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α-RESOLVABLE λ-FOLD G-DESIGNS

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ABSTRACT. A λ-fold G-design is said to be α-resolvable if its blocks can be partitioned into classes such that every class contains each vertex exactly α times. In this paper we study the existence problem of an α-resolvable λ-fold G-design of order v in the case when G is any connected subgraph of K_4 and prove that the necessary conditions for its existence are also sufficient.

1. INTRODUCTION

For any graph Γ, let $V(\Gamma)$ and $E(\Gamma)$ be the vertex set and the edge set of Γ, respectively, and $\lambda\Gamma$ be the graph Γ with each of its edges replicated λ times. Throughout the paper K_v will denote the complete graph on v vertices, while $K_v \setminus K_h$ will denote the graph with $V(K_v)$ as the vertex set and $E(K_v) \setminus E(K_h)$ as the edge set (this graph is sometimes referred to as a complete graph of order v with a *hole* of size h), and $K_{n_1,n_2,\ldots,n_t}$ will denote the complete multipartite graph with t parts of sizes $n_1, n_2, \ldots, n_t$.

Let G and H be simple finite graphs. A *λ-fold G-design* of H (or $(\lambda H, G)$-design for short) is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is a collection of isomorphic copies (called *blocks*) of the graph G, whose edges partition $E(\lambda H)$. If $\lambda = 1$, we drop the term "1-fold". If $H = K_v$, we refer to such a λ-fold G-design as one of order v. A $(\lambda H, G)$-design is *balanced* if, for every vertex x of H, the number of blocks containing x is a constant r.

A $(\lambda H, G)$-design is said to be *α-resolvable* if it is possible to partition the blocks into classes (often referred to as *α-parallel classes*) such that every vertex of H appears in exactly α blocks of each class. When $\alpha = 1$, we simply speak of resolvable designs and parallel classes. The existence problem of resolvable G-decompositions has been the subject of extensive research (see [1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 15, 16, 17, 20, 19]). The α-resolvability, with $\alpha > 1$, has been studied for $G = K_3$ by D. Jungnickel, R. C. Mullin, S. A. Vanstone [9]; Y. Zhang and B. Du [22]; $G = K_4$ by M. J. Vasiga, S. Furino

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and A. C. H. Ling [18]; $G = C_4$ by M. X. Wen and T. Z. Hong [21]; and $G = K_4 - e$ by M. Gionfriddo, G. Lo Faro, S. Milici, and A. Tripodi [5].

In this paper we shall focus on the existence of an α-resolvable λ-fold G-design when $G = P_3, P_4, K_{1,3}, K_3 + e$ (where $K_3 + e$ is a *kite*, i.e., a triangle with a tail consisting of a single edge) completely solving the spectrum problem for any connected subgraph of K_4.

In what follows, we will denote by:

- $P_k = [a_1, a_2, \dots, a_k]$, $k \geq 3$, the simple graph on the k vertices $a_1, a_2, \dots, a_k$ with $\{\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{k-1}, a_k\}\}$ as the edge set;
- $K_{1,3} = (a_1; a_2, a_3, a_4)$ the 3-star on the vertex set $\{a_1, a_2, a_3, a_4\}$ with $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$ as the edge set;
- $K_3 + e = (a_1, a_2, a_3) - a_4$ the kite on the vertex set $\{a_1, a_2, a_3, a_4\}$ with $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_3, a_4\}\}$ as the edge set.

By the definition of α-resolvability, we can derive the following necessary conditions:

$$\lambda v(v-1) \equiv 0 \pmod{2|E(G)|}\,; \tag{1.1}$$

$$\alpha v \equiv 0 \pmod{|V(G)|}\,; \tag{1.2}$$

$$\lambda |V(G)|(v-1) \equiv 0 \pmod{2\alpha|E(G)|}\,. \tag{1.3}$$

Note that any α-resolvable λ-fold G-design is balanced because every vertex of $V(G)$ appears exactly α times in each α-parallel class. Let $D(G)$ be the set of all degrees of the vertices of G. For every vertex x of an α-resolvable λ-fold G-design $\mathcal{D}$ of order v and for every $d \in D(G)$, let $r_d(x)$ denote the number of blocks of $\mathcal{D}$ containing x as a vertex of degree d. It is easy to see that the following relations hold:

$$\sum_{d \in D(G)} r_d(x) d = \lambda(v-1); \tag{1.4}$$

$$\sum_{d \in D(G)} r_d(x) = \lambda |V(G)| \frac{v-1}{2|E(G)|}. \tag{1.5}$$

From Conditions (1.1) – (1.5) we can deduce minimum values for α and λ, say α_0 and λ_0, respectively.

