



## A SHORT CONSTRUCTION OF HIGHLY CHROMATIC DIGRAPHS WITHOUT SHORT CYCLES

MICHAEL SEVERINO

**ABSTRACT.** A natural digraph analogue of the graph-theoretic concept of an ‘independent set’ is that of an ‘acyclic set’, namely a set of vertices not spanning a directed cycle. Hence a digraph analogue of a graph coloring is a decomposition of the vertex set into acyclic sets. In the spirit of a famous theorem of P. Erdős [Graph theory and probability, Canad. J. Math. **11** (1959), 34–38], it was shown probabilistically in [D. Bokal et al., The circular chromatic number of a digraph, J. Graph Theory **46** (2004), no. 3, 227–240] that there exist digraphs with arbitrarily large girth and chromatic number. Here we give a construction of such digraphs.

In [2], it is shown that the coloring theory for digraphs is similar to the coloring theory for graphs when stable sets are replaced by acyclic sets and homomorphisms are replaced by ‘acyclic homomorphisms’. One of the results therein asserts the existence of digraphs with arbitrarily large girth and (digraph) chromatic number. This, of course, is analogous to the seminal theorem of Erdős [3] on graphs with arbitrarily large girth and chromatic number, and it is likewise proved probabilistically, whence non-constructively. It is worth noting that although many results about digraph coloring theory are generalizations of results about graphs, the aforementioned result in [2] is not a generalization of Erdős’ theorem because the relationship between independent sets and cycles in graphs is different from the relationship between acyclic sets and directed cycles in digraphs.

In this note, we *construct* digraphs with arbitrarily large girth and chromatic number. In fact, the construction strengthens the result in [2] because it produces a digraph with girth  $k$  and chromatic number  $n$  for each pair  $k, n$  of integers exceeding one. It is also of interest that unlike the analogous graph constructions in [4, 5, 6], our construction is primitively recursive in  $n$ .

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Received by the editors February 14, 2014, and in revised form March 11, 2014.

2010 *Mathematics Subject Classification.* 05C20, 05C15, 68R10.

*Key words and phrases.* Digraph, chromatic number, acyclic homomorphism, girth.

This work forms part of the author’s Ph.D. dissertation [7]; research supported in part by a grant from the Simons Foundation (# 279367 to Mark Kayll) and a UM Graduate Student Summer Research Award funded by the George and Dorothy Bryan Endowment.

Although basic terminology can be found in [1], we include the main definitions for completeness. The *girth* of a digraph  $D$ , denoted  $g(D)$ , is the length of its shortest directed cycle. Following [2], we define the *chromatic number*  $\chi(D)$  of  $D$  to be the minimum number of parts in a partition of  $V(D)$  into acyclic sets, and we say that  $D$  is *n-chromatic* if  $\chi(D) = n$ . An *acyclic homomorphism* from  $D$  to  $H$  is a mapping  $\phi : V(D) \rightarrow V(H)$  such that  $uv \in A(D)$  implies that either  $\phi(u)\phi(v) \in A(H)$  or  $\phi(u) = \phi(v)$ , and for all vertices  $x \in V(H)$ , the *fiber*  $\phi^{-1}(x)$  is acyclic. It is easy to check that the composition of two acyclic homomorphisms is again an acyclic homomorphism. We use the notation  $D \rightarrow H$  to denote that there exists an acyclic homomorphism from  $D$  to  $H$  and define  $K_n^*$  to be the complete bidirected digraph with vertex set  $[n]$ . As in the case of the graph coloring analogue, an equivalent definition of the chromatic number is  $\chi(D) = \min\{n \mid D \rightarrow K_n^*\}$ . In order to confirm the correctness of our construction, we will need the fact that  $D \rightarrow H$  implies that  $g(D) \geq g(H)$ , which is a direct consequence of Propositions 1.2 and 1.3 in [2]. It is worth noticing the subtle difference between the last statement and its graph analogue, which is true only for odd girth.

**Theorem 1.** *For any given integers  $k$  and  $n$  exceeding one, there exists an  $n$ -chromatic digraph  $D$  with  $g(D) = k$ .*

*Proof.* For  $n = 2$ , the directed  $k$ -cycle will suffice. For  $n \geq 2$ , we proceed by induction on  $n$  and suppose that we have already constructed a digraph  $D_n$  with chromatic number  $n$ , girth  $k$ , and  $V(D_n) = \{d_1, d_2, \dots, d_m\}$ . We now define  $D_{n+1}$ .

For each  $i \in [m]$  let  $D_n^i$  be a digraph with vertex set  $V(D_n^i) = \{(d_1, i), (d_2, i), \dots, (d_m, i)\}$ , which is isomorphic to  $D_n$  in the natural way. Next, construct  $m$  directed paths  $P_{d_i}$ , for  $1 \leq i \leq m$ , each of length  $k - 2$ , with vertex sets  $\{(d_i, p_1), (d_i, p_2), \dots, (d_i, p_{k-1})\}$  and arc sets  $A(P_{d_i}) = \{\overrightarrow{(d_i, p_j)(d_i, p_{j+1})} \mid j \in [k - 2]\}$ . Now define  $m$  digraphs  $H(n, i)$ , for  $1 \leq i \leq m$ , with vertex sets  $V(H(n, i)) := V(D_n^i) \cup V(P_{d_i})$ , and arc sets

$$\begin{aligned} A(H(n, i)) &:= A(D_n^i) \cup A(P_{d_i}) \\ &\cup \left\{ \overrightarrow{(d, i)(d_i, p_1)} \mid d \in V(D_n) \right\} \cup \left\{ \overrightarrow{(d_i, p_{k-1})(d, i)} \mid d \in V(D_n) \right\}. \end{aligned}$$

