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CHARACTERIZING GRAPH CLASSES BY INTERSECTIONS OF NEIGHBORHOODS

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Abstract. The interplay between maxcliques (maximal cliques) and intersections of closed neighborhoods leads to new types of characterizations of several standard graph classes. For instance, being hereditary clique-Helly is equivalent to every nontrivial maxclique Q containing the intersection of closed neighborhoods of two vertices of Q, and also to, in all induced subgraphs, every nontrivial maxclique containing a simplicial edge (an edge in a unique maxclique). Similarly, being trivially perfect is equivalent to every maxclique Q containing the closed neighborhood of a vertex of Q, and also to, in all induced subgraphs, every maxclique containing a simplicial vertex. Maxcliques can be generalized to maximal cographs, yielding a new characterization of ptolemaic graphs.

1. Maximal cliques and closed neighborhoods

A clique of a graph G is a complete subgraph of G or, interchangeably, the vertex set of a complete subgraph. A clique Q is a *p-clique* if $|Q| = p$; is a *maxclique* if Q is not contained in a $(|Q|+1)$-clique; and is a *simplicial clique* if Q is contained in a unique maxclique. Simplicial 1-cliques are traditionally called *simplicial vertices*, and so simplicial 2-cliques can be called *simplicial edges*. The *open neighborhood* $N_G(v)$ of a vertex v is the set $\{w \in V(G) : wv \in E(G)\}$. The *closed neighborhood* $N_G[v]$ of v is the set $N_G(v) \cup \{v\}$ or, interchangeably, the subgraph of G induced by the set $N_G[v]$.

Lemma 1. *For every graph G and every $p \geq 1$, a maxclique Q of G contains $N_G[v_1] \cap \cdots \cap N_G[v_p]$ for distinct $v_1, \ldots, v_p \in Q$ if and only if Q contains a simplicial p-clique of G.*

For $k \geq 2$, define a *k-ocular* graph to consist of $2k$ vertices that are partitioned into $W = \{w_1, \ldots, w_k\}$ and $U = \{u_1, \ldots, u_k\}$ where W induces a k-clique and each $N_G(u_i) = \{w_j : j \neq i\}$ (the subgraph induced by U can contain any subset of edges u_iu_j); see Figure 1. This is the terminology of

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[2] when $k \geq 3$. The only 2-ocular graphs are the path $u_1, w_2, w_1, u_2 \cong P_4$ and the cycle $u_1, w_2, w_1, u_2, u_1 \cong C_4$.

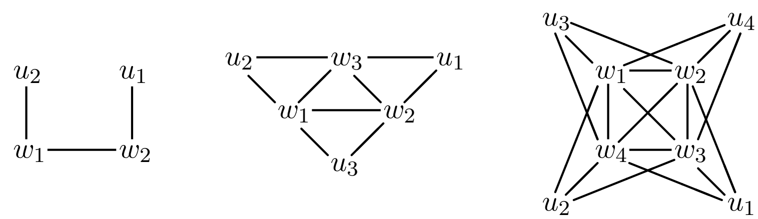

FIGURE 1. The smallest of the two 2-ocular graphs, of the four 3-ocular graphs, and of the eleven 4-ocular graphs.

For $p \geq 2$, define a graph G to be *p-clique-Helly* when, for every family $\mathcal{F}$ of maxcliques of G, if every p members of $\mathcal{F}$ have an element in common, then all the members of $\mathcal{F}$ have an element in common. If every induced subgraph of G is p-clique-Helly, then G is called *hereditary p-clique-Helly*. Reference [2] contains several characterizations of hereditary p-clique-Helly graphs, including condition (4) in Theorem 2.

Theorem 2. *The following are equivalent for every graph G and $p \geq 2$:*

(1) *If Q is a maxclique of an induced subgraph G' of G and $|Q| \geq p$, then Q contains $N_{G'}[v_1] \cap \cdots \cap N_{G'}[v_p]$ for distinct $v_1, \ldots, v_p \in Q$.*
(2) *If Q is a maxclique of an induced subgraph G' of G and $|Q| \geq p$, then Q contains a simplicial p-clique of G'.*
(3) *G is hereditary p-clique-Helly.*
(4) *G contains no induced $(p+1)$-ocular subgraph.*

Proof. Suppose $p \geq 2$. Lemma 1 implies the equivalence (1) $\Leftrightarrow$ (2). The equivalence (3) $\Leftrightarrow$ (4) is [2, Thm. 4].