For any graph $G \in \{P_3, P_4, K_{1,3}, K_3 + e\}$, similarly to Lemmas 2.1, 2.2 in [18], we have the following lemmas.

Lemma 1.1. *If an α-resolvable λ-fold G-design of order v exists, then $\alpha_0 |\, \alpha$ and $\lambda_0 |\, \lambda$.*

Lemma 1.2. *If an α-resolvable λ-fold G-design of order v exists, then a $t\alpha$-resolvable $n\lambda$-fold G-design of order v exists for any positive integers n and t where t divides $\lambda |V(G)|(v-1)/(2\alpha|E(G)|)$.*

The above two lemmas imply the following theorem (for the proof, see Theorem 2.3 in [18]).

Theorem 1.3. *If an α_0-resolvable λ_0-fold G-design of order v exists and α and λ satisfy Conditions* (1.1) – (1.5)*, then an α-resolvable λ-fold G-design of order v exists.*

Therefore, in order to show that the necessary conditions for α-resolvable designs are also sufficient, we simply need to prove the existence of an α_0-resolvable λ_0-fold G-design of order v, for any given v.

2. Auxiliary definitions

A $(\lambda K_{n_1,n_2,\ldots,n_t}, G)$-design is known as a λ-fold *group divisible design* (or G-GDD for short), of type $\{n_1, n_2, \ldots, n_t\}$ (the parts are called the *groups* of the design). We usually use "exponential" notation to describe group-types: the group-type $1^i 2^j 3^k$... denotes i occurrences of 1, j occurrences of 2, etc. When $G = K_n$ we will call it an n-GDD.

If the blocks of a λ-fold G-GDD can be partitioned into *partial* α-parallel classes, each of which contains all vertices except those of one group, we refer to the decomposition as a λ-fold (α, G)-*frame*; when $\alpha = 1$, we simply speak of λ-fold G-frames (n-frames if additionally $G = K_n$). In a λ-fold (α, G)-frame the number of partial α-parallel classes missing a specified group of size g is $\lambda g|V(G)|/(2\alpha|E(G)|)$.

An *incomplete* α-resolvable λ-fold G-design of order $v + h$, $h \geq 1$, with a hole of size h is a $(\lambda(K_{v+h} \setminus K_h), G)$-design in which there are two types of classes, $\lambda(h-1)|V(G)|/(2\alpha|E(G)|)$ partial classes which cover every vertex α times except those in the hole and $\lambda v|V(G)|/(2\alpha|E(G)|)$ *full* classes which cover every vertex of K_{v+h} α times.

3. The case $\mathbf{G} = \mathbf{P_3}$

In this section the existence of an α_0-resolvable λ_0-fold P_3-design of any order v is proved by distinguishing the following cases.

Case 1: $v \equiv 0 \pmod 6$: $\lambda_0 = 4$ and $\alpha_0 = 1$.
For a solution, see [4].

Case 2: $v \equiv 1, 5 \pmod{12}$: $\lambda_0 = 1$ and $\alpha_0 = 3$.
In Z_v develop the base blocks: $[i, 0, (v+1)/2 - i]$, $i = 1, 2, \ldots, (v-1)/4$.

Case 3: $v \equiv 2, 4, 8, 10 \pmod{12}$: $\lambda_0 = 4$ and $\alpha_0 = 3$.
In Z_v develop the base blocks: $[i, 0, 1+i]$, $i = 1, 2, \ldots, v-2$; $[v-1, 0, 1]$.

Case 4: $v \equiv 3 \pmod{12}$: $\lambda_0 = 2$ and $\alpha_0 = 1$..
For a solution see [4].

Case 5: $v \equiv 7, 11 \pmod{12}$: $\lambda_0 = 2$ and $\alpha_0 = 3$.
In Z_v develop the base blocks: $[i, 0, v - i]$, $i = 1, 2, \ldots, (v-1)/2$.

