now widely studied; see, for example, [1, 4, 6, 10, 11, 14, 15, 17, 18, 19].

Definitions and Notation. For notation and graph theory terminology, we in general follow [12]. Specifically, let G be a graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$. We assume throughout the paper that all graphs considered are *simple* graphs, i.e., finite graphs without multiple edges and no directed edges or loops. A *non-trivial graph* is a graph on at least two vertices. A *neighbor* of a vertex v in G is a vertex adjacent to v. The *open neighborhood* of v, denoted $N_G(v)$, is the set of all neighbors of v in G, while the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *closed k-neighborhood*, denoted $N_k[v]$, of v is defined in [4] as the set of all vertices within distance k from v in G; that is, $N_k[v] = \{u \mid d(u, v) \leq k\}$. When $k = 1$, $N_k[v] = N[v]$.

The *degree* of a vertex v in G, denoted $d_G(v)$, is the number of neighbors, $|N_G(v)|$, of v in G. The minimum and maximum degree among all the vertices of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The subgraph induced by a set S of vertices of G is denoted by $G[S]$. The *girth* of G, denoted $g = g(G)$, is the length of a shortest cycle in G. For sets of vertices X and Y of G, the set X *k-dominates* the set Y if every vertex of Y is within distance k from some vertex of X. In particular, if X k-dominates the set $V(G)$, then X is a k-dominating set of G.

If the graph G is clear from context, we simply write V, E, $d(v)$, $\mathrm{ecc}(v)$, $N(v)$, and $N[v]$ rather than $V(G)$, $E(G)$, $d_G(v)$, $\mathrm{ecc}_G(v)$, $N_G(v)$, and $N_G[v]$, respectively. We use the standard notation $[n] = \{1, 2, \ldots, n\}$.

Known Results. The k-domination number of G is in the class of NP-hard graph invariants to compute [7]. Because of the computational complexity of computing $\gamma_k(G)$, graph theorists have sought upper and lower bounds on $\gamma_k(G)$ in terms of simple graph parameters like order, size, and degree.

Since every k-dominating set of a spanning subgraph of a graph G is a k-dominating set of G, we recall the following observation:

Proposition 1.1 ([20]). *For $k \geq 1$, if H is a spanning subgraph of a graph G, then $\gamma_k(G) \leq \gamma_k(H)$.*

In 1975, Meir and Moon [15] established an upper bound for the k-domination number of a tree in terms of its order. They proved that for $k \geq 1$, if T is a tree of order $n \geq k+1$, then $\gamma_k(T) \leq n/(k+1)$. As a consequence of this result and Proposition 1.1, if G is a connected graph of order $n \geq k+1$, then $\gamma_k(G) \leq n/(k+1)$. A short proof of the Meir-Moon upper bound can be found in [11]; see also Proposition 24 and Corollary 12.5 in the book [9].

A complete characterization of the graphs G achieving equality in this upper bound was obtained by Topp and Volkmann [19]. Tian and Xu [18] improved the Meir-Moon upper bound and showed that for $k \geq 1$, if G is a connected graph of order $n \geq k+1$ with maximum degree Δ, then $\gamma_k(G) \leq (n - \Delta + k - 1)/k$. The Tian-Xu bound was further improved by Henning and Lichiardopol [10], who showed that for $k \geq 2$, if G is a connected graph with minimum degree $\delta \geq 2$ and maximum degree Δ of order $n \geq \Delta + k - 1$, then

$$\gamma_k(G) \leq \frac{n + \delta - \Delta}{\delta + k - 1}.$$

We recall the following well-known lower bound on the domination number of a graph in terms of its diameter.

Theorem 1.2 ([8]). *If G is a connected graph with diameter d, then $\gamma(G) \geq (d+1)/3$.*

The following two results were originally conjectured by the conjecture making program Graffiti.pc; see [2] for details.

Theorem 1.3 ([3]). *If G is a connected graph with radius r, then $\gamma(G) \geq (2r)/3$.*

Theorem 1.4 ([3]). *If G is a connected graph with girth $g \geq 3$, then $\gamma(G) \geq g/3$.*

Our Results. In this paper, we establish lower bounds for the k-domination number of a graph in terms of its diameter (Theorem 3.1), radius (Corollary 3.5), and girth (Theorem 3.6). These results generalize the results of Theorem 1.2, 1.3, and 1.4. A key tool in order to prove our results is the important lemma (Lemma 2.1) that every connected graph has a spanning tree with equal k-domination number. We also prove a key property (Lemma 2.2) of shortest cycles in a graph that enables us to establish our girth result for the k-domination number of a graph. We also show that our bounds are all sharp and provide examples following the proofs.

