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A THEOREM ON FRACTIONAL ID-(g, f)-FACTOR-CRITICAL GRAPHS

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Abstract. Let a, b and r be three nonnegative integers with $2 \leq a \leq b-r$, let G be a graph of order p satisfying the inequality $p(a+r) \geq (a+b-3)(2a+b+r)+1$, and let g and f be two integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x)-r \leq b-r$ for every $x \in V(G)$. A graph G is said to be fractional ID-(g, f)-factor-critical if $G-I$ contains a fractional (g, f)-factor for every independent set I of G. In this paper, we prove that G is fractional ID-(g, f)-factor-critical if $\mathrm{bind}(G)((a+r)p-(a+b-2)) > (2a+b+r-1)(p-1)$, which is a generalization of a previous result of Zhou.

1. Introduction

The graphs considered here are finite undirected graphs which have neither loops nor multiple edges. Let $G=(V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote its vertex set and edge set. For every $x \in V(G)$, we denote by $d_G(x)$ the degree of x and by $N_G(x)$ the set of vertices adjacent to x in G. For a subset S of $V(G)$, we write $N_G(S)=\bigcup_{x \in S} N_G(x)$, $G[S]$ for the subgraph of G induced by S, and we define $G-S=G[V(G) \setminus S]$. The minimum degree of G is denoted by $\delta(G)$, while a subset S of $V(G)$ is said to be independent if $G[S]$ has no edges. The binding number of G is denoted by $\mathrm{bind}(G)$ and defined as

$$\mathrm{bind}(G)=\min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ satisfying $g(x) \leq f(x)$ for any $x \in V(G)$. A spanning subgraph F of G is a

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(g, f)-factor if $g(x) \le d_F(x) \le f(x)$ for any $x \in V(G)$. Assume there exists a function $h : E(G) \to [0, 1]$ such that

$$g(x) \le \sum_{e \ni x} h(e) \le f(x)$$

for every vertex x of G. The spanning subgraph of G induced by the set of edges $\{e : e \in E(G), h(e) > 0\}$ is called a fractional (g, f)-factor of G with indicator function h.

Definition 1.1. *A graph G is said to be fractional ID-(g, f)-factor-critical if $G - I$ contains a fractional (g, f)-factor for every independent set I of G.*

A fractional ID-(f, f)-factor-critical graph is a fractional ID-f-factor-critical graph. If $f(x) \equiv k$, then we say a fractional ID-k-factor-critical graph instead of a fractional ID-f-factor-critical graph. For any function $f(x)$ and $S \subseteq V(G)$, we define

$$f(S) = \sum_{x \in S} f(x).$$

In particular, note that

$$d_G(S) = \sum_{x \in S} d_G(x).$$

A huge amount of work has been done concerning factors and fractional factors in graphs (see [1, 4, 5, 6, 8]). In [3] Chang, Liu, and Zhu first investigated the fractional ID-k-factor-critical graph and obtained a minimum degree condition for a graph to be a fractional ID-k-factor-critical graph. This result is summarized below:

Theorem 1.2 (Chang, Liu, and Zhu [3])**.** *Let k be a positive integer and G be a graph of order p with $p \ge 6k - 8$. If $\delta(G) \ge 2p/3$, then G is fractional ID-k-factor-critical.*

In [11] Zhou, Xu, and Sun proved the following result on the fractional ID-k-factor-critical graphs:

Theorem 1.3 (Zhou, Xu, and Sun [11])**.** *Let G be a graph, and let k be an integer with $k \ge 1$. If*

$$\alpha(G) \le \frac{4k(\delta(G) - k + 1)}{k^2 + 6k + 1},$$

then G is fractional ID-k-factor-critical.

Zhou studied the relationship between binding number and the fractional ID-k-factor-critical graph in [10] and proved the following theorem:

Theorem 1.4 (Zhou [10])**.** *Let k be an integer with $k \ge 2$, and let G be a graph of order p with $p \ge 6k - 9$. If*

$$\mathrm{bind}(G) > \frac{(3k - 1)(p - 1)}{kp - 2k + 2},$$

then G is fractional ID-k-factor-critical.