Case 6: $v \equiv 9 \pmod{12}$: $\lambda_0 = 1$ and $\alpha_0 = 1$.
For a solution see [4].

4. The case $\mathbf{G} = \mathbf{P_4}$

Here, we construct an α_0-resolvable λ_0-fold P_4-design of any order v.

Case 1: $v \equiv 0, 8 \pmod{12}$: $\lambda_0 = 3$ and $\alpha_0 = 1$.

For a solution see [4].

Case 2: $v \equiv 1 \pmod 6$: $\lambda_0 = 1$ and $\alpha_0 = 4$.

In Z_v develop the base blocks: $[i, 0, (v+2)/3 - i, (2v+1)/3]$, $i = 1, 2, \dots, (v-1)/6$.

Case 3: $v \equiv 2, 6 \pmod{12}$: $\lambda_0 = 3$ and $\alpha_0 = 2$.

Let $Z_{v/2} \times Z_2$ be the vertex set. In $Z_{v/2}$ develop the base blocks: $[i_0, 0_0, i_1, 0_1]$, $i = 1, 2, \dots, (v-2)/2$; $[i_0, 0_0, i_1, 0_1]$, $i = 1, 2, \dots, (v-2)/4$; $[((v+2)/4 + i)_1, 0_0, i_1, ((v-2)/4)_0]$, $i = 0, 1, \dots, (v-6)/4$; $[0_1, 0_0, ((v-2)/4)_1, ((v-2)/4)_0]$.

Case 4: $v \equiv 3, 5 \pmod 6$: $\lambda_0 = 3$ and $\alpha_0 = 4$.

In Z_v develop the base blocks: $[i, 0, v-i, (v-1)/2]$, $i = 1, 2, \dots, (v-3)/2$; $[(v-1)/2, 0, 1, (v+1)/2]$.

Case 5: $v \equiv 4 \pmod{12}$: $\lambda_0 = 1$ and $\alpha_0 = 1$.

For a solution see [4].

Case 6: $v \equiv 10 \pmod{12}$: $\lambda_0 = 1$ and $\alpha_0 = 2$.

Let $v = 12k + 10$ and $Z_{6k+5} \times Z_2$ be the vertex set. In Z_{6k+5} develop the base blocks: $[i_0, 0_0, i_1, 0_1]$, $i = 1, 2, \dots, 3k+2$; $[(3k+3+i)_1, 0_0, (5k+2-i)_1, (5k+3)_0]$, $i = 0, 1, \dots, k-1$; $[(5k+3)_1, 0_0, 0_1, (k+1)_0]$.

5. The case $\mathbf{G} = \mathbf{K_{1,3}}$

To solve the spectrum problem for α-resolvable λ-fold $K_{1,3}$-designs we distinguish the following cases.

Case 1: $v \equiv 0, 8 \pmod{12}$: $\lambda_0 = 6$ and $\alpha_0 = 1$.

For a solution see [4].

Case 2: $v \equiv 1 \pmod 6$: $\lambda_0 = 1$ and $\alpha_0 = 4$.

In Z_v develop the base blocks: $(0; i, (v-1)/6 + i, (v-1)/3 + i)$, $i = 1, 2, \dots, (v-1)/6$.

Case 3: $v \equiv 2 \pmod{12}$: $\lambda_0 = 6$ and $\alpha_0 = 2$.

Let $v = 12k+2$ and $Z_{12k+1} \cup \{\infty\}$ be the vertex set. In Z_{12k+1} develop the two base classes:

P_1: $\{(12k - i + 1; i, 12k - 2i + 1, 12k - 2i + 2) : i = 2, 3, \dots, 6k - 1\} \cup \{(\infty; 0, 1, 12k), (12k; 1, 12k-1, \infty), (6k+1; 0, 2, 6k)\}$;

P_2: $\{(12k - i + 1;\ i,\ 12k - 2i,\ , 12k - 2i + 1),\ i = 2, 3, \dots, 6k,\ i \neq 4k\} \cup \{(12k; 0, 4k, 12k-1), (12k; 1, 12k-2, \infty), (8k+1; 4k, 4k+1, \infty)\}$.

Case 4: $v \equiv 3, 5 \pmod 6$: $\lambda_0 = 3$ and $\alpha_0 = 4$.

