Contributions to Discrete Mathematics

Volume 9, Number 1, Pages 85–97
ISSN 1715-0868

INTERLACEMENT IN 4-REGULAR GRAPHS: A NEW APPROACH USING NONSYMMETRIC MATRICES

LORENZO TRALDI

Abstract. Let F be a 4-regular graph with an Euler system C. We introduce a simple way to modify the interlacement matrix of C so that every circuit partition P of F has an associated modified interlacement matrix $M(C,P)$. If C and C' are Euler systems of F then $M(C,C')$ and $M(C',C)$ are inverses, and for any circuit partition P, $M(C',P) = M(C',C)\cdot M(C,P)$. This machinery allows for short proofs of several results regarding the linear algebra of interlacement.

1. Interlacement and local complements

A *graph* $G = (V(G), E(G))$ is given by a finite set $V(G)$ of *vertices*, and a finite set $E(G)$ of *edges*. In a *looped simple graph* each edge is incident on one or two vertices, and different edges have different vertex-incidences; an edge incident on only one vertex is a *loop*. A *simple graph* is a looped simple graph with no loop. In general, a graph may have *parallel edges* (distinct edges with the same vertex-incidences). Edge-vertex incidences generate an equivalence relation on $E(G)\cup V(G)$; the equivalence classes are the *connected components* of G, and the number of connected components is denoted $c(G)$. Two vertices incident on a non-loop edge are *neighbors*, and if $v\in V(G)$ then $N(v) = \{\text{neighbors of } v\}$ is the *open neighborhood* of v.

Each edge consists of two distinct *half-edges*, and the edge has two distinct *directions* given by designating one half-edge as initial and the other as terminal. Each half-edge is incident on a vertex; if the edge is not a loop then the half-edges are incident on different vertices. The number of half-edges incident on a vertex v is the *degree* of v, and a *d-regular* graph is one whose vertices all have degree d. In a directed graph each vertex has an *indegree* and an *outdegree*; a *d-in, d-out* digraph is one whose vertices all have indegree d and outdegree d. A *circuit* in a graph is a sequence v_1, h_1, h_1', v_2, ..., v_k, h_k, h_k', $v_{k+1} = v_1$ such that for each i, h_{i+1} and h_i' are half-edges incident on v_{i+1}, and h_i and h_i' are the half-edges of an edge e_i; $e_i \neq e_j$ when $i \neq j$. A *directed* circuit in a directed graph is a circuit in which h_i is the initial half-edge of e_i, for every i. An *Euler circuit* is a

Received by the editors July 14, 2012, and in revised form February 4, 2014.
2010 *Mathematics Subject Classification.* 05C31.
Key words and phrases. 4-regular graph, circuit partition, Euler system, interlacement.

circuit in which every edge appears exactly once; more generally, an *Euler system* is a collection of Euler circuits, one in each connected component of the graph. A graph has Euler systems if and only if every vertex is of even degree; we refer to Fleischner's books [17, 18] for the general theory of Eulerian graphs.

In this paper, we are concerned with the theory of Euler systems in 4-regular graphs, introduced by Kotzig [25]. If v is a vertex of a 4-regular graph F and C is an Euler system of F, then the *κ-transform* $C * v$ is the Euler system obtained from C by reversing one of the two v-to-v walks within the circuit of C incident on v. *Kotzig's theorem* is that all Euler systems of F can be obtained from any one using finite sequences of κ-transformations.

The *interlacement graph* $\mathcal{I}(C)$ of a 4-regular graph F with respect to an Euler system C was introduced by Bouchet [7] and Read and Rosenstiehl [34].

Definition 1.1. *$\mathcal{I}(C)$ is the simple graph with $V(\mathcal{I}(C)) = V(F)$, in which v and w are adjacent if and only if they appear in the order $v...w...v...w...$ on one of the circuits of C.*

There is a natural way to construct $\mathcal{I}(C * v)$ from $\mathcal{I}(C)$.

Definition 1.2. *Let G be a simple graph, and suppose $v \in V(F)$. The* simple local complement G^v *is the graph obtained from G by reversing adjacencies between neighbors of v.*

That is, if w and x are distinct elements of $V(G) = V(G^v)$ then w and x are neighbors in G^v if and only if either (a) at least one of them is not a neighbor of v, and they are neighbors in G; or (b) both are neighbors of v, and they are not neighbors in G. The well-known equality $\mathcal{I}(C * v) = \mathcal{I}(C)^v$ follows from the fact that reversing one of the two v-to-v walks within the incident circuit of C has the effect of toggling adjacencies between vertices that appear once apiece on this walk.

Another way to describe simple local complementation involves the following.