In this work, we generalize the fractional ID-k-factor-critical graph to the fractional ID-(g, f)-factor-critical graph and obtain a binding number condition for a graph to be fractional ID-(g, f)-factor-critical:

Theorem 1.5. *Let a, b, and r be three integers such that $2 \le a \le b - r$ and $r \ge 0$, let G be a graph of order p, where*

$$p \ge \frac{(a+b-3)(2a+b+r)+1}{a+r},$$

and let both g and f be nonnegative integer-valued functions defined on $V(G)$, where $a \le g(x) \le f(x) - r \le b - r$ for any $x \in V(G)$. If

$$\operatorname{bind}(G) > \frac{(2a+b+r-1)(p-1)}{(a+r)p-(a+b-2)},$$

then G is fractional ID-(g, f)-factor-critical.

We obtain the following corollary by setting $r = 0$ in Theorem 1.5:

Corollary 1.6. *Let a and b be two integers with $2 \le a \le b$, and let G be a graph of order p, where*

$$p \ge \frac{(a+b-3)(2a+b)+1}{a},$$

and let g and f be nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for any $x \in V(G)$. If

$$\operatorname{bind}(G) > \frac{(2a+b-1)(p-1)}{ap-(a+b-2)},$$

then G is fractional ID-(g, f)-factor-critical.

If $g(x) \equiv f(x)$ in Corollary 1.6, then we have the following result:

Corollary 1.7. *Let a and b be two integers satisfying $2 \le a \le b$, and let G be a graph of order p with*

$$p \ge \frac{(a+b-3)(2a+b)+1}{a},$$

and let f be a nonnegative integer-valued function defined on $V(G)$, where $a \le f(x) \le b$ for any $x \in V(G)$. If

$$\operatorname{bind}(G) > \frac{(2a+b-1)(p-1)}{ap-(a+b-2)},$$

then G is fractional ID-f-factor-critical.

2. Proof of Theorem 1.4

The following result was first obtained by Anstee [2], and it is very useful for proving Theorem 1.5. An alternative proof was provided by Liu and Zhang in [7].

Lemma 2.1 (Anstee [2], Liu and Zhang [7])**.** *Let G be a graph. Then G has a fractional (g, f)-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$.

In [9] Woodall presented the following result, which will also be used in the proof of Theorem 1.5:

Lemma 2.2 (Woodall [9])**.** *Let c be a positive real number and let G be a graph of order p with* $\operatorname{bind}(G) > c$*. Then*

$$\delta(G) \geq p - \frac{p-1}{\operatorname{bind}(G)} > p - \frac{p-1}{c}.$$

Proof of Theorem 1.5. Let X be an independent set of G and $H = G - X$. In order to prove Theorem 1.5, by Definition 1.1 we only need to prove that H admits a fractional (g, f)-factor.

Suppose that H has no fractional (g, f)-factor. Then from Lemma 2.1, there exists some subset S of $V(H)$ satisfying

$$\delta_H(S,T) = f(S) + d_{H-S}(T) - g(T) \leq -1, \tag{1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq g(x)\}$.

Henceforth we write $\operatorname{bind}(G) = \lambda$. In terms of Lemma 2.2 and the hypotheses of Theorem 1.5, we obtain the inequality

$$\delta(G) \geq p - \frac{p-1}{\lambda} > \frac{(a+b-1)p + a + b - 2}{2a + b + r - 1}. \tag{2}$$

Assume, in order to derive a contradiction, that $T = \emptyset$. Then using Equation (1) we derive that

$$-1 \geq \delta_H(S,T) = f(S) \geq 0,$$

which is a contradiction. Therefore $T \neq \emptyset$.

In the following, we set $h = \min\{d_{H-S}(x) : x \in T\}$. Obviously, $0 \leq h \leq b - r$. We now must prove the following claims:

Claim 2.3. $|S| \geq \delta(G) - |X| - h$.

Proof. We choose $x_1 \in T$ with $d_{H-S}(x_1) = h$. Clearly, we have

$$\begin{aligned}\delta(G) &\leq d_G(x_1) \leq d_{G-X-S}(x_1) + |X| + |S| \\ &= d_{H-S}(x_1) + |X| + |S| = h + |X| + |S|,\end{aligned}$$

which implies

$$|S| \geq \delta(G) - |X| - h.$$

This completes the proof of Claim 2.3. $\square$

Claim 2.4. $|X| \leq p - \delta(G)$.