In Z_v develop the base blocks: $(0; i, v-i, 1+i)$, $i = 1, 2, \dots, (v-3)/2$; $(0; (v-1)/2, (v+1)/2, 1)$.

Case 5: $v \equiv 4 \pmod{12}$: $\lambda_0 = 2$ and $\alpha_0 = 1$.

For a solution see [4].

Case 6: $v \equiv 6 \pmod{12}$, then $\lambda_0 = 6$ and $\alpha_0 = 2$.

Let $v = 12k+6$ and $Z_{12k+5} \cup \{\infty\}$ be the vertex set. In Z_{12k+5} develop the two base classes:

P_1: $\{(12k-i+5;\, i,\, 12k-2i+5,\, 12k-2i+6) : i = 2,3,\dots,6k+1,\ i \neq 4k+2\} \cup \{(\infty;\, 0,\, 1,\, 4k+2), (12k+4;\, 1,\, 12k+3,\, \infty), (6k+3; 0, 2, 6k+2), (8k+3; 4k+1, 4k+2, 12k+4)\}$;

P_2: $\{(12k-i+5; i, 12k-2i+4, 12k-2i+5) : i = 2,3,\dots,6k+2\} \cup \{(12k+4; 0, 12k+3, \infty), (12k+4; 1, 12k+2, \infty)\}$.

Case 7: $v \equiv 10 \pmod{12}$: $\lambda_0 = 2$ and $\alpha_0 = 2$.

This case follows by the following lemmas.

Lemma 5.1. *There exists an incomplete* 2*-resolvable* 2*-fold* $K_{1,3}$*-design of order* 10 *with a hole of size* 4*.*

Proof. Let $V = Z_6 \cup H$ be the vertex set, where $H = \{\infty_1, \infty_2, \infty_3, \infty_4\}$ is the hole. The partial classes are:

$$\{(3;4,0,2),(4;5,1,0),(5;3,2,1)\},\{(0;4,5,1),(1;5,3,2),(2;3,4,0)\}.$$

The full classes are:

$$\begin{aligned}
&\{(\infty_1;0,1,2),(\infty_2;0,1,2),(3;4,\infty_3,\infty_4),(4;5,\infty_3,\infty_4),(5;3,\infty_1,\infty_2)\},\\
&\{(\infty_2;3,4,5),(\infty_3;3,4,5),(0;1,\infty_1,\infty_4),(1;2,\infty_1,\infty_4),(2;0,\infty_2,\infty_3)\},\\
&\{(\infty_3;0,1,2),(\infty_4;3,4,5),(3;0,\infty_1,\infty_2),(4;1,\infty_1,\infty_2),(5;2,\infty_3,\infty_4)\},\\
&\{(\infty_4;0,1,2),(\infty_1;3,4,5),(0;5,\infty_2,\infty_3),(1;3,\infty_2,\infty_3),(2;4,\infty_1,\infty_4)\}.
\end{aligned}$$

□

As a consequence of Lemma 5.1 and the existence of a 2-resolvable 2-fold $K_{1,3}$-design of order $v = 4$, the following lemma is obtained.

Lemma 5.2. *There exists a* 2*-resolvable* 2*-fold* $K_{1,3}$*-design of order* $v = 10$*.*

Lemma 5.3. *There exists a* 2*-resolvable* 2*-fold* $K_{1,3}$*-GDD of type* 6^2*.*

Proof. Take $\{a,b,c,d,e,f\}$ and $\{1,2,3,4,5,6\}$ as groups and consider the classes:

$$\begin{aligned}
&\{(a;1,2,4),(b;2,3,5),(c;3,1,6),(4;d,f,a),(5;e,d,b),(6;f,e,c)\},\\
&\{(a;2,5,6),(b;3,4,6),(c;1,5,4),(1;d,f,b),(2;e,d,c),(3;f,e,a)\},\\
&\{(d;3,4,6),(e;1,4,5),(f;2,5,6),(1;b,e,a),(2;c,f,b),(3;d,a,c)\},\\
&\{(d;1,2,5),(e;2,3,6),(f;3,1,4),(4;b,e,c),(5;c,f,a),(6;d,a,b)\}.
\end{aligned}$$

