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ON MONOCHROMATIC LINEAR RECURRENCE SEQUENCES

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Abstract. In this paper we prove some van der Waerden type theorems for linear recurrence sequences. Under the assumption $a_{i-1} \leq a_i a_{s-1}$ $(i = 2, \ldots, s)$, we extend results of G. Nyul and B. Rauf for sequences satisfying $x_i = a_1 x_{i-s} + \cdots + a_s x_{i-1}$ $(i \geq s+1)$, where $a_1, \ldots, a_s$ are positive integers. Moreover, we solve completely the same problem for sequences satisfying the binary recurrence relation $x_i = a x_{i-1} - b x_{i-2}$ $(i \geq 3)$ and $x_1 < x_2$, where a, b are positive integers with $a \geq b+1$.

1. Introduction

For integers $k \geq 2$ and $r \geq 1$, given a collection $\mathcal{S}$ of positive integer sequences of length k, we will seek to answer the following question: For any r-colouring of the positive integers, does there exist $S \in \mathcal{S}$ which is monochromatic? To answer this question negatively, we simply have to show that there is an r-colouring of the positive integers such that no $S \in \mathcal{S}$ is monochromatic.

Positive or negative answers to questions of the above form are called van der Waerden type theorems; see [3] for several families of sequences. It is obvious that if $r = 1$, the above question always has a positive answer. Furthermore, if $r_1 > r_2 \geq 1$ and the answer is positive for r_1-colourings, then the answer is positive also for r_2-colourings.

In this paper, we investigate van der Waerden type theorems for linear recurrence sequences. For a detailed list of previous references in this direction, see [4].

Let $a_1, \ldots, a_s$ be positive integers, where $s \geq 2$. For the family of positive integer sequences $(x_i)_{i=1}^k$ satisfying the linear recurrence relation

$$x_i = a_1 x_{i-s} + \cdots + a_s x_{i-1} \quad (s+1 \leq i \leq k), \tag{1.1}$$

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van der Waerden type theorems are interesting only in the cases in which $k \geq s+1$ and $r \geq 2$; consequently we will be focused on these cases in this paper.

Since Equation (1.1), when written in homogeneous form, has at least three terms with both positive and negative coefficients, the following theorem follows from a theorem of R. Rado in [5]; see [1] for a finite version and some earlier results:

Theorem 1.1. *For any 2-colouring of the positive integers, there exists a monochromatic sequence $(x_i)_{i=1}^{s+1}$ satisfying Equation (1.1).*

In [2], H. Harborth and S. Maasberg settled the Fibonacci case of $s = 2$, $a_1 = a_2 = 1$. As a generalization of their result, G. Nyul and B. Rauf [4] proved the following theorems for arbitrary order and arbitrary coefficients of the linear recurrence (1.1):

Theorem 1.2 (G. Nyul, B. Rauf [4])**.** *For any r-colouring of the positive integers, there exists a monochromatic sequence $(x_i)_{i=1}^{k}$ satisfying Equation (1.1) in the following cases:*

(1) *If $a_i = 1$ for some i, $k = s+1$, and $r \geq 2$;*
(2) *If $a_1 = a_s = 1$, $k = s+2$, and $r \geq 2$.*

Theorem 1.3 (G. Nyul, B. Rauf [4])**.** *There is an r-colouring of the positive integers such that there exists no monochromatic sequence $(x_i)_{i=1}^{k}$ satisfying Equation (1.1) in the following cases:*

(1) *If $a_i \geq 2$ for all i, $k \geq s+1$ and $r \geq p-1$;*
(2) *If $a_1 \geq 2$ or $a_s \geq 2$, $k \geq s+2$ and $r \geq p-1$;*
(3) *If $k \geq s+3$ and $r \geq p-1$;*
(4) *If $k \geq 2s+1$ and $r \geq 2$;*

where p is the smallest prime with $a_1 + \cdots + a_s + 1 \leq p$.

In statements (2) and (3), if $s = 2$ and both $a_1, a_2 \geq 2$, or if $s \geq 3$, then p can be replaced by the smallest prime p with $a_1 + \cdots + a_s \leq p$.

