Contributions to Discrete Mathematics

THE EXACT MAXIMAL ENERGY OF INTEGRAL CIRCULANT GRAPHS WITH PRIME POWER ORDER

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ABSTRACT. The energy of a graph was introduced by GUTMAN in 1978 as the sum of the absolute values of the eigenvalues of its adjacency matrix. We study the energy of integral circulant graphs, also called gcd graphs. These are Cayley graphs on cyclic groups (i.e. there adjacency matrix is circulant) each of whose eigenvalues is an integer.

Given an arbitrary prime power p^s, we determine all integral circulant graphs of order p^s having maximal energy. This enables us to compute the maximal energy $\mathcal{E}_{\max}(p^s)$ among all integral circulant graphs of order p^s.

1. INTRODUCTION

A graph is called *circulant* if it is a Cayley graph on a cyclic group, and thus its adjacency matrix is circulant. An *integral* graph has the property that all of its eigenvalues are integers. Fixing a positive integer n, it is well known (cf. [27]) that any integral circulant graph G of order n is characterized by a set $\mathcal{D}$ of positive divisors of n such that G is isomorphic to $\mathrm{ICG}(n, \mathcal{D})$, defined as the graph having the residue class ring $\mathbb{Z}/n\mathbb{Z}$ as vertex set and $\{\{a, b\} \mid a, b \in \mathbb{Z}/n\mathbb{Z}, \ \gcd(a - b, n) \in \mathcal{D}\}$ as edge set. These graphs are also known as *gcd graphs* in the literature. For $|\mathcal{D}| = 1$ we obtain the subclass of so-called *unitary Cayley graphs*. We consider only loopfree gcd graphs, i.e. we require $n \notin \mathcal{D}$. Moreover, we note that $\mathrm{ICG}(n, \mathcal{D})$ with $\mathcal{D} = \{d_1, \ldots, d_r\}$ is connected if and only if $\gcd(n, d_1, \ldots, d_r) = 1$ (cf. [27]). For a prime power $n = p^s$ this is equivalent to $1 \in \mathcal{D}$.

In recent years, quite a few structural properties of integral circulant graphs have been brought to light (cf. [1, 4, 9, 12, 13, 14, 18, 19, 27]). Some emphasis has lately been placed on researching the energy of integral circulant graphs (see [2, 6, 7, 8, 17, 21, 22, 23, 24, 26]). The *energy* $E(G)$

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of a graph G on n vertices is defined as

$$E(G) = \sum_{\lambda \in \mathrm{Spec}(G)} |\lambda| = \sum_{i=1}^{n} |\lambda_i|,$$

where the *spectrum* $\mathrm{Spec}(G) = \{\lambda_1, \ldots, \lambda_n\}$ of G denotes the set of eigenvalues λ_i of the adjacency matrix of G, counted with multiplicities. Observe that an undirected graph has real spectrum, since its adjacency matrix is symmetric.

Let us abbreviate $\mathcal{E}(n, \mathcal{D}) = E(\mathrm{ICG}(n, \mathcal{D}))$. Since $\mathrm{ICG}(n, \mathcal{D})$ is an integral graph, we have $\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D})) \subset \mathbb{Z}$, hence $\mathcal{E}(n, \mathcal{D})$ is an integer. Given a positive integer n, it is most desirable to determine both

$$\mathcal{E}_{\min}(n) := \min\left\{\mathcal{E}(n, \mathcal{D}) \,|\, \mathcal{D} \subseteq \{1 \leq d < n \,|\, d \text{ divides } n\}\right\}$$

and

$$\mathcal{E}_{\max}(n) := \max\left\{\mathcal{E}(n, \mathcal{D}) \,|\, \mathcal{D} \subseteq \{1 \leq d < n \,|\, d \text{ divides } n\}\right\},$$

which are of particular interest with respect to the question of hyperenergeticity or hypoenergeticity of certain classes of graphs.

Consider a prime power $n = p^s$ and a divisor set $\mathcal{D} = \{p^{a_1}, p^{a_2}, \ldots, p^{a_r}\}$ with arbitrary exponents $0 \leq a_1 < \cdots < a_r \leq s - 1$. In [23], Theorem 2.1, the authors proved that

$$(1.1) \qquad \mathcal{E}(p^s, \mathcal{D}) = 2(p-1)p^{s-1}\big(r - (p-1)h_{p,r}(a_1, \ldots, a_r)\big),$$

where

$$(1.2) \qquad h_{p,r}(x) = h_{p,r}(x_1, \ldots, x_r) := \sum_{k=1}^{r-1} \sum_{i=k+1}^{r} \frac{1}{p^{x_i - x_k}}$$

for $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$. Observe that $h_{p,r}$ has the symmetry property

$$(1.3) \qquad h_{p,r}(a_1, \ldots, a_r) = h_{p,r}(s - 1 - a_r, \ldots, s - 1 - a_1)$$

for all integral exponents $0 \leq a_1 < a_2 < \cdots < a_{r-1} < a_r \leq s - 1$.

A rather straightforward consequence of (1.1) is that

$$\mathcal{E}_{\min}(p^s) = 2(p-1)p^{s-1},$$

and this minimal energy is attained precisely for the singleton divisor sets $\mathcal{D} = \{p^t\}$ with $0 \leq t \leq s - 1$ (cf. [23], Theorem 3.1).

Divisor sets $\mathcal{D}$ producing graphs with maximal energy $\mathcal{E}_{\max}(p^s)$ were studied for the first time in [24]. Since, by the aforementioned result, singleton divisor sets generate integral circulant graphs having minimal energy $\mathcal{E}_{\min}(p^s)$, minimisers of some $h_{p,r}$ producing integral circulant graphs with maximal energy $\mathcal{E}_{\max}(p^s)$ necessarily satisfy $r \geq 2$ if $s \geq 2$; note that for $s = 1$ there is only one possible divisor set, namely $\mathcal{D} = \{1\}$. For that reason, we may henceforth assume without loss of generality that $2 \leq r \leq s$. Furthermore,

any minimiser $0 \le a_1 < a_2 < \cdots < a_r \le s - 1$ must have the entries $a_1 = 0$ and $a_r = s - 1$, because otherwise replacing a_1 by a smaller integer or a_r by a larger one, respectively, would obviously summandwise lessen the value of $h_{p,r}$ as defined in (1.2). Accordingly, any minimiser $a = (a_1, \ldots, a_r)$ of $h_{p,r}$ lies in the set

$$A(s,r) := \{(a_1, \ldots, a_r) \in \mathbb{Z}^r \mid 0 = a_1 < a_2 < \cdots < a_{r-1} < a_r = s - 1\},$$

and such an a is called an *admissible* exponent tuple. Hence $1 = p^{a_1} \in \mathcal{D}$ for all $a \in A(s,r)$, and by one of our introductory remarks we incidentally have that $\mathcal{E}_{\max}(p^s)$ can only be generated by connected gcd graphs.

Our strategy in [24] for finding these maximising divisor sets was to start by fixing r and discover exponent tuples $(a_1, \ldots, a_r)$ minimising $h_{p,r}$, at least approximately. Applying methods from convex optimisation it was shown that, for fixed s and r, the function $h_{p,r}$ becomes minimal if $0 = a_1 < a_2 < \cdots < a_{r-1} < a_r = s - 1$ are chosen in nearly equidistant position ([24], Corollary 3.2). Inserting the corresponding approximations $h_{p,r}(a_1, \ldots, a_r)$ into (1.1) and varying r, the order of magnitude of $\mathcal{E}_{\max}(p^s)$ was determined in the sense that explicit upper and lower bounds for $\mathcal{E}_{\max}(p^s)$ were given, which differed roughly by a factor of 2. More precisely, it was shown in [24], Theorem 4.2 that, on setting $C(p) := 1 - (\log \log p)/\log p$,

$$C(p)(p-1)p^{s-1}(s-1) \le \mathcal{E}_{\max}(p^s) \le 2C(p)(p-1)p^{s-1}s$$

for all primes $p \ge 17$. Numerical examples suggested that $\mathcal{E}_{\max}(p^s)$ approached the lower bound with increasing p, i.e. we had conjecturally $\mathcal{E}_{\max}(p^s) \approx sp^s$.