To prove (4) $\Rightarrow$ (1), suppose (4) holds, G' is an induced subgraph of G, and Q is a maxclique of G' with $|Q| > p$ (the $|Q| = p$ case being immediate). Let $Q' = \{w_1, \ldots, w_{p+1}\} \subseteq Q$ and, whenever $1 \leq i \leq p+1$, let $S_i = \bigcap_{j \neq i} N_{G'}[w_j]$. Thus $Q' \subseteq S_i$ for each i. If for each i there exists a vertex $u_i \in S_i - Q$, then $W = Q'$ and $U = \{u_1, \ldots, u_{p+1}\}$ would induce a $(p+1)$-ocular subgraph (noting that u_i and w_i are not adjacent, since Q is a maxclique). Therefore (4) implies that some $S_i = Q$; without loss of generality, say $S_{p+1} = Q$. That makes $Q = \bigcap_{i=1}^{p} N_{G'}[w_j]$ with distinct $w_1, \ldots, w_p \in Q$, showing that (1) holds.

To prove (1) $\Rightarrow$ (4), suppose (4) fails because G contains an induced $(p+1)$-ocular subgraph G' where $V(G')$ is partitioned into $W \cup U$ as in the definition of $(p+1)$-ocular. Each $N_{G'}[w_i]$ contains the p vertices u_j that have $j \neq i$. Therefore, each $u_j \in \bigcap_{i \neq j} N_{G'}[w_i]$, and so $Q = W$ can never contain the intersection of p neighborhoods $N_{G'}[w_j]$ with $w_j \in Q$, showing that (1) fails. $\square$

Hereditary 2-clique-Helly graphs are called *hereditary clique-Helly graphs* in [10] and *clique reducible graphs* in [11]. These papers contain several other characterizations, and [8] will contain more characterizations involving simplicial cliques.

Corollary 3. *The following are equivalent for every graph G:*

(1) *If Q is a maxclique of an induced subgraph G' of G and $|Q| \geq 2$, then Q contains $N_{G'}[v] \cap N_{G'}[v']$ for distinct $v, v' \in Q$.*
(2) *Every nontrivial maxclique of an induced subgraph of G contains a simplicial edge.*
(3) *G is hereditary clique-Helly.*
(4) *G contains no induced 3-ocular subgraph.*

Proof. This follows from the $p = 2$ case of Theorem 2. (The equivalence (3) $\Leftrightarrow$ (4) also appears in [10, 11].) □

A graph G is called *trivially perfect* if, for every induced subgraph G' of G, the cardinality of the largest independent set in G' equals the number of maxcliques in G'. Reference [4] contains several other characterizations, including condition (4) in Theorem 4. See [1, 9] for many additional characterizations—and names—for trivially perfect graphs; [8] will contain more characterizations involving simplicial cliques.

Theorem 4. *The following are equivalent for every graph G:*

(1) *If Q is a maxclique of an induced subgraph G' of G, then Q contains $N_{G'}[v]$ for some $v \in Q$.*
(2) *Every maxclique of an induced subgraph of G contains a simplicial vertex.*
(3) *G is trivially perfect.*
(4) *G contains no induced P_4 or C_4 subgraph.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Lemma 1. The equivalence (3) $\Leftrightarrow$ (4) is [4, Thm. 2]. The equivalence (1) $\Leftrightarrow$ (4) can be proved by a simple modification of the proof of (1) $\Leftrightarrow$ (4) in Theorem 2, taking $p = 1$ and using that P_4 and C_4 are the two 2-ocular graphs. □

Corollary 5 will be a restriction of Corollary 3 to *chordal graphs*—the graphs in which every cycle of length four or more has a chord (see [1, 9] for other many characterizations and history). Note that the existence of an edge $u_i u_j$ in a p-ocular graph (with vertex set partitioned into W and U as in the definition) would produce a chordless 4-cycle u_i, u_j, w_i, w_j, u_i. Therefore a p-ocular graph is chordal if and only if the set U is independent (meaning that there are no edges between vertices in U).

A *disk* $D_G[v, k]$ of a graph G is the set $\{x : 0 \leq d(v, x) \leq k\} \subseteq V(G)$, where $d(v, x)$ denotes the v-to-x distance in G. Define G to be *disk-Helly* when, for every family $\mathcal{F}$ of disks of G, if every two members of $\mathcal{F}$ have an element in common, then all the members of $\mathcal{F}$ have an element in common. If every induced subgraph of G is disk-Helly, then G is called *hereditary* *disk-Helly*. References [3, 5] contain several characterizations of hereditary disk-Helly graphs, including the following two: (i) being both chordal and clique-Helly, and (ii) being chordal with no induced Hajós subgraph (where the *Hajós graph*—sometimes called the 3-*sun*—is the 3-ocular graph with $\{u_1, u_2, u_3\}$ independent, shown as the center graph in Figure 1).

