Contributions to Discrete Mathematics

Volume 13, Number 2, Pages 112–115
ISSN 1715-0868

# A LOWER BOUND ON THE HYPERGRAPH RAMSEY NUMBER $R(4,5;3)$

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Abstract. The finite version of Ramsey's theorem says that for positive integers $r$, $k$, $a_1,\ldots,a_r$, there exists a least number $n = R(a_1,\ldots, a_r;k)$ so that if $X$ is an $n$-element set and all $k$-subsets of $X$ are $r$-coloured, then there exists an $i$ and an $a_i$-set $A$ so that all $k$-subsets of $A$ are coloured with the $i$th colour.

In this paper, the bound $R(4,5;3) \geq 35$ is shown by using a SAT solver to construct a red–blue colouring of the triples chosen from a 34-element set.

## 1. Introduction

In 1930, Ramsey proved the following theorem:

**Theorem 1.1** ([11])**.** *Let $r,k,a_1,\ldots,a_r$ be given positive integers. Then there is an integer $n$ with the following property. If all $k$-subsets of an $n$-set are coloured with $r$ colours, then for some $i$, $1 \leq i \leq r$, there exists an $a_i$-set entirely coloured in colour $i$ (all of its $k$-subsets have colour $i$).*

The smallest $n$ for which Ramsey's theorem holds, we call a Ramsey number and is denoted by $R(a_1,\ldots,a_r;k)$. This notation is used by the survey by Radziszowski [10]. Note that there are at least two other notations for these numbers in the literature, namely: $R_k(a_1,\ldots,a_r)$, used for example in [5], or $R^{(k)}(a_1,\ldots,a_r)$, used in [2]. Since colouring of all $k$-subsets can be viewed as colouring of the edges of complete $k$-uniform hypergraphs, numbers $R(a_1,\ldots,a_r;k)$, for $k \geq 3$, are also called hypergraph Ramsey numbers.

For $k = 2$, only ten exact values for nontrivial Ramsey numbers are known (see [10] for details). For $k = 3$, only one exact nontrivial value is known, namely $R(4,4;3) = 13$, where $R(4,4;3) \geq 13$ was proved in 1969 by Isbell [7] and equality was shown by McKay and Radziszowski [9] in 1991.

In this paper we deal with the number $R(4,5;3)$. In 1983 Isbell [8] proved that $R(4,5;3) \geq 24$, and in 1998 Exoo [4] presented a colouring which gives the bound $R(4,5;3) \geq 33$. Up to the author's knowledge

Received by the editors July 27, 2016, and in revised form November 8, 2017.
2010 *Mathematics Subject Classification.* 05C15.
*Key words and phrases.* Ramsey numbers, Hypergraphs.

the best upper bound for $R(4,5;3)$ can be obtained by using one step of the Erdős–Szekeres recursion [3] and known bounds for Ramsey numbers with smaller parameters, see [10]. These give the estimation $R(4,5;3) \le R(R(3,5;3), R(4,4;3);2) + 1 = R(5,13;2) + 1 \le 1139$. In this paper we show that $R(4,5;3) \ge 35$. This bound is shown by producing a 2-colouring of the triples of set with 34-elements. The colouring is formed by dividing all triples into classes and then by looking through 2-colouring of the classes. Similar approaches were used to establish lower bound for graph Ramsey numbers ($k = 2$). For example, Harborth and Krause [6] searched through colourings such that the colouring matrix is partitioned into cyclic orbits.

## 2. Colouring

**Theorem 2.1.** *Let $V = \{0, \dots, 33\}$. There exists a colouring $c : \binom{V}{3} \to \{red, blue\}$, such that no 4-subset of $V$ is entirely coloured in red, and no 5-subset of $V$ is entirely coloured in blue.*

*Proof.* This colouring is achieved by first dividing all such triples into 176 classes and then by giving a 2-colouring of the classes. For each two integers $a, b$ satisfying $1 \le a \le 11$ and $a + 1 \le b \le 22$, define the class:

$$C_{ab} = \Big\{\{0 + d, a + d, b + d\} : 0 \le d \le 33\Big\},$$

where addition is modulo 34. It is easy to see that there are 176 classes, each contains 34 3-sets and the classes are pairwise disjoint. Hence, we have a partition of $\binom{V}{3}$ into 176 disjoint classes. To find a proper colouring of the classes we use a SAT solver. We construct a formula with 176 variables—one for every class $C_{ab}$. For every $E \in \binom{V}{3}$, let $f(E)$ be the variable for the class that contains $E$. We assume that $E$ is blue if the variable $f(E)$ is true, and $E$ is red if $f(E)$ is false. We use the following formula:

$$\Big[\bigwedge_{S\in\binom{V}{4}} \bigvee_{E\in\binom{S}{3}} f(E)\Big] \wedge \Big[\bigwedge_{S\in\binom{V}{5}} \bigvee_{E\in\binom{S}{3}} \neg f(E)\Big].$$

The first part of the formula says that in every 4-set at least one of 3-subsets is coloured in blue (variable is true). Similarly, the second part says that in every 5-sets at least one of 3-subsets is coloured in red. It may happen that for some $S \in \binom{V}{4}$ we have two triples $E$ and $E'$ such that $f(E) = f(E')$ and we have two repeated literals in the clause $\bigvee_{E\in\binom{S}{3}} f(E)$. We simplify the clauses by removing such repetitions. Similarly it may happen that for two sets $S$ and $S' \in \binom{V}{4}$ the clauses $\bigvee_{E\in\binom{S}{3}} f(E)$ and $\bigvee_{E\in\binom{S'}{3}} f(E)$ are equivalent. We simplify the formula by removing such repetitions. Similarly we simplify the second part of the formula.

Finally, a formula with 176 variables and 9552 clauses is found. We use the SparrowToRiss [1] SAT solver and find out that the formula is satisfied. The SAT solver finishes in less than five minutes on a personal computer[1]

[1]Computer with processor Intel® Core™ i7-4790, 3.60GHz.

| a\b | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 2 | | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| 3 | | | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 4 | | | | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 5 | | | | | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 6 | | | | | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 7 | | | | | | | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 8 | | | | | | | | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| 9 | | | | | | | | | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 10 | | | | | | | | | | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 11 | | | | | | | | | | | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |

FIGURE 1. The colouring of the classes $C_{ab}$ without red 4-subset and blue 5-subset.

and returns the assignment that gives the proper colouring of the classes presented in Figure 1.

One can easily find, using a computer, that the colouring is proper, i.e. each clique $K_4$ contains at least one 3-set coloured in blue and each clique $K_5$ contains at least one 3-set coloured in red. On the website, `https://www.inf.ug.edu.pl/ramsey`, we posted a simplified formula, satisfying assignment, the corresponding colouring of all triples, and a C++ program that verifies that the colouring is proper. □

## Acknowledgement

The author would like to thank the anonymous reviewer for helpful comments that contributed to improving the final version of the paper.

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