In [25] the authors used combinatorial instead of analytic arguments to refine the earlier approximative results on minimisers of $h_{p,r}$ for fixed values of r. This shed more light on the structure of these minimisers, which can be regarded as the result of a repeated balancing process. In several cases this process allowed us to obtain accurate results after only one or two balancing steps.

In this paper, we completely settle the problem to find all divisor sets maximising the energy of an integral circulant graph of prime power order, i.e. for any prime power p^s we shall explicitly determine all divisor sets $\mathcal{D}$ satisfying $\mathcal{E}(p^s, \mathcal{D}) = \mathcal{E}_{\max}(p^s)$. At the same time this enables us to compute $\mathcal{E}_{\max}(p^s)$ precisely, thus confirming the conjecture mentioned above.

For an admissible exponent tuple $a = (a_1, \ldots, a_r)$, we denote by

$$\mathcal{D}(a) := \{p^{a_1}, \ldots, p^{a_r}\}$$

the corresponding divisor set. Our main result is the following.

Theorem 1.1. *Let p be a prime, and let s be a positive integer.*

  (i) *If s is odd, then*

$$(1.4) \qquad \mathcal{E}_{\max}(p^s) = \frac{1}{(p+1)^2}\left((s+1)(p^2-1)p^s + 2(p^{s+1}-1)\right),$$

  and the only exponent tuple a satisfying $\mathcal{E}(p^s, \mathcal{D}(a)) = \mathcal{E}_{\max}(p^s)$ is $a = (0, 2, 4, \ldots, s-3, s-1) \in A(s, r)$ in case $p \geq 3$, while we have the additional maximising tuple $a = (0, 1, 3, 5, \ldots, s-4, s-2, s-1)$ in case $p = 2$.

  (ii) *If s is even, then*

$$(1.5) \quad \mathcal{E}_{\max}(p^s) = \frac{1}{(p+1)^2}\left(s(p^2-1)p^s + 2(2p^{s+1} - p^{s-1} + p^2 - p - 1)\right),$$

  and the only exponent tuples a satisfying $\mathcal{E}(p^s, \mathcal{D}(a)) = \mathcal{E}_{\max}(p^s)$ are $a = (0, 2, 4, \ldots, s-2, s-1)$ and $a = (0, 1, 3, 5, \ldots, s-3, s-1)$.

We would like to add the following remarks with regard to Theorem 1.1:

  (a) For computational reasons we prefer to present formulae (1.4) and (1.5) in a compact form. However, being the sum of the moduli of integral eigenvalues, $\mathcal{E}_{\max}(p^s)$ is certainly a positive integer. In fact, it is an immediate consequence of identity (1.1) that $\mathcal{E}_{\max}(p^s)$ is always divisible by $2(p-1)$. This is in line with the work of Bapat and Pati [5] who showed that the energy of any graph is never an odd integer (see also Pirzada and Gutman [16]). The divisibility property of $\mathcal{E}_{\max}(p^s)$ becomes obvious by not using geometric sum formulae in (3.3) and (3.4), which leads to

$$\mathcal{E}_{\max}(p^{2m+1}) = 2(p-1)\left((m+1)p^{2m} - (p-1)\sum_{j=0}^{m-1}(j+1)p^{2j}\right)$$

  and

$$\mathcal{E}_{\max}(p^{2m}) = 2(p-1)\left(mp^{2m-1} - (p-1)\sum_{j=0}^{m-3}(j+1)p^{2j+3} + 1\right)$$

  instead of (1.4) and (1.5).

  (b) Note that the two different exponent tuples in part (ii) just occur due to the symmetry property (1.3) of the function $h_{p,r}$.

  (c) A graph G on n vertices is called *hyperenergetic* if its energy is greater than the energy of the complete graph K_n, i.e. if $E(G) > E(K_n) = 2(n-1)$. There exist several bounds for the energies of different graph classes. For an arbitrary graph G with n vertices, Koolen and

Moulton [20] showed that

$$(1.6) \qquad\qquad E(G) \le \frac{n}{2}\left(\sqrt{n}+1\right).$$

Shparlinksi [26] constructed an infinite family of circulant graphs that asymptotically achieves the upper bound in (1.6). We observe that integral circulant graphs of prime power order p^s with maximal energy, that is $\mathcal{E}_{\max}(p^s) \approx sp^s$ by Theorem 1.1, are hyperenergetic, but do not come close to the bound in (1.6). The graphs $\mathrm{ICG}(p^s, \mathcal{D})$ with minimal energy, studied in [23], are *hypoenergetic*, which means $\mathcal{E}_{\min}(p^s) < 2(p^s - 1)$. The reader finds more on hyperenergeticity as well as hypoenergeticity of gcd graphs in [23] and [24].

The proof of Theorem 1.1 uses an accordion-like compression and expansion procedure, which generates for any given admissible $a^{(0)} \in \bigcup_{r=2}^s A(s,r)$ a finite sequence $(a^{(\ell)})_{0 \le \ell \le m}$, say, of admissible $a^{(\ell)} \in \bigcup_{r=2}^s A(s,r)$ satisfying

$$\mathcal{E}(p^s, \mathcal{D}(a^{(0)})) < \mathcal{E}(p^s, \mathcal{D}(a^{(1)})) < \cdots < \mathcal{E}(p^s, \mathcal{D}(a^{(m)})),$$

where $a^{(m)}$ is one of the maximising exponent tuples to be found in Theorem 1.1. Let us point out that, contrary to earlier strategies applied in [24] and [25], we now compare the energies related to admissible exponent tuples of *different* lengths. We recommend to take a look at the illustrative Example 3.1.

With respect to possible generalisations, let us point out some existing obstacles. So far, no formula comparable with (1.1) in terms of simplicity is available for $\mathcal{E}(n, \mathcal{D})$ if n is not a prime power. In particular, the energy reveals practically no signs of multiplicativity in terms of divisors of n. As a singular result it has been observed in [23] that $\mathcal{E}_{\max}(pq) = \mathcal{E}_{\max}(p)\mathcal{E}_{\max}(q)$ for distinct odd primes p and q. However, in this case it is a straightforward exercise to determine and compare the energies for the four possible divisor sets $\{1\}, \{1,p\}, \{1,q\}, \{1,p,q\}$ by evaluating the formula

$$\mathcal{E}(n, \mathcal{D}) = \sum_{k=1}^n \left| \sum_{d \in \mathcal{D}} \mu\left(\frac{n}{(n,kd)}\right) \cdot \frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{n}{(n,kd)}\right)} \right|,$$

(cf. [18], Theorem 16) with Möbius' function μ and Euler's totient function φ. The multiplicativity already vanishes for $n = p^2 q$ or a product of three distinct primes. Moreover, no conjectures regarding the structure of the energy maximising divisor sets exist so far, not even for square-free n. For example, $\mathcal{E}_{\max}(3 \cdot 5 \cdot 7) = 520$ with unique maximiser $\{1, 15, 21, 35\}$ and $\mathcal{E}_{\max}(2 \cdot 3 \cdot 5 \cdot 7) = 1414$ with unique maximiser $\{1, 2, 3, 30, 35, 42, 70, 105\}$, while $\mathcal{E}_{\max}(p) = 2(p-1)$ with unique maximiser $\{1\}$ for each prime p.