Corollary 5. *The following are equivalent for every chordal graph G:*

(1) *If Q is a maxclique of an induced subgraph G' of G and $|Q| \geq 2$, then Q contains $N_{G'}[v] \cap N_{G'}[v']$ for distinct $v, v' \in Q$.*
(2) *Every nontrivial maxclique of an induced subgraph of G contains a simplicial edge.*
(3) *G is hereditary disk-Helly.*
(4) *G contains no induced Hajós subgraph.*

Proof. Since the Hajós graph is the only chordal 3-ocular graph, the equivalences (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (4) follow from the $p = 2$ case of Theorem 2. The equivalence (3) $\Leftrightarrow$ (4) is [5, Thm. 1.2]. □

2. Maximal cographs and closed neighborhoods

In this section, cliques—which are simply the graphs that have no induced P_3 subgraphs—will be generalized to the *complement-reducible graphs* (or *cographs*)—which are the graphs that have no induced P_4 subgraphs (or, equivalently, the graphs in which every connected induced subgraph has diameter at most two). See [1, 9] for many additional characterizations. Echoing the $\mathcal{CC}(G)$ notation in [1] for the set of all inclusion-maximal subsets of $V(G)$ that induce connected cographs, define a $\mathcal{CC}$*-subgraph* of G to be a subgraph of G that is induced by a set in $\mathcal{CC}(G)$.

For each $p \geq 1$, let $K_{p+4} - P_4$ denote the graph on the vertex set $\{w_1, \ldots, w_{p+2}, u_1, u_2\}$ that is complete except that edges w_1u_1, u_1u_2 and u_2w_2 do not occur. Equivalently, $K_{p+4} - P_4$ is the chordal $(p+2)$-ocular graph with vertices $u_3, \ldots, u_{p+2}$ deleted. The p-clique $\{w_3, \ldots, w_{p+2}\}$ is the *center* of the $K_{p+4} - P_4$ graph.

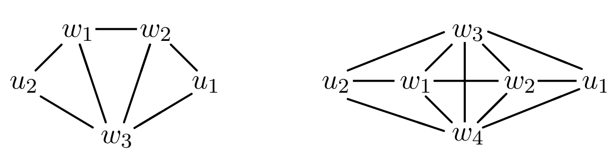

Figure 2. The graphs $K_5 - P_4$ and $K_6 - P_4$.

Lemma 6. *For every graph G and $p \geq 1$, a $\mathcal{CC}$-subgraph H contains $N_G[v_1] \cap \cdots \cap N_G[v_p]$ for distinct $v_1, \ldots, v_p$ in $V(H)$ if and only if H does not contain the center of a $K_{p+4} - P_4$ subgraph of G.*

Proof. Some $\mathcal{CC}$-subgraph of G contains $\bigcap_{i=1}^{p} N_G[v_i]$ for some set $S = \{v_1, \ldots, v_p\}$ of p vertices if and only if $\bigcap_{i=1}^{p} N_G[v_i]$ contains no induced P_4 path a, b, c, d, which in turn is equivalent to no set $S \cup \{a, b, c, d\}$ inducing a $K_{p+4} - P_4$ subgraph of G with $w_1 = b$, $w_2 = c$, $w_3 = v_1$, $\ldots$, $w_{p+2} = v_p$, $u_1 = d$, and $u_2 = a$ (in the notation in the definition of $K_{p+4} - P_4$). $\square$

Theorem 7. *The following are equivalent for every graph G and $p \geq 1$:*

(1) *If H is a $\mathcal{CC}$-subgraph of an induced subgraph G' of G, then H contains $N_{G'}[v_1] \cap \cdots \cap N_{G'}[v_p]$ for distinct $v_1, \ldots, v_p \in V(H)$.*
(2) *G contains no induced $K_{p+4} - P_4$ subgraph.*

Proof. To prove (1) $\Rightarrow$ (2), suppose $p \geq 1$ and condition (2) fails; specifically, suppose G has an induced subgraph $G' \cong K_{p+4} - P_4$ on the vertex set $\{w_1, \ldots, w_{p+2}, u_1, u_2\}$ as described in the definition of $K_{p+4} - P_4$. Take H to be the $\mathcal{CC}$-subgraph of G' that is obtained by deleting w_1 from G', and note that H contains the center $\{w_3, \ldots, w_{p+2}\}$ of G'. Lemma 6 then implies that H does not contain $\bigcap_{i=1}^{p} N_{G'}[v_i]$ for distinct vertices $v_1, \ldots, v_p$, and so condition (1) fails.

To prove (2) $\Rightarrow$ (1), suppose $p \geq 1$ and condition (1) fails; specifically, suppose H is a $\mathcal{CC}$-subgraph of an induced subgraph G' of G such that H contains distinct vertices $v_1, \ldots, v_p$ without containing $\bigcap_{i=1}^{p} N_{G'}[v_i]$. Lemma 6 implies that H contains the center of a $K_{p+4} - P_4$ subgraph of G', and so condition (2) fails. $\square$

A graph G is called *ptolemaic* if G is both chordal and contains no induced $K_5 - P_4$ subgraph (often called a *gem*); see [1] for additional characterizations. Corollary 8 corresponds to Theorem 4.

Corollary 8. *For every chordal graph G, every $\mathcal{CC}$-subgraph H of an induced subgraph G' of G contains $N_{G'}[v]$ for some $v \in V(H)$ if and only if G is ptolemaic.*

Proof. This follows from the $p = 1$ case of Theorem 7. $\square$

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