2. Preliminary Lemmas

We shall need the following two lemmas.

Lemma 2.1. *For $k \geq 1$, every connected graph G has a spanning tree T such that $\gamma_k(T) = \gamma_k(G)$.*

Proof. Let S be a minimum k-dominating set of G and note that $|S| = \gamma_k(G)$. For $i \in [k]$, let $D_i(S) = \{v \in V(G) \setminus S \mid d_G(v, S) = i\}$. Since S is a k-dominating set of G, every vertex v in G is within distance k from some vertex of S and therefore belongs to $D_i(S)$ for some $i \in [k]$. Furthermore, such a vertex is adjacent to at least one vertex of $D_{i-1}(S)$, and possibly to vertices in $D_i(S)$ and $D_{i+1}(S)$. For all $i \in [k]$ and for each vertex $v \in D_i(S)$, we delete all but one edge that joins v to a vertex of $D_{i-1}(S)$. Further, we delete all edges, if any, that join v to vertices in $D_i(S)$. Let F denote the resulting spanning subgraph of the graph G.

We claim that F is a forest. Suppose, to the contrary, that F contains a cycle C. Let v be a vertex in such a cycle C at maximum distance from a vertex of S in G, and let v_1 and v_2 be the two neighbors of v on C. Suppose that $v \in D_p(S)$ for some $p \in [k]$. Then $d_G(v, S) = p$ and $d_G(w, S) \leq p$ for every vertex w of C different from v. If v_1 or v_2 belongs to $D_p(S)$, this contradicts the way in which F was constructed, noting that no edge in F joins two vertices in the same set $D_i(S)$. Thus, both v_1 and v_2 belong to $D_{p-1}(S)$. Once again, this contradicts the way in which F was constructed, noting that exactly one edge in F joins a vertex in $D_i(S)$ to a vertex in $D_{i-1}(S)$. Therefore, F is a forest.

If F is a tree, then we let $T = F$; otherwise, if the forest F has $\ell \geq 2$ components, then we let T be obtained from F by adding to it $\ell - 1$ edges in such a way that the resulting subgraph is connected. We note that T is a tree. By construction, if $v \in D_i(S)$ for some $i \in [k]$, then there is a path from v to S of length i in T, and so $d_T(v, S) \leq d_G(v, S)$. Since T is a spanning tree of G, $d_G(v, S) \leq d_T(v, S)$ for every vertex $v \in V(G)$. Consequently, the spanning tree T of G is distance-preserving from the set S in the sense that $d_G(v, S) = d_T(v, S)$ for every vertex $v \in V(G)$. Since S is a k-dominating set of G, the set S is therefore a k-dominating set of T, and so $\gamma_k(T) \leq |S| = \gamma_k(G)$. However, by Observation 1.1, $\gamma_k(G) \leq \gamma_k(T)$. Consequently, $\gamma_k(T) = \gamma_k(G)$. □

Lemma 2.2. *Let G be a connected graph that contains a cycle, and let C be a shortest cycle in G. If v is a vertex of G outside C that k-dominates at least $2k$ vertices of C, then there exist two vertices $u, w \in V(C)$ that are both k-dominated by v such that a shortest (u, v)-path does not contain w, and a shortest (v, w)-path does not contain u.*

Proof. Since v is not on C, it has a distance of at least 1 to every vertex of C. Let u be a vertex of C at minimum distance from v in G and let Q be the set of vertices on C that are k-dominated by v in G. Thus $Q \subseteq V(C)$ and, by assumption, $|Q| \geq 2k$. Among all vertices in Q, let $w \in Q$ be chosen to have maximum distance from u on the cycle C. Since there are $2k-1$ vertices within distance $k-1$ from u on C, the vertex w has distance at least k from u on the cycle C. Let P_u be a shortest (u,v)-path and let P_w be a shortest (v,w)-path in G. If $w \in V(P_u)$, then $d_G(v,w) < d_G(v,u)$, contradicting our choice of the vertex u. Therefore, $w \notin V(P_u)$.