Definition 1.3. *The* Boolean adjacency matrix *of a graph G is the symmetric $V(G) \times V(G)$ matrix $\mathcal{A}(G)$ with entries in $GF(2)$ given by: a diagonal entry is 1 if and only if the corresponding vertex is looped in G, and an off-diagonal entry is 1 if and only if the corresponding vertices are neighbors in G.*

Definition 1.4. *Suppose G is a simple graph and*

$$\mathcal{A}(G) = \begin{bmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & M_{11} & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{bmatrix},$$

with the first row and column corresponding to v. Then G^v is the simple graph whose adjacency matrix is

$$\mathcal{A}(G^v) = \begin{bmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \overline{M}_{11} - I & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{bmatrix},$$

where I is an identity matrix and the overbar indicates toggling of all entries.

Kotzig's theorem tells us that the Euler systems of a 4-regular graph F form an orbit under κ-transformations. It follows that the interlacement graphs of Euler systems of F form an orbit under simple local complementation. From a combinatorial point of view this "naturality" of interlacement graphs is intuitively satisfying: the Euler systems of F must share some structural features, as they coexist in F, and these shared structural features are reflected in shared structural features of their interlacement graphs. Many researchers have studied simple local complementation in the decades since Kotzig founded the theory; the associated literature is large and quite fascinating. We do not presume to summarize this body of work, but we might mention that intrinsic properties distinguish the simple graphs that arise as interlacement graphs from those that do not [12, 16] and that 4-regular graphs with isomorphic interlacement graphs are closely related to each other [19].

In contrast, the algebraic properties of interlacement graphs are *not* intuitively satisfying. The adjacency matrices of the various interlacement graphs associated to F have little in common, aside from the fact that they are symmetric matrices of the same size. To say the same thing in a different way, simple local complementation changes fundamental algebraic properties of the adjacency matrix. For instance the ranks of $\mathcal{A}(G)$ and $\mathcal{A}(G^v)$ may be quite different; this rank change is caused by the $-I$ in Definition 1.4.

The purpose of this paper is to present *modified interlacement matrices*, whose algebraic properties are in many ways more natural than those of interlacement matrices. We present the theory of these matrices in Section 2, and then briefly summarize the connections between this theory and earlier work in Section 3.

2. Modified interlacement and local complements

Our modifications involve the following notions. If v is a vertex of a 4-regular graph F then Kotzig [25] observed that there are three *transitions* at v, i.e., three different pairings of the four incident half-edges into disjoint pairs. If C is an Euler system of F then we can classify these three transitions according to their relationship with C, as in [39, 40]. One transition appears in C; we label this one ϕ, for *follow*. Of the other two transitions, one is consistent with an orientation of the incident circuit of C, and the other

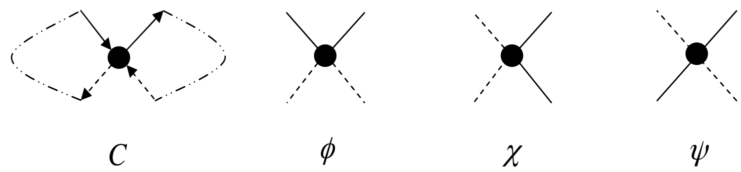

FIGURE 1. The three transitions at v are labeled according to their relationship with an Euler system C.

is not; we label them χ and ψ, respectively. (It does not matter which orientation of this circuit of C is used.)

See Figure 1, where circuits are indicated with this convention: when a circuit traverses a vertex, the dash pattern is maintained. In more complex diagrams like Figure 3 it is sometimes necessary to change the dash pattern in the middle of an edge, in order to make sure that two different dash patterns appear at each vertex.

There are $3^{|V(G)|}$ different ways to choose ϕ, χ or ψ at each vertex of F. Each system of choices determines a *circuit partition* or *Eulerian partition* of F, i.e., a partition of $E(F)$ into edge-disjoint circuits. The circuit partitions that include precisely $c(F)$ circuits are the Euler systems of F. Circuit partitions have received a great deal of attention since they were introduced by Kotzig [25], who called them *ξ-decompositions*. Building on earlier work of Martin [31], Las Vergnas [26, 27, 28] introduced the idea of using the generating function $\sum x^{|P|}$ that records the sizes of the circuit partitions of F as a structural invariant of F. This idea has subsequently appeared in knot theory (where it underlies the Kauffman bracket [23]) and in general graph theory (where it motivates the interlace polynomials of Arratia, Bollobás and Sorkin [2, 3, 4]).

Here is the central definition of the paper.