Proof. Obviously, $d_G(x) \geq \delta(G)$ for any $x \in V(G)$. Consequently, $d_G(x) \geq \delta(G)$ for any $x \in X$. Because X is an independent set of G we have

$$p \geq d_G(x) + |X| \geq \delta(G) + |X|$$

for all $x \in X$, which implies

$$|X| \leq p - \delta(G).$$

This proves Claim 2.4. □

We now consider the following two cases regarding the value of h:

CASE 1: $h = 0$:

In this case, we first prove the following claim:

Claim 2.5. $\lambda \leq a + b - 1$.

Proof. Suppose that $\lambda > a + b - 1$. In view of Equation (2) and $2 \leq a \leq b - r$, we obtain

$$\delta(G) \geq p - \frac{p-1}{\lambda} > \frac{(a+b-2)p}{a+b-1} \geq \frac{(a+b)p}{2a+b+r}.$$

Combining this with Equation (1), the inequality $p \geq |X| + |S| + |T|$, and Claims 2.3 and 2.4, we have:

$$\begin{aligned}
-1 &\geq \delta_H(S,T) = f(S) + d_{H-S}(T) - g(T) \\
&\geq (a+r)|S| - (b-r)|T| \\
&\geq (a+r)|S| - (b-r)(p - |X| - |S|) \\
&= (a+b)|S| - (b-r)p + (b-r)|X| \\
&\geq (a+b)(\delta(G) - |X|) - (b-r)p + (b-r)|X| \\
&= (a+b)\delta(G) - (b-r)p - (a+r)|X| \\
&\geq (a+b)\delta(G) - (b-r)p - (a+r)(p - \delta(G)) \\
&= (2a+b+r)\delta(G) - (a+b)p > 0,
\end{aligned}$$

which is a contradiction. This completes the proof of Claim 2.5. □

Now set $Y = \{x : x \in T, d_{H-S}(x) = 0\}$. Note that $Y \neq \emptyset$ and $N_G(V(G) \setminus (X \cup S)) \cap Y = \emptyset$, which gives $|N_G(V(G) \setminus (X \cup S))| \leq p - |Y|$. Thus,

$$\text{bind}(G) = \lambda \leq \frac{|N_G(V(G) \setminus (X \cup S))|}{|V(G) \setminus (X \cup S)|} \leq \frac{p - |Y|}{p - |X| - |S|},$$

that is,

$$|S| \geq \left(1 - \frac{1}{\lambda}\right) p - |X| + \frac{1}{\lambda}|Y|. \tag{3}$$

It then follows from Equation (1) and the inequality $|X|+|S|+|T|\le p$ that:

$$\begin{aligned}-1 &\ge \delta_H(S,T)=f(S)+d_{H-S}(T)-g(T)\\ &\ge (a+r)|S|+|T|-|Y|-(b-r)|T|\\ &= (a+r)|S|-(b-r-1)|T|-|Y|\\ &\ge (a+r)|S|-(b-r-1)(p-|X|-|S|)-|Y|\\ &= (a+b-1)|S|-(b-r-1)p+(b-r-1)|X|-|Y|.\end{aligned}$$

Invoking Equation (3) then gives that:

$$\begin{aligned}&(a+b-1)|S|-(b-r-1)p+(b-r-1)|X|-|Y|\\ &\ge (a+b-1)\left(\left(1-\frac{1}{\lambda}\right)p-|X|+\frac{|Y|}{\lambda}\right)+(b-r-1)(|X|-p)-|Y|\\ &= (a+r)p-\frac{(a+b-1)p}{\lambda}-(a+r)|X|+\left(\frac{a+b-1}{\lambda}-1\right)|Y|.\end{aligned}$$