Suppose that $u \in V(P_w)$. Since C is a shortest cycle in G, the distance between u and w on C is the same as the distance between u and w in G. Thus, $d_G(u,w) = d_C(u,w)$, implying that $d_G(v,w) = d_G(v,u) + d_G(u,w) \geq 1 + d_G(u,w) = 1 + d_C(u,w) \geq 1 + k$, a contradiction. Therefore, $u \notin V(P_w)$. □

3. Lower Bounds

In this section we provide various lower bounds on the k-domination number for general graphs. We first prove a generalization of Theorem 1.2 by establishing a lower bound on the k-domination number of a graph in terms of its diameter.

Theorem 3.1. *For $k \geq 1$, if G is a connected graph with diameter d then*

$$\gamma_k(G) \geq \frac{d+1}{2k+1}.$$

Proof. Let $P\colon u_0u_1 \dots u_d$ be a diametral path in G, joining two peripheral vertices $u = u_0$ and $v = u_d$ of G. Then P has length $\mathrm{diam}(G) = d$. We will show that every vertex of G k-dominates at most $2k+1$ vertices of P.

Suppose, to the contrary, that there exists a vertex $q \in V(G)$ that k-dominates at least $2k+2$ vertices of P; note that it is possible that $q \in V(P)$. Let Q be the set of vertices on the path P that are k-dominated by the vertex q in G. By supposition, $|Q| \geq 2k+2$. Let i and j be the smallest and largest integers, respectively, such that $u_i \in Q$ and $u_j \in Q$. We note that $Q \subseteq \{u_i, u_{i+1}, \dots, u_j\}$. Thus, $2k+2 \leq |Q| \leq j-i+1$. Since P is a shortest (u,v)-path in G, we therefore note that $d_G(u_i,u_j) = d_P(u_i,u_j) = j-i \geq 2k+1$.

Let P_i be a shortest (u_i,q)-path in G and let P_j be a shortest (q,v_i)-path in G. Since the vertex q k-dominates both u_i and u_j in G, both paths P_i and P_j have length at most k. Therefore, the (u_i,u_j)-path obtained by following the path P_i from u_i to q, and then proceeding along the path P_j from q to u_j, has length at most $2k$, implying that $d_G(u_i,u_j) \leq 2k$, a contradiction. Therefore, every vertex of G k-dominates at most $2k+1$ vertices of P.

Now let S be a minimum k-dominating set of G so that $|S| = \gamma_k(G)$. Each vertex of S k-dominates at most $2k+1$ vertices of P, and so S k-dominates at most $|S|(2k+1)$ vertices of P. However, since S is a k-dominating set of G, every vertex of P is k-dominated the set S, and so S k-dominates $|V(P)| = d+1$ vertices of P. Therefore, $|S|(2k+1) \geq d+1$, or, equivalently, $\gamma_k(G) \geq (d+1)/(2k+1)$. □

That the lower bound of Theorem 3.1 is tight may be seen by taking G to be a path, $v_1v_2\dots v_n$, of order $n = \ell(2k+1)$ for some $\ell \geq 1$. Let $d = \mathrm{diam}(G)$, so $d = n-1 = \ell(2k+1)-1$. By Theorem 3.1, $\gamma_k(G) \geq (d+1)/(2k+1) = \ell$. The set

$$S = \bigcup_{i=0}^{\ell-1} \{v_{k+1+i(2k+1)}\}$$

is a k-dominating set of G, and so $\gamma_k(G) \leq |S| = \ell$. Consequently, $\gamma_k(G) = \ell = (d+1)/(2k+1)$. We state this formally as follows.

Proposition 3.2. *If $G = P_n$ where $n \equiv 0 \bmod (2k+1)$, then*

$$\gamma_k(G) = \frac{\mathrm{diam}(G)+1}{2k+1}.$$

More generally, by applying Theorem 3.1, the k-domination number of a path P_n on $n \geq 3$ vertices is easy to compute.