We can observe that these general results cover all possibilities when $s = 2$ and $a_1 = a_2 = 1$, or when $s = 3$ and $a_1 = a_2 = a_3 = 1$. For any other values of $a_1, \ldots, a_s$, our question remains unanswered for a finite number of pairs (k, r), even in the multibonacci case, when $s \geq 4$ and $a_1 = \cdots = a_s = 1$.

By constructing a variant of a colouring used in [4], in Theorem 2.1 we will extend the above results in a special case: under the assumption that $a_{i-1} \leq a_i a_{s-1}$ $(i = 2, \ldots, s)$. This solves the problem completely for an infinite number of orders and coefficients, including the multibonacci case (Corollary 2.2).

If a, b are positive integers with $a \geq b+1$, then by applying Rado's theorem on regularity of homogeneous systems of linear equations and an explicit colouring, we give a complete van der Waerden type theorem (Theorem 2.3) for positive integer sequences $(x_i)_{i=1}^{k}$ satisfying the relation:

$$x_i = a x_{i-1} - b x_{i-2} \quad (3 \leq i \leq k) \quad \text{and} \quad x_1 < x_2. \tag{1.2}$$

We note that for $a = 2$ and $b = 1$, the sequences $(x_i)_{i=1}^k$ satisfying Equation (1.2) are precisely the strictly increasing arithmetic progressions. Therefore Theorem 2.3 can be viewed as an extension of the classical theorem of B.L. van der Waerden [6].

2. Main results

We now summarize the main van der Waerden type results of this paper about linear recurrence sequences satisfying Equations (1.1) or (1.2):

Theorem 2.1. *Suppose that $s \geq 2$ and $a_1, \ldots, a_s$ are positive integers such that $a_{i-1} \leq a_i a_{s-1}$ for $i = 2, \ldots, s$. If $k \geq s+3$ and $r \geq 2$, then there is an r-colouring of the positive integers such that there exists no monochromatic sequence $(x_i)_{i=1}^k$ satisfying Equation (1.1).*

Together with the above Theorem 1.2 of G. Nyul and B. Rauf, Theorem 2.1 completely answers the original question under the additional assumption $a_1 = a_s = 1$, and in particular in the multibonacci case.

Corollary 2.2. *Let $s \geq 2$, $k \geq s+1$, $r \geq 2$, and $a_1, \ldots, a_s$ be positive integers such that $a_{i-1} \leq a_i a_{s-1}$ for $i = 2, \ldots, s$ and $a_1 = a_s = 1$. For any r-colouring of the positive integers, there exists a monochromatic sequence $(x_i)_{i=1}^k$ satisfying Equation (1.1) if and only if $k = s+1$ or $k = s+2$.*

Theorem 2.3. *Suppose that a, b are positive integers with $a \geq b+1$. Then:*

1. *If $a = b+1$, $k \geq 3$, and $r \geq 1$, for any r-colouring of the positive integers, then there exists a monochromatic sequence $(x_i)_{i=1}^k$ satisfying Equation (1.2);*
2. *If $a \geq b+2$, $k \geq 4$, and $r \geq 2$, then there is an r-colouring of the positive integers such that there exists no monochromatic sequence $(x_i)_{i=1}^k$ satisfying Equation (1.2).*

3. Proofs

Proof of Theorem 2.1. We construct an appropriate 2-colouring of the positive integers as follows: Let the integers in the interval

$$\left[(a_{s-1}+a_s)^i, (a_{s-1}+a_s)^{i+1}\right[$$

be red or blue if $i \geq 0$ is even or odd, respectively.

Using recurrence relation (1.1) and the assumption of the theorem, we have that

$$\begin{aligned} x_{s+1} < x_{s+2} &= a_1 x_2 + \cdots + a_{s-1} x_s + a_s x_{s+1} \\ &\leq a_2 a_{s-1} x_2 + \cdots + a_s a_{s-1} x_s + a_s x_{s+1} \\ &< a_{s-1}(a_1 x_1 + \cdots + a_s x_s) + a_s x_{s+1} \\ &= (a_{s-1}+a_s) x_{s+1}. \end{aligned}$$

This means that x_{s+1} and x_{s+2} are in the same or in consecutive intervals $[(a_{s-1}+a_s)^i, (a_{s-1}+a_s)^{i+1}[$. If they share their colours, they have to belong to the same interval.