Definition 2.1. *Let C be an Euler system of a 4-regular graph F, and let P be a circuit partition of F. Then the* modified interlacement matrix of C with respect to P *is the matrix $M(C,P)$ obtained from $\mathcal{A}(\mathcal{I}(C))$ by making the following changes:*

(1) *If P involves the ϕ transition with respect to C at a vertex v, then change the diagonal entry corresponding to v to 1 and change every other entry in that column to 0.*
(2) *If P involves the ψ transition with respect to C at a vertex v, then change the diagonal entry corresponding to v to 1.*

Definition 2.1 might seem complicated and unmotivated, but it has the surprising virtue that the modified interlacement matrices of different Euler systems with respect to a given circuit partition are related to each other through elementary row operations. Consequently, these modified interlacement matrices share many algebraic properties – for instance, they all have the same rank and the same right nullspace – and familiar ideas of elementary linear algebra can be used to explain these properties.

Definition 2.2. *Let G be a graph, and let M be a matrix whose rows and columns are indexed by $V(G)$. Suppose $v \in V(G)$ and M is*

$$M = \begin{bmatrix} d_{vv} & \rho_1 & \rho_2 \\ \kappa_1 & M_{11} & M_{12} \\ \kappa_2 & M_{21} & M_{22} \end{bmatrix},$$

where the first row and column correspond to v, the rows and columns of M_{11} correspond to vertices in $N(v)$, and the rows and columns of M_{22} correspond to vertices in $V(G) - N(v) - \{v\}$. Then the modified local complement *of M with respect to v is the matrix obtained from M by adding the v row to every row corresponding to a neighbor of v:*

$$M^v_{\text{mod}} = \begin{bmatrix} d_{vv} & \rho_1 & \rho_2 \\ \kappa_1' & M_{11}' & M_{12}' \\ \kappa_2 & M_{21} & M_{22} \end{bmatrix}.$$

Theorem 2.3. *Let C be an Euler system of a 4-regular graph F, let P be a circuit partition of F, and let $v \in V(F)$. Consider $M(C,P)$ to have row and column indices from $V(\mathcal{I}(C))$. Then*

$$M(C,P)^v_{\text{mod}} = M(C*v,P).$$

Proof. Let $\vec{v} \in GF(2)^{V(F)}$ be the vector whose only nonzero coordinate corresponds to v, and let $\vec{N}(v) \in GF(2)^{V(F)}$ be the vector whose w coordinate is 1 if and only if w neighbors v in $\mathcal{I}(C)$ and $\mathcal{I}(C*v)$.

We first verify that $M(C*v,P)$ and $M(C,P)^v_{\text{mod}}$ have the same v column. As illustrated in Figure 2, if P involves the ϕ (resp. χ) (resp. ψ) transition at v with respect to C, then P involves the ψ (resp. χ) (resp. ϕ) transition with respect to $C*v$. If P involves the ϕ transition at v with respect to C, then according to Definition 2.1, the v column of $M(C,P)$ is $\vec{v}$ and the v column of $M(C,P)^v_{\text{mod}}$ is $\vec{v}+\vec{N}(v)$. As P involves the ψ transition at v with respect to $C*v$, the v column of $M(C*v,P)$ is also $\vec{v}+\vec{N}(v)$. If P involves the χ transition at v with respect to C, then $M(C,P)$ and $M(C,P)^v_{\text{mod}}$ have the same v column, namely $\vec{N}(v)$. This is also the v column of $M(C*v,P)$, since P involves the χ transition at v with respect to $C*v$. If P involves the ψ transition at v with respect to C, then according to Definition 2.1, the v column of $M(C,P)$ is $\vec{v}+\vec{N}(v)$, so the v column of $M(C,P)^v_{\text{mod}}$ is $\vec{v}$. As P involves the ϕ transition at v with respect to $C*v$, the v column of $M(C*v,P)$ is also $\vec{v}$.

Now consider one of the columns involved in M_{11}. This column corresponds to a vertex w that neighbors v in $\mathcal{I}(C)$ and $\mathcal{I}(C*v)$. Let $\vec{N}(w) \in GF(2)^{V(F)}$ be the vector whose x coordinate is 1 if and only if w neighbors