Proposition 3.3. *For $k \geq 1$ and $n \geq 3$,*

$$\gamma_k(P_n) = \left\lceil \frac{n}{2k+1} \right\rceil.$$

For $k \geq 1$ and $n \geq 3$, every vertex of a cycle C_n k-dominates exactly $2k+1$ vertices. Thus, if S is a minimum k-dominating set of G, then the set S k-dominates at most $|S|(2k+1)$ vertices of P, implying that $|S|(2k+1) \geq n$, or, equivalently, $\gamma_k(C_n) = |S| \geq n/(2k+1)$. Conversely, by Proposition 1.1 and Proposition 3.3, $\gamma_k(C_n) \leq \gamma_k(P_n) = \lceil n/(2k+1) \rceil$. Consequently, we have the following result.

Proposition 3.4. *For $k \geq 1$ and $n \geq 3$,*

$$\gamma_k(C_n) = \left\lceil \frac{n}{2k+1} \right\rceil.$$

For $k \geq 1$ and $n \geq 3$, where $n \equiv 0 \bmod (2k+1)$, consider a path $P\colon v_1v_2\dots v_n$. By replacing each vertex v_i, for $2 \leq i \leq n-1$, on the path P with a clique V_i of size at least $\delta \geq 1$, adding all edges between v_1 and vertices in V_2, adding all edges between v_n and vertices in V_{n-1}, and adding all edges between vertices in V_i and V_{i+1} for $2 \leq i \leq n-2$, we obtain a graph with minimum degree at least δ achieving the lower bound of Theorem 3.1.

From Theorem 3.1, we have the following lower bound on the k-domination number of a graph in terms of its radius. We remark that when $k = 1$, Corollary 3.5 is precisely Theorem 1.3. Therefore, Corollary 3.5 is a generalization of Theorem 1.3.

Corollary 3.5. *For $k \geq 1$, if G is a connected graph with radius r, then*

$$\gamma_k(G) \geq \frac{2r}{2k+1}.$$

Proof. By Lemma 2.1, the graph G has a spanning tree T such that $\gamma_k(T) = \gamma_k(G)$. Since adding edges to a graph cannot increase its radius, $\text{rad}(G) \le \text{rad}(T)$. Since T is a tree, we note that $\text{diam}(T) \ge 2\text{rad}(T) - 1$. Applying Theorem 3.1 to the tree T, we have that

$$\gamma_k(G) = \gamma_k(T) \ge \frac{\text{diam}(T) + 1}{2k+1} \ge \frac{2\text{rad}(T)}{2k+1} \ge \frac{2\text{rad}(G)}{2k+1}.$$

□

That the lower bound of Corollary 3.5 is tight may be seen by taking G to be a path, P_n, of order $n = 2\ell(2k+1)$ for some integer $\ell \ge 1$. Let $d = \text{diam}(G)$ and let $r = \text{rad}(G)$ so that $d = 2\ell(2k+1) - 1$ and $r = \ell(2k+1)$. In particular, we note that $d = 2r - 1$. By Proposition 3.3, $\gamma_k(G) = (d+1)/(2k+1) = (2r)/(2k+1)$. As before, by replacing each internal vertex on the path with a clique of size at least $\delta \ge 1$, we can obtain a graph with minimum degree at least δ achieving the lower bound of Corollary 3.5.

We next prove a generalization of Theorem 1.4 by establishing a lower bound on the k-domination number of a graph in terms of its girth. We remark that when $k = 1$, Theorem 3.6 is precisely Theorem 1.4.

Theorem 3.6. *For $k \ge 1$, if G is a connected graph with girth $g < \infty$, then*

$$\gamma_k(G) \ge \frac{g}{2k+1}.$$

Proof. The lower bound is trivial if $g \le 2k+1$. We may therefore assume that $g \ge 2k+2$. Let C be a shortest cycle in G, so that C has length g. We note that the distance between two vertices in $V(C)$ is exactly the same in C as in G. We consider two cases, depending on the value of the girth.

Case 1: $2k+2 \le g \le 4k+2$:
In this case, we need to show that $\gamma_k(G) \ge \lceil g/(2k+1) \rceil = 2$. Suppose, to the contrary, that $\gamma_k(G) = 1$. Then, G contains a vertex v that is within distance k from every vertex of G. In particular, $d(u,v) \le k$ for every vertex $u \in V(C)$. If $v \in V(C)$, then since C is a shortest cycle in G, we note that $d_C(u,v) = d_G(u,v) \le k$ for every vertex $u \in V(C)$. However, the lower bound condition on the girth, namely $g \ge 2k+2$, implies that no vertex on the cycle C is within distance k in C from every vertex of C, which is a contradiction. Therefore, $v \notin V(C)$.

