



FROBENIUS PARTITION THEORETIC INTERPRETATIONS OF SOME BASIC SERIES IDENTITIES

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ABSTRACT. Using generalized Frobenius partitions we interpret five basic series identities of Rogers combinatorially. This extends the recent work of Goyal and Agarwal and yields five new 3-way combinatorial identities.

1. INTRODUCTION, DEFINITIONS AND THE MAIN RESULTS

The following two “sum-product” basic series identities are known as the Rogers–Ramanujan Identities:

$$(1) \quad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-1})^{-1} (1 - q^{5n-4})^{-1},$$

$$(2) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-2})^{-1} (1 - q^{5n-3})^{-1},$$

where $(q; q)_n$ is a rising q -factorial which in general is defined as follows:

$$(a; q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}$$

where a is any constant. If n is any positive integer, then obviously

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}),$$

and

$$(a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2) \dots$$

Received by the editors August 25, 2011, and in revised form April 28, 2012.

2000 *Mathematics Subject Classification.* 05A15, 05A17, 11P81.

Key words and phrases. Frobenius partitions, colored partitions, basic series identities, combinatorial identities.

The first author was supported by CSIR Research Grant No. 09/135(0636)/2011-EMR-I. The second author was supported by Emeritus Scientist CSIR, Research Grant No. 21(0879)/11/EMR-II.

They were first discovered by Rogers [14] and rediscovered by Ramanujan who published them in his paper [13]. MacMahon [12] gave the following combinatorial interpretations of (1) and (2), respectively:

Theorem 1. *The number of partitions of n into parts with minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm 1 \pmod{5}$.*

Theorem 2. *The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm 2 \pmod{5}$.*

Many more identities like (1) and (2) such as given in Slater’s compendium [16] have been interpreted combinatorially using ordinary partitions by several authors (for example, see Connor [11], Subbarao [17], Subbarao and Agarwal [8] and Agarwal and Andrews [5]). In the early nineteen eighties Agarwal and Andrews introduced a new class of partitions called “ $(n + t)$ -color partitions” or partitions with “ $(n + t)$ copies of n ”. Using these new partitions many more basic series identities (also called q -identities) have been interpreted combinatorially in [1, 2, 3, 4, 6]. In a recent paper [7] Goyal and Agarwal gave an n -color partition theoretic interpretations of the following q -identities of Rogers [14].

$$(3) \quad \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q^3, -q^5, -q^7; q^{10})_{\infty}}{(q^4, q^6; q^{10})_{\infty}},$$

$$(4) \quad \sum_{n=0}^{\infty} \frac{q^{3n^2-2n}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q, -q^5, -q^9; q^{10})_{\infty}}{(q^2, q^8; q^{10})_{\infty}},$$

$$(5) \quad \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q^3, -q^7, -q^{11}; q^{14})_{\infty}}{(q^2, q^6, q^8, q^{12}; q^{14})_{\infty}},$$

$$(6) \quad \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q^5, -q^7, -q^9; q^{14})_{\infty}}{(q^4, q^6, q^8, q^{10}; q^{14})_{\infty}},$$

$$(7) \quad \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1} (q^4; q^4)_n} = \frac{(-q, -q^7, -q^{13}; q^{14})_{\infty}}{(q^2, q^4, q^{10}, q^{12}; q^{14})_{\infty}},$$

where $(a_1, a_2, \dots, a_t; z)_{\infty}$ is defined as

$$(a_1, a_2, \dots, a_t; z)_{\infty} = \prod_{j=1}^t (a_j; z)_{\infty}.$$

We remark that identities (3) and (4) were also derived by Bailey [10] and appear in [16]. Identities (5), (6), and (7) are also referred as to Rogers-Selberg identities (see [14, 15, 16]).

In this paper we give new combinatorial interpretations of (3)–(7) in terms of F -partitions. We shall prove bijectively five new combinatorial identities involving certain F -partition functions and n -color partition functions. This results in five new 3-way combinatorial identities. Before we state our main results we recall the following definitions.

Definition 1. A partition with “ $(n+t)$ copies of n ”, (also called an $(n+t)$ -color partition), $t \geq 0$, is a partition in which a part of size n , $n \geq 0$, can occur in $(n+t)$ different colors denoted by subscripts n_1, n_2, \dots, n_{n+t} .

For example, the partitions of 2 with “ $(n+1)$ copies of n ” are

$$\begin{aligned} 2_1, & \quad 2_1 + 0_1, & 1_1 + 1_1, & \quad 1_1 + 1_1 + 0_1, \\ 2_2, & \quad 2_2 + 0_1, & 1_2 + 1_1, & \quad 1_2 + 1_1 + 0_1, \\ 2_3, & \quad 2_3 + 0_1, & 1_2 + 1_2, & \quad 1_2 + 1_2 + 0_1. \end{aligned}$$

Note that zeros are permitted if and only if $t \geq 1$.