x in $\mathcal{I}(C)$, and let $\vec{N}'(w) \in GF(2)^{V(F)}$ be the vector whose x coordinate is 1 if and only if w neighbors x in $\mathcal{I}(C * v) = \mathcal{I}(C)^v$. Then Definition 1.4 tells us that $\vec{N}'(w) = \vec{N}(v) + \vec{N}(w) - \vec{w}$. As indicated in Figure 3, if P involves the ϕ (resp. χ) (resp. ψ) transition at w with respect to C, then P involves the ϕ (resp. ψ) (resp. χ) transition at w with respect to $C * v$. If P involves the ϕ transition at w with respect to C and $C * v$, then $M(C, P)$ and $M(C * v, P)$ have the same w column, namely $\vec{w}$. The vw entry of $M(C, P)$ is 0, so the w column of $M(C, P)^v_{\text{mod}}$ is also $\vec{w}$. If P involves the χ transition at w with respect to C then the w column of $M(C, P)$ is $\vec{N}(w)$. P involves the ψ transition at w with respect to $C * v$, so the w column of $M(C * v, P)$ is $\vec{N}'(w) + \vec{w} = \vec{N}(v) + \vec{N}(w)$. The vw entry of $M(C, P)$ is 1, so the w column of $M(C, P)^v_{\text{mod}}$ is also $\vec{N}(v) + \vec{N}(w)$. A similar argument shows that if P involves the ψ transition at w with respect to C, then the w columns of $M(C, P)^v_{\text{mod}}$ and $M(C * v, P)$ both equal $\vec{N}'(w)$. In every case, then, the w columns of $M(C, P)^v_{\text{mod}}$ and $M(C * v, P)$ are the same.

Finally, consider one of the columns involved in M_{12}. This column corresponds to a vertex w that does not neighbor v in $\mathcal{I}(C)$ and $\mathcal{I}(C * v)$. It follows that $M(C, P)$, $M(C * v, P)$ and $M(C, P)^v_{\text{mod}}$ all have the same w column. □

Corollary 2.4. *Suppose C and C' are two Euler systems of F. Then $M(C', C)$ is nonsingular and for every circuit partition P,*

$$M(C', P) = M(C', C) \cdot M(C, P).$$

Proof. Consider the double matrix

$$\begin{bmatrix} I & M(C, P) \end{bmatrix}$$

where $I = M(C, C)$ is the identity matrix. According to Kotzig's theorem, it is possible to obtain C' from C using a finite sequence of κ-transformations. Theorem 2.3 tells us that after applying the corresponding sequence of modified local complementations we will have obtained the double matrix

$$\begin{bmatrix} M(C', C) & M(C', P) \end{bmatrix}.$$

If E is the product of the elementary matrices corresponding to the row operations involved in the modified local complementations, then $M(C', C) = E \cdot I$ and $M(C', P) = E \cdot M(C, P)$. □

We refer to the formula $M(C', P) = M(C', C) \cdot M(C, P)$ as *naturality* of the modified interlacement matrices.

Corollary 2.5. *If C and C' are Euler systems of F then*

$$M(C, C') = M(C', C)^{-1}.$$

It follows from Corollary 2.4 that all the modified interlacement matrices of a circuit partition P have the same right nullspace, i.e., the space

$$\ker M(C, P) = \{n \in GF(2)^{V(F)} \mid M(C, P) \cdot n = \mathbf{0}\}$$

does not vary with C. As we will see in Theorem 2.8, $\ker M(C,P)$ coincides with the *core space* of P, defined as follows by Jaeger [21].

Definition 2.6. *If γ is a circuit of F then the* core vector *$core(\gamma)$ is the element of $GF(2)^{V(F)}$ whose v coordinate is 1 if and only if γ is singly incident at v, i.e., γ includes precisely two of the four half-edges incident at v. The* core space *$core(P)$ is the subspace of $GF(2)^{V(F)}$ spanned by the core vectors of circuits of P.*

Observe that $core(\gamma) = \mathbf{0}$ if and only if γ is an Euler circuit of a connected component of F.

Here is a useful construction.

Lemma 2.7. *Let P be a circuit partition of a 4-regular graph F, which is not an Euler system. Then there is an Euler system C of F with the following properties:*

(1) *P involves only ϕ and χ transitions with respect to C.*
(2) *There is a circuit $\gamma_0 \in P$ and a vertex v_0 incident on γ_0, such that P involves the χ transition with respect to C at v_0, and P involves the ϕ transition with respect to C at every other vertex incident on γ_0.*
(3) *The core vector $core(\gamma_0)$ is $\vec{v}_0 + \vec{N}(v_0)$, where $\vec{v}_0 \in GF(2)^{V(F)}$ is the vector whose only nonzero coordinate corresponds to v_0, and $\vec{N}(v_0) \in GF(2)^{V(F)}$ is the vector whose w coordinate is 1 if and only if w neighbors v_0 in $\mathcal{I}(C)$.*

Proof. We build an Euler system C from P as follows. For each circuit of P, arbitrarily choose a preferred orientation. Find a vertex where two distinct circuits of P are incident, and let P' be the circuit partition obtained by uniting the two incident circuits into one circuit, as indicated in Figure 4. If this new circuit is not an Euler circuit of a connected component of F, then there must be a vertex at which two distinct circuits of P' are incident, one of the two circuits being the new one and the other being a circuit of P; unite these two into one circuit as in Figure 4. Continue this process until an Euler circuit of a connected component of F is obtained. If F has another connected component that contains two distinct circuits of P, repeat the process in that connected component. After $|P| - c(F)$ steps, each step uniting two distinct circuits at least one of which is an element of P, we must end with an Euler system C. Observe that at every vertex where two circuits are united during the construction, P involves the χ transition with respect to C; at every other vertex, P involves the ϕ transition with respect to C.