2. Comparison of certain admissible exponent tuples

Given $2 \leq r \leq s$, we define for each $a \in A(s, r)$ its *delta vector*

$$\delta_r(a) := (d_1, d_2, \ldots, d_{r-1}) \in \mathbb{N}^{r-1}$$

by setting $d_j := a_{j+1} - a_j$ $(1 \leq j \leq r - 1)$. Obviously, we have $\sum_{j=1}^{r-1} d_j = s - 1$. Thus, introducing

$$D(s, r) := \left\{ (d_1, \ldots, d_{r-1}) \in \mathbb{N}^{r-1} \,\middle|\, \sum_{j=1}^{r-1} d_j = s - 1 \right\},$$

the function

$$\delta_r : \begin{cases} A(s, r) & \longrightarrow & D(s, r) \\ (a_1, a_2, \ldots, a_r) & \mapsto & (a_2 - a_1, a_3 - a_2, \ldots, a_r - a_{r-1}) \end{cases}$$

is 1–1 with its inverse

$$\delta_r^{-1} : \begin{cases} D(s, r) & \longrightarrow & A(s, r) \\ (d_1, d_2, \ldots, d_{r-1}) & \mapsto & \left(0, d_1, d_1 + d_2, \ldots, \sum_{i=1}^{r-2} d_i, s - 1\right). \end{cases}$$

As an immediate consequence of [23], Theorem 2.1, we have the following observation, which will be used several times in the sequel.

Lemma 2.1. *Let p be a prime, and let $s \geq 2$ be a fixed integer. Let $a \in A(s, r)$ and $a' \in A(s, r')$. If*

$$(2.1) \qquad\qquad h_{p,r}(a) - h_{p,r'}(a') > \frac{r - r'}{p - 1},$$

then $\mathcal{E}(p^s, \mathcal{D}(a')) > \mathcal{E}(p^s, \mathcal{D}(a))$.

Proof. It follows from (1.1) that

$$\frac{\mathcal{E}(p^s, \mathcal{D}(a')) - \mathcal{E}(p^s, \mathcal{D}(a))}{2(p - 1)p^{s-1}} = r' - r - (p - 1)\big(h_{p,r'}(a') - h_{p,r}(a)\big),$$

which is positive by (2.1). $\qquad\square$

We denote by $\|d\|_\infty := \max\{d_j \,|\, 1 \leq j \leq r - 1\}$ the standard maximum norm of $d = (d_1, \ldots, d_{r-1}) \in D(s, r)$.

Proposition 2.2. *Let p be a prime, and let $s \geq 2$ be a fixed integer. Assume that $d = (d_1, \ldots, d_{r-1}) \in D(s, r)$ for some $2 \leq r \leq s$ satisfies one of the following two conditions*

  (i) $\|d\|_\infty \geq 4$,
  (ii) $\|d\|_\infty = 3$ and $d_j \geq 2$ for $1 \leq j \leq r - 1$.

Let $1 \leq u \leq r - 1$ be such that $d_u = \|d\|_\infty$, and define

$$d' = (d'_1, \ldots, d'_r) := (d_1, \ldots, d_{u-1}, 2, d_u - 2, d_{u+1}, \ldots, d_{r-1}).$$

Then $d' \in D(s, r+1)$ and $\mathcal{E}(p^s, \mathcal{D}(\delta_{r+1}^{-1}(d'))) > \mathcal{E}(p^s, \mathcal{D}(\delta_r^{-1}(d)))$.

Proof. By the fact that $\sum d_j = s - 1$ for $d \in D(s, r)$, we have

$$1 \leq u \leq r - 1 \leq s - \|d\|_\infty \leq s - 3,$$

thus $r + 1 \leq s - 1$ and $\sum d'_j = s - 1$, which implies $d' \in D(s, r+1)$. For $a = (a_1, \ldots, a_r) := \delta_r^{-1}(d) \in A(s, r)$, we obtain that $a' = (a'_1, \ldots, a'_r, a'_{r+1}) := \delta_{r+1}^{-1}(d') \in A(s, r+1)$ satisfies

$$a'_j = \begin{cases} a_j & \text{for } 1 \leq j \leq u, \\ a_u + 2 & \text{for } j = u + 1, \\ a_{j-1} & \text{for } u + 2 \leq j \leq r + 1. \end{cases}$$

By definition of the functions $h_{p,r}$ and $h_{p,r+1}$ in (1.2), we have

$$h_{p,r+1}(a') - h_{p,r}(a) = \sum_{k=1}^{r} \sum_{i=k+1}^{r+1} \frac{1}{p^{a'_i - a'_k}} - \sum_{k=1}^{r-1} \sum_{i=k+1}^{r} \frac{1}{p^{a_i - a_k}}.$$

Since all entries of a are also entries of a', each difference $a_i - a_k$ occurring in the second double sum also occurs in the first one. Hence the corresponding summands cancel out. However, in comparison with a the tuple a' has the additional entry $a'_{u+1} = a_u + 2$. Therefore, $h_{p,r+1}(a') - h_{p,r}(a)$ consists of all summands in which a'_{u+1} shows up, thus

$$\begin{aligned}
h_{p,r+1}(a') - h_{p,r}(a) &= \sum_{k=1}^{u} \frac{1}{p^{a'_{u+1} - a'_k}} + \sum_{i=u+2}^{r+1} \frac{1}{p^{a'_i - a'_{u+1}}} \\
&= \sum_{k=1}^{u} \frac{1}{p^{(a_u+2) - a_k}} + \sum_{i=u+2}^{r+1} \frac{1}{p^{a_{i-1} - (a_u+2)}} \\
&= \sum_{k=1}^{u} \frac{1}{p^{a_u - a_k + 2}} + \sum_{i=u+1}^{r} \frac{1}{p^{a_i - a_u - 2}}.
\end{aligned}$$

(2.2)

Case 1: $d_u \geq 4$.
  Since $a_1 < a_2 < \cdots < a_u$, we have $a_u - a_k \geq u - k$ for $k = 1, 2, \ldots, u$, and since $a_{u+1} - a_u = d_u \geq 4$, we have $a_i - a_u \geq i - u + 3$ for $i = u+1, \ldots, r$.

By (2.2), this implies

$$h_{p,r+1}(a') - h_{p,r}(a) \leq \sum_{k=1}^{u} \frac{1}{p^{u-k+2}} + \sum_{i=u+1}^{r} \frac{1}{p^{i-u+1}}$$

$$< 2 \sum_{j=2}^{\infty} \frac{1}{p^j} = \frac{2}{p(p-1)} \leq \frac{1}{p-1}$$

for each prime p.

Case 2: $d_u = 3$ and $d_j \geq 2$ for $j \neq u$.

Since $a_{u+1} - a_u = d_u = 3$ and $a_{j+1} - a_j = d_j \geq 2$ for all j, we have $a_u - a_k \geq 2(u-k)$ for $k = 1, 2, \ldots, u$ and $a_i - a_u \geq 2(i-u) + 1$ for $i = u+1, \ldots, r$. By (2.2), this implies

$$h_{p,r+1}(a') - h_{p,r}(a) \leq \sum_{k=1}^{u} \frac{1}{p^{2(u-k+1)}} + \sum_{i=u+1}^{r} \frac{1}{p^{2(i-u)-1}}$$

$$< \sum_{j=1}^{\infty} \frac{1}{p^{2j}} + \frac{1}{p} \sum_{j=0}^{\infty} \frac{1}{p^{2j}} = \frac{1}{p-1} .$$

In both cases we have $h_{p,r}(a) - h_{p,r+1}(a') > -\frac{1}{p-1}$, hence Lemma 2.1 proves our Proposition.