Claim 2.5 and the fact that $Y\neq\emptyset$ imply together the inequality

$$\begin{aligned}&(a+r)p-\frac{(a+b-1)p}{\lambda}-(a+r)|X|+\left(\frac{a+b-1}{\lambda}-1\right)|Y|\\ &\ge (a+r)p-\frac{(a+b-1)p}{\lambda}-(a+r)|X|+\frac{a+b-1}{\lambda}-1;\end{aligned}$$

applying Claim 2.4 then yields the following:

$$\begin{aligned}&(a+r)p-\frac{(a+b-1)p}{\lambda}-(a+r)|X|+\frac{a+b-1}{\lambda}-1\\ &\ge (a+r)p-\frac{(a+b-1)p}{\lambda}-(a+r)(p-\delta(G))+\frac{a+b-1}{\lambda}-1\\ &= -\frac{(a+b-1)p}{\lambda}+(a+r)\delta(G)+\frac{a+b-1}{\lambda}-1.\end{aligned}$$

Using Equation (2) allows us to conclude

$$\begin{aligned}&-\frac{(a+b-1)p}{\lambda}+(a+r)\delta(G)+\frac{a+b-1}{\lambda}-1\\ &\ge -\frac{(a+b-1)p}{\lambda}+(a+r)\left(p-\frac{p-1}{\lambda}\right)+\frac{a+b-1}{\lambda}-(a+b-1)\\ &= -\frac{(2a+b+r-1)(p-1)}{\lambda}+(a+r)p-(a+b-1),\end{aligned}$$

which implies

$$\lambda\le\frac{(2a+b+r-1)(p-1)}{(a+r)p-(a+b-2)},$$

contradicting the hypotheses of Theorem 1.5.

Case 2: $1\le h\le b-r$:

According to Equation (1), Claims 2.3 and 2.4, and the inequality $p \geq |S| + |T| + |X|$, we obtain:

$$\begin{aligned}
-1 &\geq \delta_H(S,T) = f(S) + d_{H-S}(T) - g(T) \\
&\geq (a+r)|S| - (b-r-h)|T| \\
&\geq (a+r)|S| - (b-r-h)(p - |X| - |S|) \\
&= (a+b-h)|S| + (b-r-h)|X| - (b-r-h)p \\
&\geq (a+b-h)(\delta(G) - |X| - h) + (b-r-h)|X| - (b-r-h)p \\
&= (a+b-h)\delta(G) - (a+r)|X| - h(a+b-h) - (b-r-h)p \\
&\geq (a+b-h)\delta(G) - (a+r)(p - \delta(G)) - h(a+b-h) - (b-r-h)p \\
&= (2a+b+r-h)\delta(G) - h(a+b-h) - (a+b-h)p,
\end{aligned}$$

that is,

$$\delta(G) \leq \frac{(a+b-h)(p+h)-1}{2a+b+r-h}. \tag{4}$$

If $h = 1$ in Equation (4), then we have

$$\delta(G) \leq \frac{(a+b-1)(p+1)-1}{2a+b+r-1},$$

which contradicts Equation(2). Hence we assume $2 \leq h \leq b - r$. Let

$$F(h) = \frac{(a+b-h)(p+h)-1}{2a+b+r-h}.$$

Using

$$p \geq \frac{(a+b-3)(2a+b+r)+1}{a+r},$$

we calculate $F'(h) < 0$, implying that $F(h)$ attains its maximum value at $h = 2$. Therefore we have

$$\delta(G) \leq F(2) = \frac{(a+b-2)(p+2)-1}{2a+b+r-2}. \tag{5}$$

Since

$$p \geq \frac{(a+b-3)(2a+b+r)+1}{a+r},$$

we prove easily that

$$\frac{(a+b-2)(p+2)-1}{2a+b+r-2} \leq \frac{(a+b-1)p+a+b-2}{2a+b+r-1}.$$

Combining this with Equation (5), we obtain

$$\delta(G) \leq \frac{(a+b-1)p+a+b-2}{2a+b+r-1},$$

which contradicts Equation (2). This completes the proof of Theorem 1.5.

$\square$

Finally, we present the following problem:

Problem. Is it possible to weaken the binding number condition

$$\mathrm{bind}(G) > \frac{(2a+b+r-1)(p-1)}{(a+r)p-(a+b-2)}$$

for the existence of fractional ID-(g, f)-factor-critical graphs in Theorem 1.5?

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