Let v_0 be the vertex at which two circuits are united in the last step of the construction. Suppose that in the last step, a circuit $\gamma_0 \in P$ is united with some other circuit at v_0. As $\gamma_0 \in P$, γ_0 must not have been involved in any earlier step of the construction. Consequently, every vertex of γ_0 other than v_0 is a vertex where P involves the ϕ transition with respect to C. Also, one

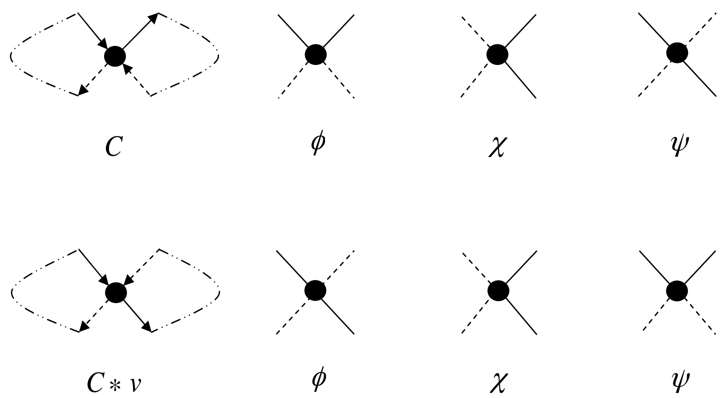

FIGURE 2. When C is replaced with $C * v$, the ϕ and ψ transition labels are interchanged at v.

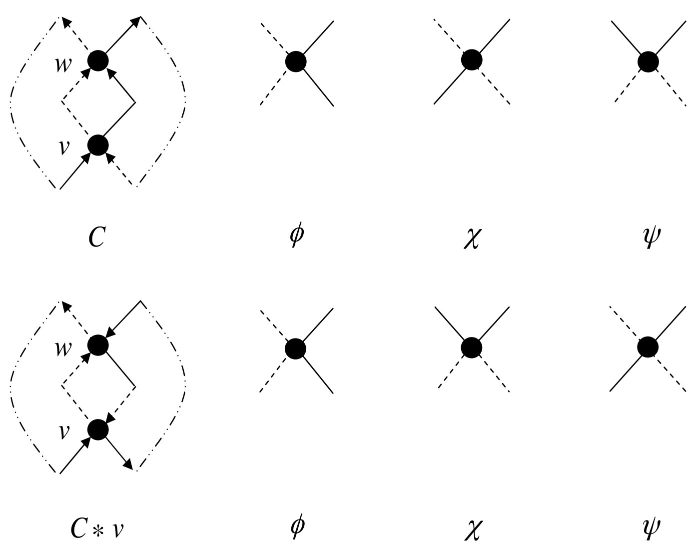

FIGURE 3. When C is replaced with $C * v$, the χ and ψ transition labels are interchanged at vertices interlaced with v.

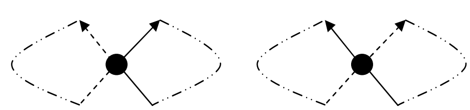

FIGURE 4. Two circuits are united, respecting orientations.

of the v_0-to-v_0 walks within the incident circuit of C simply follows γ_0, so a vertex $w \neq v_0$ neighbors v_0 in $\mathcal{I}(C)$ if and only if w appears precisely once on γ_0. □

Theorem 2.8. *Let P be a circuit partition of a 4-regular graph F, and let C be an Euler system of F. Then*

$$core(P) = \ker M(C, P).$$

Proof. As $\ker M(C, P)$ does not vary with C, we need only prove that the theorem holds for one choice of C. If P itself is an Euler system, then the theorem holds because $M(P, P)$ is the identity matrix and every core vector of a circuit of P is $\mathbf{0}$.

The proof proceeds by induction on $|P| > c(F)$. Let C, v_0 and γ_0 be as in the lemma. Then the v_0 row of $M(C, P)$ is $\mathbf{0}$, and the v_0 column of $M(C, P)$ is $\vec{N}(v_0)$.