$\square$

Before we proceed, we introduce a helpful tool to visualise the calculation of differences $h_{p,r}(a) - h_{p,r'}(a')$. Given $a = (a_1, \ldots, a_r) \in A(s,r)$ for some $2 \leq r \leq s$, we define the *delta tableau* $\mathfrak{A} = (a_{k,i})_{1 \leq k < i \leq r}$ of a as a triangular array of integers, corresponding to a strictly upper triangular matrix $(a_{k,i})_{1 \leq k,i \leq r}$ with $a_{k,i} := a_i - a_k$ for $1 \leq k < i \leq r$ and no entries attributed to the positions $k \geq i$ below or on the diagonal of the matrix.

Example 2.3. Let us look at an example of the kind that occurs in the proof of Proposition 2.4. The entries a_j and a'_j of

$$a = (0, 3, 5, 6, 8, 10, 12, 15, 16, 20, 23) \in A(24, 11),$$
$$a' = (0, 3, 5, 7, 9, 11, 13, 15, 16, 20, 23) \in A(24, 11),$$

pairwise coincide for $1 \leq j \leq 3 =: u$ and for $v := 8 \leq j \leq 11 =: r$. The delta tableaux $\mathfrak{A}$ and $\mathfrak{A}'$ of a and a', respectively, are depicted by the following two diagrams:

3	5	6	8	10	12	15	16	20	23
	2	3	5	7	9	12	13	17	20
		1	3	5	7	10	11	15	18
			2	4	6	9	10	14	17
				2	4	7	8	12	15
					2	5	6	10	13
						3	4	8	11
							1	5	8
								4	7
									3

3	5	7	9	11	13	15	16	20	23
	2	4	6	8	10	12	13	17	20
		2	4	6	8	10	11	15	18
			2	4	6	8	9	13	16
				2	4	6	7	11	14
					2	4	5	9	12
						2	3	7	10
							1	5	8
								4	7
									3

If we wish to evaluate $h_{p,11}(a) - h_{p,11}(a')$, we have to subtract summands of type $p^{-a'_{k,i}}$ from summands of type $p^{-a_{k,i}}$. We observe that the entries inside the dark-coloured areas (three triangles and the upper right rectangle) of $\mathfrak{A}$ one by one coincide with those of $\mathfrak{A}'$, which means that the corresponding summands annihilate each other. Consequently, we only have to consider the entries in each of the two remaining light-coloured rectangles of $\mathfrak{A}$ and $\mathfrak{A}'$, where differences $a_{k,i} - a'_{k,i}$ are -1 between the upper left rectangles and $+1$ between the lower right rectangles. Note that the three vertical blocks in the tableaux have column indices i running in the intervals $2 \le i \le u$ and $u+1 \le i \le v-1$ and $v \le i \le r$, while the row indices of the three horizontal blocks have ranges $1 \le k \le u$ and $u+1 \le k \le v-1$ and $v \le k \le r-1$.

The following result is a corollary to Proposition 3.1 in [25]. Yet, in view of the new perspective of this exposition and for the convenience of the reader, we provide a short proof of it.

Proposition 2.4. *Let p be a prime. For some integers $2 \le r \le s$ and $1 \le u < v \le r - 1$, let $d = (d_1, \ldots, d_{r-1}) \in D(s, r)$ have the following properties:*

(i) $(d_u, d_v) \in \{(1, 3), (3, 1)\}$,
(ii) $d_j = 2$ *for $u < j < v$.*

Then $d' = (d'_1, \ldots, d'_{r-1})$, defined by

$$d'_j := \begin{cases} 2 & \text{for } j = u \text{ and } j = v, \\ d_j & \text{otherwise,} \end{cases}$$

satisfies $d' \in D(s, r)$ and $\mathcal{E}(p^s, \mathcal{D}(\delta_r^{-1}(d'))) > \mathcal{E}(p^s, \mathcal{D}(\delta_r^{-1}(d)))$.

Proof. By condition (i) and the symmetry property (1.3) of $h_{p,r}$ we may assume without loss of generality that $d_u = 1$ and $d_v = 3$, i.e.

$$d = (d_1, \ldots, d_{u-1}, 1, 2, \ldots, 2, 3, d_{v+1}, \ldots, d_{r-1}) \in D(s, r),$$

and

$$d' = (d_1, \ldots, d_{u-1}, 2, 2, \ldots, 2, 2, d_{v+1}, \ldots, d_{r-1}).$$

Clearly, $\sum d'_j = \sum d_j = s - 1$, hence $d' \in D(s, r)$. For $a = (a_1, \ldots, a_r) := \delta_r^{-1}(d) \in A(s, r)$, this means that $a' = (a'_1, \ldots, a'_r) := \delta_r^{-1}(d') \in A(s, r)$ satisfies

$$(2.3) \qquad a'_j = \begin{cases} a_j & \text{for } 1 \le j \le u, \\ a_j + 1 & \text{for } u + 1 \le j \le v, \\ a_j & \text{for } v + 1 \le j \le r. \end{cases}$$

By definition of the function $h_{p,r}$ in (1.2), we have

$$h_{p,r}(a) - h_{p,r}(a') = \sum_{k=1}^{r-1} \sum_{i=k+1}^{r} \left(\frac{1}{p^{a_i - a_k}} - \frac{1}{p^{a'_i - a'_k}} \right).$$

By use of (2.3), comparison of the delta tableaux $\mathfrak{A} = (a_{k,i})_{1 \le k < i \le r}$ and $\mathfrak{A}' = (a'_{k,i})_{1 \le k < i \le r}$ (cf. Example 2.3), where $a_{k,i} := a_i - a_k$ and $a'_{k,i} := a'_i - a'_k$

for $1 \le k < i \le r$, reveals that

$$h_{p,r}(a) - h_{p,r}(a') = \sum_{k=1}^{u} \sum_{i=u+1}^{v} \left(\frac{1}{p^{a_{k,i}}} - \frac{1}{p^{a'_{k,i}}} \right)$$

$$+ \sum_{k=u+1}^{v} \sum_{i=v+1}^{r} \left(\frac{1}{p^{a_{k,i}}} - \frac{1}{p^{a'_{k,i}}} \right)$$

$$= \sum_{k=1}^{u} \sum_{i=u+1}^{v} \left(\frac{1}{p^{a_{k,i}}} - \frac{1}{p^{a_{k,i}+1}} \right)$$

$$+ \sum_{k=u+1}^{v} \sum_{i=v+1}^{r} \left(\frac{1}{p^{a_{k,i}}} - \frac{1}{p^{a_{k,i}-1}} \right)$$

$$= \left(1 - \frac{1}{p} \right) \sum_{k=1}^{u} \sum_{i=u+1}^{v} \frac{1}{p^{a_{k,i}}} + (1-p) \sum_{k=u+1}^{v} \sum_{i=v+1}^{r} \frac{1}{p^{a_{k,i}}}.$$

Since assumption (ii) implies that

$$(2.4) \qquad a_j = a_{u+1} + 2(j - (u+1)), \text{ for } u+1 \le j \le v,$$

it follows that

$$h_{p,r}(a) - h_{p,r}(a') = \left(1 - \frac{1}{p} \right) \sum_{k=1}^{u} p^{a_k} \sum_{i=u+1}^{v} \frac{1}{p^{a_{u+1}+2(i-(u+1))}}$$

$$+ (1-p) \sum_{k=u+1}^{v} p^{a_{u+1}+2(k-(u+1))} \sum_{i=v+1}^{r} \frac{1}{p^{a_i}}$$

$$(2.5) \qquad = \left(1 - \frac{1}{p} \right) \frac{1}{p^{a_{u+1}}} \sum_{k=1}^{u} p^{a_k} \sum_{i=0}^{v-u-1} \frac{1}{p^{2i}}$$

$$- (p-1) p^{a_{u+1}} \sum_{k=0}^{v-u-1} p^{2k} \sum_{i=v+1}^{r} \frac{1}{p^{a_i}}.$$