Similarly, it follows that x_{s+2} and x_{s+3} should be elements of the same interval. However, this is impossible since

$$(a_{s-1}+a_s)x_{s+1} < a_1x_3 + \cdots + a_{s-1}x_{s+1} + a_sx_{s+2} = x_{s+3}.$$

□

Proof of Theorem 2.3. We begin the proof with the remark that the assumption $a \geq b+1$ guarantees that each element of a sequence $(x_i)_{i=1}^k$ satisfying Equation (1.2) is a positive integer if the initial values satisfy $0 < x_1 < x_2$, since

$$x_{i-1} + b(x_{i-1} - x_{i-2}) \leq ax_{i-1} - bx_{i-2} = x_i$$

for $3 \leq i \leq k$.

(1): To prove the first part of the theorem, we will need the following result of R. Rado [5]; see also [3]. Suppose that C is an integer matrix with columns $\mathbf{c}_1, \ldots, \mathbf{c}_n$ where $n \geq 2$. We then say that the homogeneous system of equations $C\mathbf{x} = \mathbf{0}$ is regular if $C\mathbf{x} = \mathbf{0}$ has a monochromatic solution for any $r \geq 1$ and r-colouring of the positive integers. The theorem of R. Rado in [5] states that $C\mathbf{x} = \mathbf{0}$ is regular if and only if C satisfies the so-called columns condition: $\{1, \ldots, n\}$ can be partitioned into subsets $I_1, \ldots, I_t$ such that $\sum_{i \in I_1} \mathbf{c}_i = \mathbf{0}$ and $\sum_{i \in I_j} \mathbf{c}_i$ is a linear combination of columns $\mathbf{c}_\ell$ ($\ell \in I_1 \cup \cdots \cup I_{j-1}$) for all $j = 2, \ldots, t$.

If $a = b+1$, then the system of equations $x_i = (b+1)x_{i-1} - bx_{i-2}$, for all $3 \leq i \leq k$, together with the additional equation $x_2 - x_1 = y$, can be written into homogeneous form having $(k-1) \times (k+1)$ coefficient matrix

$$\begin{pmatrix} b & -(b+1) & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & -(b+1) & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & b & -(b+1) & 1 & 0 & 0 \\ 0 & \ldots & 0 & 0 & b & -(b+1) & 1 & 0 \\ 1 & -1 & 0 & \ldots & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote by $\mathbf{c}_i$ its ith column, where $i = 1, \ldots, k+1$. Then this matrix satisfies the columns condition, since

$$\sum_{i=1}^{k} \mathbf{c}_i = \mathbf{0},$$

and

$$\sum_{i=1}^{k} (b^{i-1} + \cdots + b^{k-1})\mathbf{c}_i = \mathbf{c}_{k+1}.$$

Therefore the system of equations is regular by Rado's theorem. In other words, it has a monochromatic solution in positive integers for any $r \geq 1$

and r-colouring of the positive integers, while $x_1 < x_2$ follows from the last equation.

(2): In the case of $a \geq b+2$, consider the following 2-colouring of the positive integers: colour the integers in the interval $[a^i, a^{i+1}[$ by red or blue if $i \geq 0$ is even or odd, respectively.

From the remark at the beginning of the proof, we immediately have $x_2 < x_3 < ax_2$ and $x_3 < x_4 < ax_3$. Similarly to the proof of Theorem 2.1, we can deduce that monochromaticity of x_2, x_3, x_4 would imply that x_2, x_3, x_4 all belong to the same interval $[a^i, a^{i+1}[$. However, this is impossible because

$$ax_2 < (a+2)x_2 \leq (a^2 - b - ab)x_2 < (a^2 - b)x_2 - abx_1 = x_4.$$

□

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