Let γ_1 be the other circuit of P incident at v_0, and let P' be the circuit partition obtained from P by uniting γ_0 and γ_1 at v_0 as indicated in Figure 4. The only difference between the transitions that appear in P and the transitions that appear in P' occurs at v_0, where P involves the χ transition with respect to C and P' involves the ϕ transition with respect to C. Hence, the only difference between $M(C, P)$ and $M(C, P')$ is that the v_0 column of $M(C, P')$ is $\vec{v}_0$ and the v_0 column of $M(C, P)$ is $\vec{N}(v_0)$. That is,

$$M(C, P) = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & I & A \\ \mathbf{0} & \mathbf{0} & B \end{bmatrix} \text{ and } M(C, P') = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & A \\ \mathbf{0} & \mathbf{0} & B \end{bmatrix},$$

where the first row and column correspond to v_0, I is an identity matrix involving the rows and columns corresponding to neighbors of v_0 in $\mathcal{I}(C)$, and B is a square matrix involving the rows and columns corresponding to vertices $v \neq v_0$ that are not neighbors of v_0 in $\mathcal{I}(C)$.

Notice that $M(C, P')$ is row equivalent to

$$\begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & I & A \\ \mathbf{0} & \mathbf{0} & B \end{bmatrix},$$

which differs from $M(C, P)$ only in one entry. Consequently, the ranks of $M(C, P')$ and $M(C, P)$ do not differ by more than 1.

Considering the v_0 row of $M(C, P')$, we see that every element of the kernel of $M(C, P')$ must have its v_0 coordinate equal to 0. Clearly, then $\ker M(C, P') \subseteq \ker M(C, P)$. Note also that $core(\gamma_0) = \vec{v}_0 + \vec{N}(v_0) \notin \ker M(C, P')$ and $core(\gamma_0) \in \ker M(C, P)$. The ranks of $M(C, P)$ and $M(C, P')$ do not differ by more than 1, so

$$\ker M(C, P) = \ker M(C, P') \oplus [core(\gamma_0)],$$

where $[core(\gamma_0)]$ denotes the one-dimensional subspace spanned by $core(\gamma_0)$.

As $|P'| = |P| - 1$, we may assume inductively that $\ker M(C,P') = core(P')$. The core vectors of the circuits of P' coincide with the core vectors of the circuits of P, except for the fact that the core vector of the circuit obtained by uniting γ_0 and γ_1 is $core(\gamma_0) + core(\gamma_1)$. Consequently $\ker M(C,P) = \ker M(C,P') \oplus [core(\gamma_0)]$ is spanned by the core vectors of the circuits of P other than γ_0 and γ_1, together with $core(\gamma_0) + core(\gamma_1)$ and $core(\gamma_0)$. It follows that $\ker M(C,P) = core(P)$. □

Theorem 2.8 yields a useful formula with an interesting history; we call it the *circuit-nullity formula* [38]. Many special cases and different versions of the formula have been discovered independently during the last 100 years [5, 6, 10, 13, 15, 21, 22, 24, 29, 30, 32, 33, 35, 36, 37, 41]. In the notation of Theorem 2.8, the formula is

$$\dim \ker M(C,P) = |P| - c(F). \tag{2.1}$$

The proof is simple: Theorem 2.8 implies $\dim \ker M(C,P) = \dim core(P)$, and Lemma 2.9 implies $\dim core(P) = |P| - c(F)$.

Lemma 2.9. *Let S be a family of edge-disjoint circuits in a 4-regular graph F, and let $E'(S) = \{e \in E(F) \mid e$ does not appear in any element of $S\}$. Then the core vectors of the elements of S are linearly independent if and only if $E'(S)$ contains an edge from each connected component of F.*

Proof. If F has a connected component from which $E'(S)$ contains no edge, then the core vectors of the elements of S contained in that component sum to $\mathbf{0}$, so the core vectors of elements of S are linearly dependent.

Suppose $E'(S)$ contains an edge from every connected component of F. In order to prove that the core vectors of the elements of S are linearly independent, we must show that for every nonempty subset T of S, the sum of the core vectors of the elements of T is nonzero. As $T \subseteq S$, no element of T is an Euler circuit for the corresponding connected component of F, so the core vectors of the elements of T are all nonzero. If there is no vertex of F at which two different circuits of T are incident, then the core vectors of the elements of T are all orthogonal to each other, so their sum is certainly nonzero.