The identity $a_v = a_{u+1} + 2(v - (u+1))$, being a special case of (2.4), yields

$$\frac{1}{p^{a_{u+1}}} \sum_{i=0}^{v-u-1} \frac{1}{p^{2i}} = \frac{1}{p^{a_{u+1}}} \cdot \frac{1}{p^{2(v-u-1)}} \sum_{i=0}^{v-u-1} p^{2i} = \frac{1}{p^{a_v}} \sum_{i=0}^{v-u-1} p^{2i}.$$

Inserting this into (2.5), we obtain

$$(2.6)$$

$$h_{p,r}(a) - h_{p,r}(a') = \left(\frac{1}{p^{a_v+1}} \sum_{k=1}^{u} p^{a_k} - p^{a_{u+1}} \sum_{i=v+1}^{r} \frac{1}{p^{a_i}} \right) (p-1) \sum_{i=0}^{v-u-1} p^{2i}$$

$$= \left(\frac{1}{p^{a_v+1}} \sum_{k=1}^{u} p^{a_k} - p^{a_{u+1}} \sum_{i=v+1}^{r} \frac{1}{p^{a_i}} \right) \frac{p^{2(v-u)}-1}{p+1}.$$

Since $d_u = 1$ and $d_v = 3$ by assumption, we have $a_{u+1} = a_u + 1$ and $a_{v+1} = a_v + 3$. Therefore, $\sum_{k=1}^{u} p^{a_k} \geq p^{a_u} = p^{a_{u+1}-1}$ and

$$\sum_{i=v+1}^{r} \frac{1}{p^{a_i}} < \frac{1}{p^{a_{v+1}}} \sum_{i=0}^{\infty} \frac{1}{p^i} = \frac{p}{(p-1)p^{a_{v+1}}} = \frac{1}{(p-1)p^{a_{v+1}+2}} \cdot$$

With this, (2.6) yields

$$h_{p,r}(a) - h_{p,r}(a') > \left(p^{a_{u+1}-a_v-2} - \frac{p^{a_{u+1}-a_v-2}}{p-1} \right) \frac{p^{2(v-u)} - 1}{p+1}$$

$$= \frac{p-2}{p^2-1} \cdot p^{a_{u+1}-a_v-2}(p^{2(v-u)} - 1) \,,$$

hence $h_{p,r}(a) - h_{p,r}(a') > 0$ for each prime p. By Lemma 2.1, we finally obtain the desired conclusion. $\qquad\square$

In Proposition 2.4, tableaux of the same size were "subtracted" from each other. Now we study the case where a given tableau has to be compared with a smaller one. For that purpose, we introduce the (u,v)-*derivative* of an admissible tuple.

Definition 2.5. *Let $a \in A(s,r)$ for some $3 \leq r \leq s$, and let $1 \leq u < v \leq r-1$ be arbitrary integers. Then $\partial_{u,v}(a) = (a'_1, \ldots, a'_{r-1}) \in A(s, r-1)$, defined by setting*

$$a'_j := \begin{cases} a_j & \text{for } 1 \leq j \leq u, \\ a_j + 1 & \text{for } u+1 \leq j \leq v-1, \\ a_{j+1} & \text{for } v \leq j \leq r-1, \end{cases}$$

is called the (u,v)-derivative of a.

Lemma 2.6 (Tableau Reduction Lemma)**.** *Let p be a prime, and let $a = (a_1, \ldots, a_r) \in A(s,r)$ be admissible for some $3 \leq r \leq s$. Moreover, let $1 \leq u < v \leq r-1$ be arbitrary integers. If $a_{j+1} - a_j = 2$ for $u+1 \leq j \leq v-1$, then*

$$\begin{aligned}
(2.7) \qquad h_{p,r}(a) - h_{p,r-1}(\partial_{u,v}(a)) &= \frac{1}{p+1}\left(p + \frac{1}{p^{2(v-u-1)}} \right)\left(\frac{U}{p^{a_{u+1}}} + p^{a_v} V \right) \\
&\quad + \frac{1}{p^2-1}\left(1 - \frac{1}{p^{2(v-u-1)}} \right),
\end{aligned}$$

where

$$U := \sum_{k=1}^{u} p^{a_k} \quad and \quad V := \sum_{i=v+1}^{r} \frac{1}{p^{a_i}}.$$

Proof. By definition of the functions $h_{p,r}$ and $h_{p,r-1}$ in (1.2), we have for $a' = (a'_1, \ldots, a'_{r-1}) := \partial_{u,v}(a)$

$$\Delta h := h_{p,r}(a) - h_{p,r-1}(\partial_{u,v}(a)) = \sum_{k=1}^{r-1} \sum_{i=k+1}^{r} \frac{1}{p^{a_i - a_k}} - \sum_{k=1}^{r-2} \sum_{i=k+1}^{r-1} \frac{1}{p^{a'_i - a'_k}}.$$

Using (2.5), we obtain the following two delta tableaux $\mathfrak{A} = (a_{k,i})_{1 \le k < i \le r}$ and $\mathfrak{A}' = (a'_{k,i})_{1 \le k < i \le r-1}$, where $a_{k,i} := a_i - a_k$ $(1 \le k < i \le r)$ and $a'_{k,i} := a'_i - a'_k$ for $1 \le k < i \le r-1$:

Observe that the entries inside the dark-coloured areas (three triangles and the upper right rectangle) of $\mathfrak{A}$ one by one equal the corresponding entries of $\mathfrak{A}'$. For that reason, we only have to consider the entries in each of the two remaining light-coloured rectangles of $\mathfrak{A}$ and $\mathfrak{A}'$ and finally the

entries in the white-coloured column inside $\mathfrak{A}$. Hence

$$\Delta h = \left(\sum_{k=1}^{u} \sum_{i=u+1}^{v} + \sum_{k=u+1}^{v} \sum_{i=v+1}^{r} \right) \frac{1}{p^{a_{k,i}}} + \sum_{k=u+1}^{v-1} \frac{1}{p^{a_{k,v}}}$$
$$- \left(\sum_{k=1}^{u} \sum_{i=u+1}^{v-1} + \sum_{k=u+1}^{v-1} \sum_{i=v}^{r-1} \right) \frac{1}{p^{a'_{k,i}}}.$$

Then (2.5) yields

(2.8)

$$\Delta h = \sum_{k=1}^{u} \left(\sum_{i=u+1}^{v-1} \left(\frac{1}{p^{a_{k,i}}} - \frac{1}{p^{a'_{k,i}}} \right) + \frac{1}{p^{a_{k,v}}} \right)$$
$$+ \sum_{k=u+1}^{v-1} \left(\sum_{i=v+1}^{r} \frac{1}{p^{a_{k,i}}} - \sum_{i=v}^{r-1} \frac{1}{p^{a'_{k,i}}} \right) + \sum_{i=v+1}^{r} \frac{1}{p^{a_{v,i}}} + \sum_{k=u+1}^{v-1} \frac{1}{p^{a_{k,v}}}$$
$$= \sum_{k=1}^{u} \left(\sum_{i=u+1}^{v-1} \left(\frac{1}{p^{a_{k,i}}} - \frac{1}{p^{a_{k,i+1}}} \right) \right) + \sum_{k=1}^{u} \frac{1}{p^{a_{k,v}}}$$
$$+ \sum_{k=u+1}^{v-1} \left(\sum_{i=v+1}^{r} \frac{1}{p^{a_{k,i}}} - \sum_{i=v+1}^{r} \frac{1}{p^{a_{k,i-1}}} \right) + \sum_{i=v+1}^{r} \frac{1}{p^{a_{v,i}}} + \sum_{k=u+1}^{v-1} \frac{1}{p^{a_{k,v}}}$$
$$= \frac{p-1}{p} \sum_{k=1}^{u} p^{a_k} \sum_{i=u+1}^{v-1} \frac{1}{p^{a_i}} + \frac{1}{p^{a_v}} \sum_{k=1}^{u} p^{a_k}$$
$$+ (1-p) \sum_{k=u+1}^{v-1} p^{a_k} \sum_{i=v+1}^{r} \frac{1}{p^{a_i}} + p^{a_v} \sum_{i=v+1}^{r} \frac{1}{p^{a_i}} + \sum_{k=u+1}^{v-1} \frac{1}{p^{a_{k,v}}}$$
$$= \frac{p-1}{p} U \sum_{i=u+1}^{v-1} \frac{1}{p^{a_i}} + \frac{U}{p^{a_v}} - (p-1)V \sum_{k=u+1}^{v-1} p^{a_k} + p^{a_v} V + \sum_{k=u+1}^{v-1} \frac{1}{p^{a_{k,v}}}.$$