□

Lemma 5.4. *For every* $v \equiv 10 \pmod{12}$*, there exists a* 2*-resolvable* 2*-fold* $K_{1,3}$*-design of order* v*.*

Proof. Let v=$12k+10$. The case $v = 10$ follows by Lemma 5.2. For $k \geq 1$, start from a 2-frame of type 1^{2k+1} with groups G_i, $i = 1,2,\dots,2k+1$, expand each vertex six times and add a set H of size 4 such that $H \cap (\cup_{i=1}^{2k+1} G_i) = \emptyset$. For $i = 1,2,\dots,2k+1$, let P_i be the partial class which misses the group G_i and for each block $b \in P_i$ place on $b \times \{1, 2, \ldots, 6\}$ a copy of a 2-resolvable 2-fold $K_{1,3}$-GDD of type 6^2, which exists by Lemma 5.3; this gives four partial classes missing $G_i \times \{1, 2, \ldots, 6\}$, say $P_{i,1}, P_{i,2}, P_{i,3}, P_{i,4}$. For $i = 1, 2, \ldots, 2k+1$, place on $H \cup (G_i \times \{1, 2, \ldots, 6\})$ a copy $\mathcal{D}_i$ of an incomplete 2-resolvable 2-fold $K_{1,3}$-design of order 10 with a hole of size 4, which exists by Lemma 5.1. Finally, filling in the hole H with a copy $\mathcal{D}$ of a 2-resolvable 2-fold $(K_{1,3})$-design of order 4 gives a 2-fold $(K_{1,3})$-design of order v which is also 2-resolvable. Indeed, for every $i = 1, 2, \ldots, 2k+1$ combining $P_{i,1}, P_{i,2}, P_{i,3}, P_{i,4}$ with the full classes of $\mathcal{D}_i$ gives four 2-parallel classes, while combining the two classes of $\mathcal{D}$ with the union of the partial classes of $\mathcal{D}_i$, $i = 1, 2, \ldots, 2k+1$ gives the remaining ones. □

6. The case $\mathbf{G = K_3 + e}$

For $G = K_3 + e$ we have the following cases with the corresponding solutions.

Case 1: $v \equiv 0 \pmod 4$: $\lambda_0 = 2$ and $\alpha_0 = 1$.

For a solution see [4].

Case 2: $v \equiv 1 \pmod 8$: $\lambda_0 = 1$ and $\alpha_0 = 4$.

In Z_{8k+1} develop the base blocks ([13]): $(4k-i, 2k+1+i,\ 0)-(2k-2i)$, $i = 0, 1, \ldots, k-1$.

Case 3: $v \equiv 2 \pmod 4$: $\lambda_0 = 4$ and $\alpha_0 = 2$.

Let $Z_{2k+1} \times Z_2$ be the vertex set. In Z_{2k+1} develop the base blocks: $(i_j, (2k+1-i)_j, 0_{j+1}) - i_{j+1}$, $i = 1, 2, \ldots, k$, $j \in Z_2$; $(i_j, (2k-1-i)_j, 0_{j+1}) - (i+1)_{j+1}$, $i = 1, 2, \ldots, k-1$, $j \in Z_2$; $(1_0, 1_1, 0_0) - 2_1$, $(1_1, 1_0, 0_1) - 0_0$, $(0_0, 2_0, 0_1) - 2_1$.

Case 4: $v \equiv 3 \pmod 4$: $\lambda_0 = 4$ and $\alpha_0 = 4$.

In Z_{4k+3} develop the base blocks: $(i, 4k+3-i,\ 0) - (1+i)$, $i = 1, 2, \ldots, 2k$; $(2k+1, 2k+2,\ 0) - 1$.

Case 5: $v \equiv 5 \pmod 8$: $\lambda_0 = 2$ and $\alpha_0 = 4$.

In Z_{8k+5} develop the base blocks ([14]): $(i, 4k+3-i,\ 0)-(4k-1+2i)$, $i = 1, 2, \ldots, 2k+1$.

7. Main result

Theorem 1.3 along with the results of the previous sections allows us to obtain our main result.

Theorem 7.1. *For any graph $G \in \{P_3, P_4, K_{1,3}, K_3+e\}$, the necessary conditions* (1.1) − (1.5) *for the existence of α-resolvable λ-fold G-designs are also sufficient.*

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