Suppose instead that two different circuits $\gamma_1, \gamma_2 \in T$ are incident at v. Let T' be a set of circuits obtained by uniting γ_1 and γ_2 into a single circuit γ, as in Figure 4; then $core(\gamma) = core(\gamma_1) + core(\gamma_2)$. The other circuits of T and T' are the same, so the sum of the core vectors of the elements of T is the same as the sum of the core vectors of the elements of T'. As $E'(S) \subseteq E'(T')$ and $|S| > |T'|$, we may presume inductively that the core vectors of the elements of T' are linearly independent, and so their sum is certainly nonzero. □

3. Discussion

As mentioned in the introduction, the theory outlined in Section 2 includes modified versions of ideas that have been known for decades. It seems that interlacement first appeared in Brahana's version of the circuit-nullity formula [13], but the notion did not achieve broad recognition until the 1970s when interlacement was rediscovered in two areas of combinatorics: Bouchet [7] introduced the *alternance* graph of an Euler circuit of a 4-regular graph, and Cohn and Lempel [15] used *link relation matrices* to state a special case of the circuit-nullity formula in the context of the theory of permutations. Shortly thereafter, Rosenstiehl and Read [34] coined the term *interlacement*, and used the technique to analyze the problem of identifying the double occurrence words that correspond to plane curves in general position (i.e., the only singularities are double points). Since then, interlacement has given rise to the theory of circle graphs [9, 10, 12, 16, 19] and the more general theory of graph equivalence under simple local complementation (see for instance Bouchet's work on isotropic systems [8, 11]), and these ideas have inspired the work of Arratia, Bollobás and Sorkin on the interlace polynomials of graphs [2, 3, 4]. Also, as noted above several special cases and different versions of the circuit-nullity formula have appeared in the literature of combinatorics and low-dimensional topology during the last thirty years, for instance in the work of Mellor [32], Soboleva [36] and Zulli [41] on polynomial invariants of knots and links.

Only symmetric matrices appear in the references mentioned above. The same is true of our earlier work [39]. The *relative interlacement matrices* we considered there are defined using a symmetric form of Definition 2.1, in which part 1 is replaced by the stipulation that the off-diagonal entries of the rows and columns corresponding to ϕ vertices are changed to 0. (The resulting matrix has the same $GF(2)$-nullity as $M(C,P)$, so the circuit-nullity formula is valid under either definition.) Corollary 20 of [39] states that under very particular circumstances, there is a multiplicative relationship between the relative interlacement matrices of a circuit partition with respect to two Euler systems. We developed the modified interlacement machinery hoping to extend this multiplicative naturality to arbitrary circuit partitions and Euler systems, as in Corollary 2.4. We are grateful to R. Brijder for pointing out that matrices like the modified interlacement matrices appear in the discussion of interlace polynomials given by Aigner and van der Holst [1]. Their Theorem 4 includes an implicit form of the circuit-nullity formula (2.1), but none of the other results we have presented appear there.

We should also note that versions of Corollary 2.5 have appeared in the earlier literature: Jaeger [21] proved the special case in which the Euler systems do not involve the same transition at any vertex, Bouchet [10] provided a different proof of the even more special case in which the Euler systems involve only each others' χ transitions, and a general result for relative interlacement matrices appeared in [39]. Without naturality, though, the proofs of these results are considerably more intricate than that of Corollary 2.5.

Acknowledgments

We would like to thank R. Brijder, H. J. Hoogeboom, D. P. Ilyutko, V. O. Manturov and L. Zulli for many discussions of their work on interlacement, including [14], [20] and [41]. We are also indebted to two anonymous readers for comments on earlier versions of the paper.