Definition 2. The weighted difference of two elements m_i and n_j , $m \geq n$, is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

Definition 3 ([9]). A two-rowed array of non-negative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

with each row aligned in non-increasing order is called a generalized Frobenius partition or more simply an F -partition of ν if

$$\nu = r + \sum_{i=1}^r a_i + \sum_{i=1}^r b_i.$$

In [7] the following colored partition theoretic interpretations of (3)–(7) were proved:

Theorem 3. *Let $A_1(\nu)$ denote the number of n -color partitions of ν such that even parts appear with even subscripts and odd parts with odd subscripts, and all subscripts are greater than 2. If m_i is either the smallest part or the only part in the partition, then $m \equiv i \pmod{4}$, and the weighted difference of any two consecutive parts is non-negative, and is congruent to $0 \pmod{4}$. Let*

$$B_1(\nu) = \sum_{k=0}^{\nu} C_1(\nu - k) D_1(k),$$

where $C_1(\nu)$ is the number of partitions of ν into parts of size congruent to $\pm 4 \pmod{10}$, and $D_1(\nu)$ denotes the number of partitions of ν into distinct parts of size congruent to $\pm 3, 5 \pmod{10}$. Then

$$A_1(\nu) = B_1(\nu), \text{ for all } \nu.$$

Theorem 4. Let $A_2(\nu)$ denote the number of n -color partitions of ν such that even parts appear with even subscripts and odd parts with odd subscripts. If m_i is either the smallest part or the only part in the partition, then $m \equiv i \pmod{4}$, and the weighted difference of any two consecutive parts is at least 4 and is congruent to $0 \pmod{4}$. Let

$$B_2(\nu) = \sum_{k=0}^{\nu} C_2(\nu - k)D_2(k),$$

where $C_2(\nu)$ is the number of partitions of ν into parts of size congruent to $\pm 2 \pmod{10}$, and $D_2(\nu)$ denotes the number of partitions of ν into distinct parts of size congruent to $\pm 1, 5 \pmod{10}$. Then

$$A_2(\nu) = B_2(\nu), \text{ for all } \nu.$$

Theorem 5. Let $A_3(\nu)$ denote the number of n -color partitions of ν such that even parts appear with even subscripts and odd parts with odd subscripts that are greater than 1. If m_i is either the smallest part or the only part in the partition, then $m \equiv i \pmod{4}$, and the weighted difference of any two consecutive parts is non-negative and is congruent to $0 \pmod{4}$. Let

$$B_3(\nu) = \sum_{k=0}^{\nu} C_3(\nu - k)D_3(k),$$

where $C_3(\nu)$ is the number of partitions of ν into parts of size congruent to $\pm 2, \pm 6 \pmod{14}$, and $D_3(\nu)$ denotes the number of partitions of ν into distinct parts of size congruent to $\pm 3, 7 \pmod{14}$. Then

$$A_3(\nu) = B_3(\nu), \text{ for all } \nu.$$

Theorem 6. Let $A_4(\nu)$ denote the number of n -color partitions of ν such that even parts appear with even subscripts and odd parts with odd subscripts, and all subscripts are greater than 3. If m_i is either the smallest part or the only part in the partition, then $m \equiv i \pmod{4}$ and the weighted difference of any two consecutive parts is at least -4 and is congruent to $0 \pmod{4}$. Let

$$B_4(\nu) = \sum_{k=0}^{\nu} C_4(\nu - k)D_4(k),$$

where $C_4(\nu)$ is the number of partitions of ν into parts of size congruent to $\pm 4, \pm 6 \pmod{14}$, and $D_4(\nu)$ denotes the number of partitions of ν into distinct parts of size congruent to $\pm 5, 7 \pmod{14}$. Then

$$A_4(\nu) = B_4(\nu), \text{ for all } \nu.$$

Theorem 7. Let $A_5(\nu)$ denote the number of partitions of ν with “ $(n + 2)$ -copies of n ” such that even parts appear with even subscripts and odd parts with odd subscripts, all subscripts are greater than 1, for some i , i_{i+2} is a

part, and the weighted difference of any two consecutive parts is non-negative and is congruent to $0 \pmod{4}$. Let

$$B_5(\nu) = \sum_{k=0}^{\nu} C_5(\nu - k)D_5(k),$$

where $C_5(\nu)$ is the number of partitions of ν into parts of size congruent to $\pm 2, \pm 4 \pmod{14}$, and $D_5(\nu)$ denotes the number of partitions of ν into distinct parts of size congruent to $\pm 1, 7 \pmod{14}$. Then

$$A_5(\nu) = B_5(\nu), \text{ for all } \nu.$$

In our next section, we shall prove the following combinatorial identities:

Theorem 8. Let $E_1(\nu)$ denote the number of F -partitions of ν such that

- (8.a) $a_i \geq b_i + 2$,
- (8.b) $b_r \equiv 0 \pmod{2}$,
- (8.c) $b_i \equiv a_{i+1} + 1 \pmod{2}$ and $b_i \geq a_{i+1} + 1$,

and let $A_1(\nu)$ denote the number of n -color partitions of ν such that

- (8.d) even parts appear with even subscripts and odd parts with odd subscripts, and all subscripts are greater than 2,
- (8.e) if m_i is either the smallest part or the only part in the partition, then $m \equiv i \pmod{4}$, and
- (8.f) the weighted difference of any two consecutive parts is non-negative, and is congruent to $0 \pmod{4}$.