By Lemma 2.2, there exists two vertices $u, w \in V(C)$ such that a shortest (v,u)-path does not contain w and a shortest (v,w)-path does not contain u. We will show that we can choose u and w to be adjacent vertices on C.

Let w be a vertex of C at maximum distance, say d_w, from v in G. Let w_1 and w_2 be the two neighbors of w on the cycle C. If $d_G(v,w_1) = d_w$, then we can take $u = w_1$, and the desired property (that a shortest (v,u)-path does not contain w and a shortest (v,w)-path does not contain u) holds. Hence, we may assume that $d_G(v,w_1) \ne d_w$. By our choice of the vertex w, we note that $d_G(v,w_1) \le d_w$, implying that $d_G(v,w_1) = d_w - 1$. Similarly, we may assume that $d_G(v, w_2) = d_w - 1$. Let P_w be a shortest (v, w)-path. At most one of w_1 and w_2 belong to the path P_w. After renaming w_1 and w_2, if necessary, we may assume that w_1 does not belong to the path P_w. In this case, letting $u = w_1$ and letting P_u be a shortest (v, u)-path, we note that $w \notin V(P_u)$. Since we have already observed that $u \notin V(P_w)$, this shows that u and w can indeed be chosen to be neighbors on C.

Let x be the last vertex in common with the (v, u)-path, P_u, and the (v, w)-path, P_w; note that it is possible that $x = v$. Then the cycle obtained from the (x, u)-section of P_u by proceeding along the edge uw to w, and then following the (w, x)-section of P_w back to x, has length at most $d_G(v, u)+1+d_G(v, w) \le 2k+1$, contradicting the fact that the girth satisfies $g \ge 2k+2$. Therefore, $\gamma_k(G) \ge 2$, as desired.

CASE 2: $g \ge 4k + 3$:
Let S be a minimum k-dominating set of G so that $|S| = \gamma_k(G)$. Let $K = S \cap V(C)$ and let $L = S \setminus V(C)$. Then $S = K \cup L$. If $L = \emptyset$, then $S = K$ and the set K is a k-dominating set of C; by Proposition 3.4 it follows that

$$\gamma_k(G) = |S| = |K| \ge \gamma_k(C_g) = \left\lceil \frac{g}{2k+1} \right\rceil,$$

and the theorem holds. Hence we may assume that $|L| \ge 1$, for otherwise the desired result holds. We wish to show that $|K| + |L| = |S| \ge \lceil g/(2k+1) \rceil$. Suppose, to the contrary, that

$$|K| \le \left\lceil \frac{g}{1+2k} \right\rceil - 1 - |L|.$$

As observed earlier, the distance between two vertices in $V(C)$ is exactly the same in C as in G. This implies that each vertex of K, since $K \subseteq V(C)$, is within distance k from exactly $2k + 1$ vertices of C. Thus, the set K k-dominates at most

$$\begin{aligned}
|K|(2k+1) &\le \left(\left\lceil \frac{g}{2k+1} - 1 - |L| \right\rceil \right) (2k+1) \\
&\le \left(\frac{g+2k}{2k+1} - 1 - |L| \right) (2k+1) \\
&= g - 1 - |L|(2k+1)
\end{aligned}$$

vertices from C. Consequently, since $|V(C)| = g$, there are at least $|L|(2k+1)+1$ vertices of C which are not k-dominated by vertices of K, and therefore must be k-dominated by vertices from L. Thus, by the Pigeonhole Principle, there is at least one vertex, call it v, in L that k-dominates at least $2k + 2$ vertices in C. By Lemma 2.2, there exist two vertices $u, w \in V(C)$ that are both k-dominated by v and such that a shortest (u, v)-path, P_u, from u to v, does not contain w and a shortest (w, v)-path, P_w, from w to v, does not contain u. Analogously as in the proof of Lemma 2.2, we can choose the vertex u to be a vertex of C at minimum distance from v in G. Thus, the vertex u is the only vertex on the cycle C that belongs to the path P_u. Combining the paths P_u and P_w produces a (u,w)-walk of length at most $d_G(u,v) + d_G(v,w) \leq 2k$, implying that $d_G(u,w) \leq 2k$. Since C is a shortest cycle in G, we therefore have that $d_C(u,w) = d_G(u,w) \leq 2k$.