Our assumption $a_{j+1} - a_j = 2$ for $u+1 \le j \le v-1$ implies that

(2.9) $\qquad a_j = a_{u+1} + 2(j - (u+1)), \text{ for } u+1 \le j \le v.$

It follows that

$$\sum_{k=u+1}^{v-1} p^{a_k} = p^{a_{v-1}} \sum_{k=u+1}^{v-1} \frac{1}{p^{a_{v-1}-a_k}} = p^{a_{v-1}} \sum_{k=0}^{v-u-2} \frac{1}{p^{2k}}$$
$$= p^{a_{v-1}} \frac{p^2}{p^2-1} \left(1 - \frac{1}{p^{2(v-u-1)}} \right),$$
$$\sum_{i=u+1}^{v-1} \frac{1}{p^{a_i}} = \frac{1}{p^{a_{u+1}}} \sum_{k=u+1}^{v-1} p^{a_i-a_{u+1}} = \frac{1}{p^{a_{u+1}}} \sum_{i=0}^{v-u-2} \frac{1}{p^{2i}}$$
$$= \frac{1}{p^{a_{u+1}}} \cdot \frac{p^2}{p^2-1} \left(1 - \frac{1}{p^{2(v-u-1)}} \right),$$

and
$$\sum_{i=u+1}^{v-1} \frac{1}{p^{a_{k,v}}} = \sum_{i=u+1}^{v-1} \frac{1}{p^{2(v-k)}} = \sum_{k=1}^{v-u-1} \frac{1}{p^{2k}} = \frac{1}{p^2-1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right).$$

If we insert these identities into (2.8) and use $a_v = a_{u+1} + 2(v-u-1)$, being a special case of (2.9), we obtain

$$\Delta h = \left(\frac{1}{p^{a_v}} + \frac{1}{p^{a_{u+1}}} \cdot \frac{p}{p+1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)\right) U$$

$$+ \left(p^{a_v} - p^{a_{v-1}} \frac{p^2}{p+1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)\right) V$$

$$+ \frac{1}{p^2-1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)$$

(2.10)
$$= \left(\frac{1}{p^{2(v-u-1)}} + \frac{p}{p+1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)\right) \frac{U}{p^{a_{u+1}}}$$

$$+ \left(1 - \frac{p^{2-(a_v-a_{v-1})}}{p+1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)\right) p^{a_v} V$$

$$+ \frac{1}{p^2-1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right).$$

In case $v \geq u+2$ we know by (2.9) that $a_v - a_{v-1} = 2$, and (2.7) follows. To complete the proof we observe that identity (2.7) is a trivial consequence of (2.10) in case $v = u + 1$. □

Proposition 2.7. *Let p be a prime. For some integers $3 \leq r \leq s$ and $1 \leq u < v \leq r - 1$, let $d = (d_1, \ldots, d_{r-1}) \in D(s,r)$ have the following properties:*

  (i) $d_u = d_v = 1$,
  (ii) $d_j = 2$ for $u < j < v$.
Then $d' = (d'_1, \ldots, d'_{r-2})$, defined by

$$d'_j := \begin{cases} d_j & \text{for } 1 \leq j \leq u-1, \\ 2 & \text{for } u \leq j \leq v-1, \\ d_{j+1} & \text{for } v \leq j \leq r-2, \end{cases}$$

satisfies $d' \in D(s, r-1)$ and $\mathcal{E}(p^s, \mathcal{D}(\delta_{r-1}^{-1}(d'))) > \mathcal{E}(p^s, \mathcal{D}(\delta_r^{-1}(d)))$ with precisely one exceptional case, namely $p = 2$, $u = 1$ and $v = r - 1$, i.e. $p = 2$ and $d = (1, 2, 2, \ldots, 2, 1)$, where $d' = (2, 2, \ldots, 2, 2) \in D(s, r-1)$ and $\mathcal{E}(2^s, \mathcal{D}(\delta_{r-1}^{-1}(d'))) = \mathcal{E}(2^s, \mathcal{D}(\delta_r^{-1}(d)))$.

Proof. Since $\sum d'_j = \sum d_j = s - 1$, we have $d' \in D(s, r-1)$. We set $a = (a_1, \ldots, a_r) := \delta_r^{-1}(d) \in A(s,r)$. Then it is easy to check that $a' := \delta_{r-1}^{-1}(d') \in A(s, r-1)$ is the (u,v)-derivative of a. By virtue of condition (ii),

we can apply the Tableau Reduction Lemma 2.6. Before evaluating (2.7), we observe that

$$U = \sum_{k=1}^{u} p^{a_k} \geq p^{a_u} \quad \text{and} \quad V = \sum_{i=v+1}^{r} \frac{1}{p^{a_i}} \geq \frac{1}{p^{a_{v+1}}},$$

where equality in both cases simultaneously holds if and only if $u = 1$ and $v = r - 1$. Inserting this into (2.7) and making use of (i), that is $a_{u+1} - a_u = d_u = 1$ and $a_{v+1} - a_v = d_v = 1$ by definition of δ_r, we obtain

$$h_{p,r}(a) - h_{p,r-1}(a') = h_{p,r}(a) - h_{p,r-1}(a')$$

$$\geq \frac{1}{p+1}\left(p + \frac{1}{p^{2(v-u-1)}}\right)\left(\frac{p^{a_u}}{p^{a_{u+1}}} + \frac{p^{a_v}}{p^{a_{v+1}}}\right)$$

$$+ \frac{1}{p^2 - 1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)$$

$$= \frac{2}{p(p+1)}\left(p + \frac{1}{p^{2(v-u-1)}}\right) + \frac{1}{p^2 - 1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)$$

$$= \frac{2p-1}{p^2 - 1} + \frac{p-2}{p^{2v-2u-1}(p^2 - 1)}$$

$$\geq \frac{1}{p-1}$$

for all primes p, but equality in the final step only for $p = 2$. Altogether, we have shown that $h_{p,r}(a) - h_{p,r-1}(a') > 1/(p-1)$ in all cases with the unique exception excluded in the Proposition, where equality holds. The proof is completed by Lemma 2.1. $\qquad\square$

Proposition 2.8. *Let p be a prime. For some integers $4 \leq r \leq s$ and $1 \leq u < v \leq r - 2$, let $d' = (d'_1, \ldots, d'_{r-2}) \in D(s, r-1)$ have the following properties:*

(i) $d'_u = d'_v = 3$,
(ii) $d'_j = 2$ for $u < j < v$.

Then $d = (d_1, \ldots, d_{r-1})$, defined by

$$d_j := \begin{cases} d'_j & \text{for } 1 \leq j \leq u-1 \text{ or } u+1 \leq j \leq v-1, \\ 2 & \text{for } u \leq j \leq v+1, \\ d'_{j-1} & \text{for } v+2 \leq j \leq r-1, \end{cases}$$

satisfies $d \in D(s, r)$ and $\mathcal{E}(p^s, \mathcal{D}(\delta_r^{-1}(d))) > \mathcal{E}(p^s, \mathcal{D}(\delta_{r-1}^{-1}(d')))$.