References

1. M. Aigner and H. van der Holst, *Interlacement polynomials*, Linear Algebra Appl. **377** (2004), 11–30.
2. R. Arratia, B. Bollobás, and G. B. Sorkin, *The interlace polynomial: A new graph polynomial*, Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms (San Francisco, CA, 2000), ACM, New York, 2000, pp. 237–245.
3. ______, *The interlace polynomial of a graph*, J. Combin. Theory Ser. B **92** (2004), 199–233.
4. ______, *A two-variable interlace polynomial*, Combinatorica **24** (2004), 567–584.
5. I. Beck, *Cycle decomposition by transpositions*, J. Combin. Theory Ser. A **23** (1977), 198–207.
6. I. Beck and G. Moran, *Introducing disjointness to a sequence of transpositions*, Ars Combin. **22** (1986), 145–153.
7. A. Bouchet, *Caractérisation des symboles croisés de genre nul*, C. R. Acad. Sci. Paris Sér. A-B **274** (1972), A724–A727.
8. ______, *Isotropic systems*, European J. Combin. **8** (1987), 231–244.
9. ______, *Reducing prime graphs and recognizing circle graphs*, Combinatorica **7** (1987), 243–254.
10. ______, *Unimodularity and circle graphs*, Discrete Math. **66** (1987), 203–208.
11. ______, *Graphic presentation of isotropic systems*, J. Combin. Theory Ser. B **45** (1988), 58–76.
12. ______, *Circle graph obstructions*, J. Combin. Theory Ser. B **60** (1994), 107–144.
13. H. R. Brahana, *Systems of circuits on two-dimensional manifolds*, Ann. Math. **23** (1921), 144–168.
14. R. Brijder and H. J. Hoogeboom, *Nullity invariance for pivot and the interlace polynomial*, Linear Algebra Appl. **435** (2011), 277–288.
15. M. Cohn and A. Lempel, *Cycle decomposition by disjoint transpositions*, J. Combin. Theory Ser. A **13** (1972), 83–89.
16. H. de Fraysseix, *A characterization of circle graphs*, European J. Combin. **5** (1984), 223–238.
17. H. Fleischner, *Eulerian graphs and related topics. part 1, vol. 1*, Ann. Discrete Math., 45, North-Holland Publishing Co., Amsterdam, 1990.
18. ______, *Eulerian graphs and related topics. part 1. vol. 2*, Ann. Discrete Math., 50, North-Holland Publishing Co., Amsterdam, 1991.
19. L. Ghier, *Double occurrence words with the same alternance graph*, Ars Combin. **36** (1993), 57–64.
20. D. P. Ilyutko and V. O. Manturov, *Graph-links*, Introductory Lectures on Knot Theory (Trieste, Italy, 2009), World Scientific, New Jersey-London, 2009, pp. 135–161.
21. F. Jaeger, *On some algebraic properties of graphs*, Progress in Graph Theory (Waterloo, ON, 1982), Academic Press, Toronto, 1984, pp. 347–366.
22. J. Jonsson, *On the number of Euler trails in directed graphs*, Math. Scand. **90** (2002), 191–214.
23. L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.
24. J. Keir and R. B. Richter, *Walks through every edge exactly twice* II, J. Graph Theory **21** (1996), 301–309.

25. A. Kotzig, *Eulerian lines in finite 4-valent graphs and their transformations*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 219–230.
26. M. Las Vergnas, *On Eulerian partitions of graphs*, Graph Theory and Combinatorics (Proc. Conf., Open Univ., Milton Keynes, 1978), Res. Notes in Math., 34, Pitman, Boston, Mass.-London, 1979, pp. 62–75.
27. ______, *Eulerian circuits of 4-valent graphs imbedded in surfaces*, Algebraic Methods in Graph Theory (Colloq. Math. Soc. János Bolyai, 25, Szeged, 1978) vol. I, II, North-Holland, Amsterdam-New York, 1981, pp. 451–477.
28. ______, *Le polynôme de Martin d'un graphe Eulérien*, Ann. Discrete Math. **17** (1983), 397–411.
29. J. Lauri, *On a formula for the number of Euler trails for a class of digraphs*, Discrete Math. **163** (1997), 307–312.
30. N. Macris and J. V. Pulé, *An alternative formula for the number of Euler trails for a class of digraphs*, Discrete Math. **154** (1996), 301–305.
31. P. Martin, *Enumérations eulériennes dans les multigraphes et invariants de Tutte-Grothendieck*, Ph.D. thesis, Grenbole, 1977.
32. B. Mellor, *A few weight systems arising from intersection graphs*, Michigan Math. J. **51** (2003), 509–536.
33. G. Moran, *Chords in a circle and linear algebra over $GF(2)$*, J. Combin. Theory Ser. A **37** (1984), 239–247.
34. R. C. Read and P. Rosenstiehl, *On the Gauss crossing problem*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), vol. II, Colloq. Math. Soc. János Bolyai, 18, North-Holland, Amsterdam-New York, 1978, pp. 843–876.
35. R. B. Richter, *Walks through every edge exactly twice*, J. Graph Theory **18** (1994), 751–755.
36. E. Soboleva, *Vassiliev knot invariants coming from Lie algebras and 4-invariants*, J. Knot Theory Ramifications **10** (2001), 161–169.
37. S. Stahl, *On the product of certain permutations*, European J. Combin. **8** (1987), 69–72.
38. L. Traldi, *Binary nullity, Euler circuits and interlacement polynomials*, European J. Combin. **32** (2011), 944–950.
39. ______, *On the linear algebra of local complementation*, Linear Algebra Appl. **436** (2012), 1072–1089.
40. ______, *On the interlace polynomials*, J. Combin. Theory Ser. B **103** (2013), 184–208.
41. L. Zulli, *A matrix for computing the Jones polynomial of a knot*, Topology **34** (1995), 717–729.

Lafayette College
Easton, Pennsylvania 18042
E-mail address: traldil@lafayette.edu