The cycle C yields two (w,u)-paths. Let P_{wu} be the (w,u)-path on the cycle C of shorter length (starting at w and ending at u). Thus, P_{wu} has length $d_C(u,w) \leq 2k$. Note that the path P_{wu} belongs entirely on the cycle C. Let $x \in V(C)$ be the last vertex in common with the (w,v)-path, P_w, and the (w,u)-path, P_{wu}; note that it is possible that $x = w$. However, observe that $x \neq u$, since $u \notin V(P_w)$. Let y be the first vertex in common with the (x,v)-subsection of the path P_w and with the (u,v)-path P_u; note that it is possible that $y = v$. However, observe that $y \neq x$ since $x \notin V(P_u)$ and $V(P_u) \cap V(C) = \{u\}$. Using the (x,u)-subsection of the path P_{wu}, the (x,y)-subsection of the path P_w, and the (u,y)-subsection of the path P_u produces a cycle in G of length at most $d_G(u,v) + d_G(w,v) + d_G(u,w) \leq k+k+2k = 4k$, contradicting the fact that the girth $g \geq 4k+3$. Therefore, $\gamma_k(G) = |S| = |K| + |L| \geq \lceil g/(2k+1) \rceil$, as desired. □

The lower bound of Theorem 3.6 is tight, as may be seen by taking G to be a cycle C_n, where $n \equiv 0 \bmod (2k+1)$. We note that G has girth $g = n$ and, by Proposition 3.4, $\gamma_k(G) = n/(2k+1) = g/(2k+1)$.

4. Direct Product Graphs

The *direct product graph*, $G \times H$, of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and with edges $(g_1,h_1)(g_2,h_2)$, where $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Let $A \subseteq V(G \times H)$. The *projection of* A onto G is defined as

$$P_G(A) = \{g \in V(G) \colon (g,h) \in A \text{ for some } h \in V(H)\}.$$

Similarly, the projection of A onto H is defined as

$$P_H(A) = \{g \in V(H) \colon (g,h) \in A \text{ for some } h \in V(G)\}.$$

For a detailed discussion on direct product graphs, we refer the reader to the handbook on graph products [5]. There have been various studies on the domination number of direct product graphs. For example, Mekiš [16] proved the following lower bound on the domination number of direct product graphs.

Theorem 4.1 ([16])**.** *If G and H are connected graphs, then*

$$\gamma(G \times H) \geq \gamma(G) + \gamma(H) - 1.$$

Staying within the theme of our previous results, we now prove a projection lemma which will enable us generalize the result of Theorem 4.1 on the domination number to the k-domination number.

Lemma 4.2 (Projection Lemma)**.** *Let G and H be connected graphs. If D is a k-dominating set of $G \times H$, then $P_G(D)$ is a k-dominating set of G and $P_H(D)$ is a k-dominating set of H.*

Proof. Let $D \subseteq V(G \times H)$ be a k-dominating set of $G \times H$. We show firstly that $P_G(D)$ is a k-dominating set of G. Let g be a vertex in $V(G)$. If $g \in P_G(D)$, then g is clearly k-dominated by $P_G(D)$. Hence, we may assume that $g \in V(G) \setminus P_G(D)$. Let h be an arbitrary vertex in $V(H)$. Since $g \notin P_G(D)$, the vertex $(g, h) \notin D$. However, the set D is a k-dominating set of $G \times H$, and so (g, h) is within distance k from D in G; that is, $d_{G\times H}((g, h), D) \le k$. Let $(g_0, h_0), (g_1, h_1), \ldots, (g_r, h_r)$ be a shortest path from (g, h) to D in $G \times H$, where $(g, h) = (g_0, h_0)$ and $(g_r, h_r) \in D$. By assumption, $1 \le r \le k$. For $i \in \{0, \ldots, r-1\}$, the vertices (g_i, h_i) and (g_{i+1}, h_{i+1}) are adjacent in $G \times H$. Hence, by the definition of the direct product graph, the vertices g_i and g_{i+1} are adjacent in G, implying that $g_0 g_1 \ldots g_r$ is a (g_0, g_r)-walk in G of length r. This in turn implies that there is a (g_0, g_r)-path in G of length r. Recall that $g = g_0$ and $1 \le r \le k$. Since $(g_r, h_r) \in D$, the vertex $g_r \in P_G(D)$. Hence, there is a path from g to a vertex of $P_G(D)$ in G of length at most k. Since g is an arbitrary vertex in $V(G)$, the set $P_G(D)$ is therefore a k-dominating set of G. Analogously, the set $P_H(D)$ is a k-dominating set of H. □