Proof. Since $\sum d_j = \sum d'_j = s - 1$, we have $d \in D(s, r)$. Let $a' = (a'_1, \ldots, a'_{r-1}) := \delta_{r-1}^{-1}(d') \in A(s, r-1)$, and define $a = (a_1, \ldots, a_r) := \delta_r^{-1}(d) \in A(s, r)$. It is straightforward to check that $a' = \partial_{u,v}(a)$. By condition (ii), we can thus apply the Tableau Reduction Lemma 2.6, which

yields

$$h_{p,r}(a) - h_{p,r-1}(a') = \frac{1}{p+1}\left(p + \frac{1}{p^{2(v-u-1)}}\right)\left(\frac{U}{p^{a_{u+1}}} + p^{a_v}V\right)$$
$$+ \frac{1}{p^2-1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right),$$

where

$$U = \sum_{k=1}^{u} p^{a_k} < \sum_{k=0}^{\infty} p^{a_u-k} = \frac{p^{1+a_u}}{p-1}$$

and

$$V = \sum_{i=v+1}^{r} \frac{1}{p^{a_i}} < \sum_{i=0}^{\infty} \frac{1}{p^{a_{v+1}+i}} = \frac{p^{1-a_{v+1}}}{p-1}.$$

Putting this together and applying $a_{u+1} - a_u = d_u = d'_u - 1 = 2$ and $a_{v+1} - a_v = d_v = d'_v - 1 = 2$ by virtue of (i), we have

$$h_{p,r}(a) - h_{p,r-1}(a') < \frac{1}{p^2-1}\left(p + \frac{1}{p^{2(v-u-1)}}\right)\left(\frac{p^{a_u+1}}{p^{a_{u+1}}} + \frac{p^{a_v}}{p^{a_{v+1}-1}}\right)$$
$$+ \frac{1}{p^2-1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)$$
$$= \frac{2}{p(p^2-1)}\left(p + \frac{1}{p^{2(v-u-1)}}\right)$$
$$+ \frac{1}{p^2-1}\left(1 - \frac{1}{p^{2(v-u-1)}}\right)$$
$$= \frac{1}{p^2-1}\left(3 - \frac{p-2}{p^{2v-2u-1}}\right)$$
$$\leq \frac{1}{p-1}$$

for all primes p. By Lemma 2.1, this concludes the proof. □

Proposition 2.9. *Let p be a prime. If $d = (d_1, \ldots, d_{r-1}) \in D(s,r)$ for some integers $4 \leq r \leq s$ with $d_u = 1$ for some u satisfying $2 \leq u \leq r-2$ and $d_j = 2$ for $j \neq u$, then $\mathcal{E}(p^s, \mathcal{D}(\delta_r^{-1}(d))) < \mathcal{E}(p^s, \mathcal{D}(\delta_r^{-1}(d')))$, where $d' := (2, 2, \ldots, 2, 2, 1) \in D(s,r)$.*

Proof. This is a special case of [25], Proposition 3.2. Instead of becoming acquainted with the notation there, the reader might be well advised to look at the involved delta tableaux and do some calculations similar to those above. □

3. Proof of the Theorem

As mentioned at the end of the introduction, the proof of Theorem 1.1 uses an accordion-like compression and expansion procedure. Starting with an arbitrary $a^{(0)} \in \bigcup_{r=2}^{s} A(s,r)$, a finite sequence $(a^{(\ell)})_{0 \leq \ell \leq m}$, say, of admissible $a^{(\ell)} \in \bigcup_{r=2}^{s} A(s,r)$ satisfying

$$(3.1) \qquad \mathcal{E}(p^s, \mathcal{D}(a^{(0)})) < \mathcal{E}(p^s, \mathcal{D}(a^{(1)})) < \cdots < \mathcal{E}(p^s, \mathcal{D}(a^{(m)})),$$

can be generated, where $a^{(m)}$ is one of the maximising exponent tuples to be found in Theorem 1.1. It simplifies the matter if we perform our compression and expansion procedure on the delta vectors $d^{(\ell)} := \delta_{r(\ell)}(a^{(\ell)}) \in D(s, r(\ell))$, $0 \leq \ell \leq m$, for suitable lengths parameters $r(\ell)$. By this (3.1) is replaced by

$$
\begin{aligned}
(3.2) \qquad & \mathcal{E}(p^s, \mathcal{D}(\delta_{r(0)}^{-1}(d^{(0)}))) < \mathcal{E}(p^s, \mathcal{D}(\delta_{r(1)}^{-1}(d^{(1)}))) < \cdots \\
& \qquad\qquad\qquad \cdots < \mathcal{E}(p^s, \mathcal{D}(\delta_{r(m)}^{-1}(d^{(m)}))).
\end{aligned}
$$

Each transformation $d^{(\ell)} \longrightarrow d^{(\ell+1)}$ corresponds to the application of one of the Propositions in Section 2. In order to indicate which Proposition is used and which effect it has, we label the transformation arrows according to the table below:

Transf.	Prop. #	Effect	Condition
$\xrightarrow{\text{Ia}}$	2.2(i)	Replaces entry $d_u \geq 4$ by successive entries $2, d_u - 2$	./.
$\xrightarrow{\text{Ib}}$	2.2(ii)	Replaces $d_u = 3$ by successive entries $2, 1$	$2 \leq d_j \leq 3$ for all j
$\xrightarrow{\text{II}}$	2.4	Replaces $d_u = 1, d_v = 3$ or $d_u = 3, d_v = 1$ $(u < v)$ by $2, 2$	$d_j = 2$ for $u < j < v$
$\xrightarrow{\text{III}}$	2.7	Replaces $d_u = d_v = 1$ $(u < v)$ by 2	$d_j = 2$ for $u < j < v$
$\xrightarrow{\text{IV}}$	2.8	Replaces $d_u = d_v = 3$ $(u < v)$ by $2, 2, 2$	$d_j = 2$ for $u < j < v$
$\xrightarrow{\text{V}}$	2.9	Shifts $d_u = 1$ $(u < r - 1)$ to the right	$d_j = 2$ for $j \neq u$

TABLE 1. List of transformations

The following proof of Theorem 1.1 demonstrates in which order the transformations can be applied to an initial delta vector $d^{(0)}$. Observe that the

sequence (3.2) is by no means unique, but the final delta vector $d^{(m)}$ very well is (with the solitary ambiguity for $p = 2$ mentioned in Theorem 1.1(i)). Example 3.1 right behind the proof will illustrate the transformation process.

Proof of Theorem 1.1. In [23], Theorem 3.2, the maximal energies $\mathcal{E}_{\max}(p^s)$ along with the corresponding divisor sets were determined for $s \leq 4$. Hence we may assume $s \geq 5$. Let a be an exponent tuple having the property that $\mathcal{D}(a)$ maximises the energy of $\mathrm{ICG}(p^s, \mathcal{D})$, i.e. $\mathcal{E}(p^s, \mathcal{D}(a)) = \mathcal{E}_{\max}(p^s)$. In Section 1 we saw that necessarily $a \in A(s, r)$ for some r satisfying $2 \leq r \leq s$. Define $d = (d_1, \ldots, d_{r-1}) := \delta_r(a) \in D(s, r)$.

It follows from Proposition 2.2(i) that $d_j \in \{1, 2, 3\}$ for $1 \leq j \leq r - 1$. Thus we can visualise d as an $(r-1)$-tuple possibly containing some entries 1 and 3 as well as entries 2 filling the gaps. If d contained two or more entries from the set $\{1, 3\}$, this would contradict at least one of the Propositions 2.4, 2.7 or 2.8, except for the special case $d = (1, 2, 2, \ldots, 2, 1)$ for $p = 2$ from Proposition 2.7. Therefore, only three situations remain: (i) $d = (2, 2, \ldots, 2)$, (ii) $d = (1, 2, 2, \ldots, 2, 1)$ and $p = 2$, (iii) d has a single entry 1 or 3 with all other entries equal to 2.