Finally, we define  $D_{n+1}$  to be the digraph with

$$V(D_{n+1}) := \bigcup_{i=1}^m V(H(n, i))$$

and

$$A(D_{n+1}) := \bigcup_{i=1}^m A(H(n, i)) \cup \left\{ \overrightarrow{(d_i, p_\ell)(d_j, p_h)} \mid d_i d_j \in A(D_n), \ell, h \in [k - 1] \right\}.$$

We first show that the girth of  $D_{n+1}$  is  $k$ . Since the girth of each  $H(n, i)$  is  $k$  and there are no arcs from  $D_n^i$  to  $D_n^j$  for  $j \neq i$ , any cycle containing a vertex

from some  $D_n^i$  has length exceeding  $k - 1$  provided that the subdigraph  $\Sigma$  of  $D_{n+1}$  induced by the vertices of the  $P_{d_i}$ 's has girth exceeding  $k - 1$ . Since there exists an acyclic homomorphism  $\psi : V(\Sigma) \rightarrow V(D_n)$  (sending every vertex in  $P_{d_i}$  to  $d_i$ ), we have  $g(\Sigma) \geq g(D_n) = k$ . Therefore,  $g(D_{n+1}) = k$ .

It is clear that  $\chi(D_{n+1}) \geq \chi(D_n) = n$ , since  $D_n$  is isomorphic to a subgraph of  $D_{n+1}$ . If  $D_{n+1}$  is  $n$ -chromatic, then there exists an acyclic homomorphism  $\sigma : V(D_{n+1}) \rightarrow V(K_n^*)$ . To set up the contradiction we are about to derive, fix a  $\sigma$  'color'  $\alpha \in V(K_n^*)$ . Since  $D_n^i$  is isomorphic to  $D_n$ , the function  $\sigma$  maps  $V(D_n^i)$  onto  $V(K_n^*)$ , for all  $i \in [m]$ . Every vertex in  $D_n^i$  is in a cycle with the vertices of  $P_{d_i}$ , which implies that there exists a vertex  $v_i \in P_{d_i}$  such that  $\sigma(v_i) \neq \alpha$ . The subgraph  $\Lambda$  of  $D_{n+1}$  induced by  $\{v_1, v_2, \dots, v_m\}$  is isomorphic to  $D_n$ . This contradicts the fact that  $D_n$  has chromatic number  $n$ , since  $\sigma$ , now seen to avoid  $\alpha$  on  $V(\Lambda)$ , effectively maps  $V(\Lambda)$  to  $V(K_{n-1}^*)$  acyclically. Thus  $\chi(D_{n+1}) \geq n + 1$ . We now show that  $\chi(D_{n+1}) = n + 1$  by giving an acyclic homomorphism from  $D_{n+1}$  to  $K_{n+1}^*$ . Let  $\zeta$  be an acyclic homomorphism from  $D_n$  to  $K_n^*$ . Define a mapping  $\phi : V(D_{n+1}) \rightarrow V(K_{n+1}^*)$  as follows. For vertices  $(d_i, p) \in V(P_{d_i})$ , define  $\phi((d_i, p)) = \zeta(d_i)$  and for vertices  $(d_j, i) \in V(D_n^i)$ , let

$$\phi((d_j, i)) = \begin{cases} \zeta(d_j), & \text{if } \zeta(d_j) \neq \zeta(d_i), \\ n + 1, & \text{otherwise.} \end{cases}$$

As the target digraph of  $\phi$  is complete, to show that  $\phi$  is an acyclic homomorphism, it will suffice to show that each fiber of  $\phi$  is acyclic. The fibers of  $\phi|_{D_n^i}$  are acyclic, for all  $i \in [m]$  because they are identical, up to relabeling, to the fibers of  $\zeta$ . This implies that the fibers of  $\phi|_{H(n,i)}$  are acyclic since  $\phi(V(P_{d_i})) \cap \phi(V(D_n^i)) = \emptyset$ . Hence it suffices to show that the restriction of  $\phi$  to  $\Sigma (= D_{n+1} [\cup_{i=1}^m V(P_{d_i})])$  is an acyclic homomorphism. Let  $\psi : V(\Sigma) \rightarrow V(D_n)$  be defined as above and notice that  $\phi|_{\Sigma} = \zeta \circ \psi$ , since  $\phi((d_i, p)) = \zeta(d_i)$ , for all vertices  $(d_i, p) \in V(P_{d_i})$ . As  $\zeta$  and  $\psi$  are acyclic homomorphisms, so too is their composition  $\phi|_{\Sigma}$ . Therefore,  $\phi$  is an acyclic homomorphism which finally implies that  $\chi(D_{n+1}) = n + 1$ .  $\square$

#### ACKNOWLEDGMENTS

The author would like to thank Mark Kayll for his participation during the development of the results in this paper. His participation is gratefully acknowledged.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MONTANA, MISSOULA, MT 59812  
*E-mail address:* michael.severino@umontana.edu