Using our Projection Lemma, we are now in a position to generalize Theorem 4.1.

Theorem 4.3. *If G and H are connected graphs, then*

$$\gamma_k(G \times H) \ge \gamma_k(G) + \gamma_k(H) - 1.$$

Proof. Let $D \subseteq V(G \times H)$ be a minimum k-dominating set of $G \times H$. Suppose, to the contrary, that

$$(*) \qquad |D| \le \gamma_k(G) + \gamma_k(H) - 2.$$

By Lemma 4.2, $P_G(D)$ is a k-dominating set of G and $P_H(D)$ is a k-dominating set of H. Therefore, we have that $|D| \ge |P_G(D)| \ge \gamma_k(G)$ and $|D| \ge |P_H(D)| \ge \gamma_k(H)$. If $\gamma_k(G) = 1$, then by $(*)$ we have,

$$\gamma_k(H) - 1 \ge |D| \ge \gamma_k(H),$$

which is a contradiction. Therefore, $\gamma_k(G) \ge 2$. Analogously, $\gamma_k(H) \ge 2$.

Recall that $|P_G(D)| \ge \gamma_k(G)$. We now remove vertices from the set $P_G(D)$ until we obtain a set, D_G say, of cardinality exactly $\gamma_k(G) - 1$. Thus, D_G is a proper subset of $P_G(D)$ of cardinality $\gamma_k(G) - 1$. Since D_G is not a k-dominating set of G, there exists a vertex $g \in V(G)$ that is not k-dominated by the set D_G in G, that is, $d_G(g, D_G) > k$. Let $D_G = \{g_1, \ldots, g_t\}$, where $t = \gamma_k(G) - 1 \ge 1$. For each $i \in [t]$, there exists a (not necessarily unique) vertex $h_i \in V(H)$ such that $(g_i, h_i) \in D$, as $D_G \subseteq P_G(D)$. We now consider the set

$$D_0 = \{(g_1, h_1), \ldots, (g_t, h_t)\},$$

and note that $D_0 \subset D$ and $|D_0| = \gamma_k(G) - 1$. By $(*)$, we note that

$$\begin{aligned} |P_H(D \setminus D_0)| &\leq |D \setminus D_0| \\ &= |D| - |D_0| \\ &\leq (\gamma_k(G) + \gamma_k(H) - 2) - (\gamma_k(G) - 1) \\ &= \gamma_k(H) - 1 \\ &< \gamma_k(H). \end{aligned}$$

Thus there exists a vertex $h \in V(H)$ that is not k-dominated by the set $P_H(D \setminus D_0)$ in H, that is, $d_H(h, P_H(D \setminus D_0)) > k$.

We now consider the vertex $(g, h) \in V(G \times H)$. Since D is a k-dominating set of $G \times H$, the vertex (g, h) is k-dominated by some vertex, say (g^*, h^*), of D in $G \times H$. An analogous proof as in the proof of Lemma 4.2 shows that $d_G(g, g^*) \leq k$ and $d_H(h, h^*) \leq k$. If $(g^*, h^*) \in D \setminus D_0$, then $h^* \in P_H(D \setminus D_0)$, implying that $d_H(h, P_H(D \setminus D_0)) \leq d_H(h, h^*) \leq k$, a contradiction. Hence, $(g^*, h^*) \in D_0$. This in turn implies that $g^* \in P_G(D_0) = G_D$. Thus, $d_G(g, D_G) \leq d_G(g, g^*) \leq k$, contradicting the fact that $d_G(g, D_G) > k$. Therefore, the assumption that $|D| \leq \gamma_k(G) + \gamma_k(H) - 2$ must be false, and the result follows. □

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