Case 1: s is odd.

Since $\sum_{j=1}^{r-1} d_j = s - 1$ is even, $d = (2, 2, \ldots, 2) \in D(s, (s+1)/2)$ or, only in the case $p = 2$, $d = (1, 2, 2, \ldots, 2, 1) \in D(s, (s+3)/2)$. By use of (1.1) and comparison of the delta tableaux corresponding to $\delta_{(s+1)/2}^{-1}((2, 2, \ldots, 2))$ and $\delta_{(s+3)/2}^{-1}((1, 2, 2, \ldots, 2, 1))$, it is an easy exercise to check that

$$\mathcal{E}(2^s, \mathcal{D}(\delta_{(s+1)/2}^{-1}(2, 2, \ldots, 2))) = \mathcal{E}(2^s, \mathcal{D}(\delta_{(s+3)/2}^{-1}(1, 2, 2, \ldots, 2, 1))).$$

It remains to prove the formula for $\mathcal{E}_{\max}(p^s)$. We have

$$h_{p,(s+1)/2}(0, 2, 4, \ldots, s - 3, s - 1) = \sum_{k=1}^{(s-1)/2} \sum_{i=k+1}^{(s+1)/2} \frac{1}{p^{2(i-k)}}$$

$$(3.3) \qquad\qquad = \sum_{j=1}^{(s-1)/2} \frac{1}{p^{2j}} \left(\frac{s+1}{2} - j \right)$$

$$= \frac{(s-1)p^{s+1} - (s+1)p^{s-1} + 2}{2(p^2 - 1)^2 p^{s-1}},$$

and hence by (1.1)

$$\mathcal{E}_{\max}(p^s) = \mathcal{E}(p^s, \mathcal{D}((0, 2, 4, \ldots, s - 3, s - 1)))$$

$$= 2(p-1)p^{s-1} \left(\frac{s+1}{2} - (p-1)h_{p,(s+1)/2}(0, 2, 4, \ldots, s - 3, s - 1) \right)$$

$$= \frac{1}{(p+1)^2} \left((s+1)(p^2 - 1)p^s + 2(p^{s+1} - 1) \right),$$

which proves (i).

Case 2: s is even.

Since $\sum_{j=1}^{r-1} d_j = s - 1$ is odd, d must have a single entry 1 or 3 with all other entries equal to 2. By Proposition 2.2(ii) the unique odd entry is 1, and by Proposition 2.9 and the symmetry property (1.3), we necessarily have $d = (2, 2, \ldots, 2, 2, 1) \in D(s, (s + 2)/2)$ or $d = (1, 2, 2, \ldots, 2, 2) \in D(s, (s+2)/2)$ with equal corresponding energy. With regard to $\mathcal{E}_{\max}(p^s)$, we have in this case

(3.4)

$$
h_{p,(s+2)/2}(0, 2, \ldots, s - 2, s - 1) = \sum_{k=1}^{(s-2)/2} \sum_{i=k+1}^{s/2} \frac{1}{p^{2(i-k)}} + \sum_{k=1}^{s/2} \frac{1}{p^{s-1-2(k-1)}}
$$

$$
= \sum_{j=1}^{(s-2)/2} \frac{1}{p^{2j}} \left(\frac{s}{2} - j \right) + \frac{1}{p^{s-1}} \sum_{k=0}^{(s-2)/2} p^{2k}
$$

$$
= \frac{(s - 2)p^s - sp^{s-2} + 2}{2(p^2 - 1)^2 p^{s-2}} + \frac{p^s - 1}{(p^2 - 1)p^{s-1}},
$$

and hence by (1.1)

$$
\mathcal{E}_{\max}(p^s) = \mathcal{E}(p^s, \mathcal{D}((0, 2, 4, \ldots, s - 2, s - 1)))
$$

$$
= 2(p - 1)p^{s-1} \left(\frac{s + 2}{2} - (p - 1)h_{p,(s+2)/2}(0, 2, 4, \ldots, s - 2, s - 1) \right)
$$

$$
= \frac{1}{(p + 1)^2} \left(s(p^2 - 1)p^s + 2(2p^{s+1} - p^{s-1} + p^2 - p - 1) \right),
$$

and this proves (ii). $\qquad\square$

Example 3.1. Let

$$
a^{(0)} := (0, 5, 6, 9, 12, 14, 15, 16, 22, 23, 24, 27, 29) \in A(30, 13),
$$

and hence

$$
d^{(0)} := \delta_{13}(a^{(0)}) = (5, 1, 3, 3, 2, 1, 1, 6, 1, 1, 3, 2) \in D(30, 13).
$$

Using the abbreviation $\partial(\ell) := \delta_{r(\ell)}^{-1}(d^{(\ell)})$ and applying the transformations of Table 1, we obtain step by step:

ℓ	$d^{(\ell)}$	$r(\ell)$	$\mathcal{E}(2^{30}, \mathcal{D}(\partial(\ell)))$	$\mathcal{E}(3^{30}, \mathcal{D}(\partial(\ell)))$
0	$(\underline{5}, 1, 3, 3, 2, 1, 1, 6, 1, 1, 3, 2)$	13	9 167 691 382	2 293 430 091 118 444
	$\downarrow^{\text{Ia}}$			
1	$(\mathbf{2}, \mathbf{3}, 1, 3, 3, 2, 1, 1, \underline{6}, 1, 1, 3, 2)$	14	9 761 773 390	2 479 571 746 112 800
	$\downarrow^{\text{Ia}}$			
2	$(2, 3, 1, 3, 3, 2, 1, 1, \mathbf{2}, \underline{\underline{4}}, 1, 1, 3, 2)$	15	10 226 403 150	2 655 370 924 580 476
	$\downarrow^{\text{Ia}}$			
3	$(2, 3, 1, 3, 3, 2, 1, \underline{1}, 2, \mathbf{2}, \mathbf{2}, \underline{\underline{1}}, 1, 3, 2)$	16	10 429 199 182	2 770 612 868 608 768
	$\downarrow^{\text{III}}$			
4	$(2, 3, \underline{1}, \underline{3}, 3, 2, 1, \mathbf{2}, 2, 2, 2, 1, 3, 2)$	15	10 869 926 478	2 937 991 189 453 948
	$\downarrow^{\text{II}}$			
5	$(2, 3, \mathbf{2}, \mathbf{2}, 3, 2, \underline{1}, 2, 2, 2, 2, \underline{1}, 3, 2)$	15	11 022 317 518	3 022 615 444 978 108
	$\downarrow^{\text{III}}$			
6	$(2, 3, 2, 2, \underline{3}, 2, \mathbf{2}, 2, 2, 2, 2, \underline{3}, 2)$	14	11 182 822 222	3 112 785 070 640 560
	$\downarrow^{\text{IV}}$			
7	$(2, \underline{3}, 2, 2, \mathbf{2}, 2, 2, 2, 2, 2, 2, \mathbf{2}, \mathbf{2}, 2)$	15	11 438 333 038	3 216 413 472 521 788
	$\downarrow^{\text{Ib}}$			
8	$(2, \underline{\underline{1}}, \mathbf{2}, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$	16	11 483 072 286	3 218 955 338 350 144
	$\downarrow^{\text{V}}$			
9	$(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \mathbf{1})$	16	11 572 550 770	3 234 206 533 320 112

According to Theorem 1.1(ii) we have

$$\mathcal{E}_{\max}(2^{30}) = \mathcal{E}(2^{30}, \mathcal{D}((0, 2, 4, \ldots, 28, 29))) = 11\ 572\ 550\ 770,$$

and

$$\mathcal{E}_{\max}(3^{30}) = \mathcal{E}(3^{30}, \mathcal{D}((0, 2, 4, \ldots, 28, 29))) = 3\ 234\ 206\ 533\ 320\ 112.$$

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