Then

$$E_1(\nu) = A_1(\nu), \text{ for all } \nu.$$

Example 1. $E_1(17) = 7$, since the relevant F -partitions are:

$$\begin{pmatrix} 10 \\ 6 \end{pmatrix}, \begin{pmatrix} 12 \\ 4 \end{pmatrix}, \begin{pmatrix} 14 \\ 2 \end{pmatrix}, \begin{pmatrix} 16 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 & 2 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 10 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 3 \\ 4 & 0 \end{pmatrix}.$$

Also, $A_1(17) = 7$, since the relevant partitions are:

$$17_5, 17_9, 17_{13}, 17_{17}, 14_4 + 3_3, 14_8 + 3_3, 13_5 + 4_4,$$

Theorem 9. Let $E_2(\nu)$ denote the number of F -partitions of ν such that

- (9.a) $a_i \geq b_i$,
- (9.b) $b_r \equiv 0 \pmod{2}$,
- (9.c) $b_i \equiv a_{i+1} + 1 \pmod{2}$ and $b_i \geq a_{i+1} + 3$,

and let $A_2(\nu)$ denote the number of n -color partitions of ν such that

- (9.d) even parts appear with even subscripts and odd parts with odd subscripts,
- (9.e) if m_i is either the smallest part or the only part in the partition, then $m \equiv i \pmod{4}$, and
- (9.f) the weighted difference of any two consecutive parts is at least 4, and is congruent to $0 \pmod{4}$.

Then

$$E_2(\nu) = A_2(\nu), \text{ for all } \nu.$$

Theorem 10. Let $E_3(\nu)$ denote the number of F -partitions of ν such that

- (10.a) $a_i > b_i$,
- (10.b) $b_r \equiv 0 \pmod{2}$,
- (10.c) $b_i \equiv a_{i+1} + 1 \pmod{2}$ and $b_i \geq a_{i+1} + 1$,

and let $A_3(\nu)$ denote the number of n -color partitions of ν such that

- (10.d) even parts appear with even subscripts and odd parts with odd subscripts that are greater than 1,
- (10.e) if m_i is either the smallest part or the only part in the partition, then $m \equiv i \pmod{4}$, and
- (10.f) the weighted difference of any two consecutive parts is non-negative, and is congruent to $0 \pmod{4}$. Then

$$E_3(\nu) = A_3(\nu), \text{ for all } \nu.$$

Theorem 11. Let $E_4(\nu)$ denote the number of F -partitions of ν such that

- (11.a) $a_i \geq b_i + 3$,
- (11.b) $b_r \equiv 0 \pmod{2}$,
- (11.c) $b_i \equiv a_{i+1} + 1 \pmod{2}$ and $b_i \geq a_{i+1} - 1$,

and let $A_4(\nu)$ denote the number of n -color partitions of ν such that

- (11.d) even parts appear with even subscripts and odd parts with odd subscripts, and all subscripts are greater than 3,
- (11.e) if m_i is either the smallest part or the only part in the partition, then $m \equiv i \pmod{4}$, and
- (11.f) the weighted difference of any two consecutive parts is at least -4 , and congruent to $0 \pmod{4}$.

Then

$$E_4(\nu) = A_4(\nu), \text{ for all } \nu.$$

Theorem 12. Let $E_5(\nu)$ denote the number of F -partitions of ν such that

- (12.a) $a_r \equiv 0 \pmod{2}$,
- (12.b) $a_i \leq b_i + 1$,
- (12.c) $a_i \equiv b_{i+1} + 3 \pmod{2}$ and $a_i \geq b_{i+1} + 3$,

and let $A_5(\nu)$ denote the number of partitions of ν with “ $(n + 2)$ -copies of n ” such that

- (12.d) even parts appear with even subscripts and odd parts with odd subscripts, and all subscripts are greater than 1,
- (12.e) the weighted difference of any two consecutive parts is non-negative and is congruent to $0 \pmod{4}$, and
- (12.f) for some i , i_{i+2} is a part.

Then

$$E_5(\nu) = A_5(\nu), \text{ for all } \nu.$$

2. PROOFS

Proof of Theorem 8. We establish a one-one correspondence between the F -partitions enumerated by $E_1(\nu)$ and n -color partitions enumerated by $A_1(\nu)$. We do this by mapping each column

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

of Frobenius symbol to a single part m_i of n -color partition. The mapping ϕ is given by

$$(8) \quad \phi : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a+b+1)_{(a-b+1)}, \text{ where } a \geq b+2.$$

The inverse mapping ϕ^{-1} is given by

$$(9) \quad \phi^{-1} : m_i \rightarrow \begin{pmatrix} (m+i-2)/2 \\ (m-i)/2 \end{pmatrix}.$$

Clearly $(a+b+1)$ and $(a-b+1)$ have same parity. Therefore, (8) implies (8.d). Now for any two adjacent columns

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

in the Frobenius symbol with

$$\phi \begin{pmatrix} a \\ b \end{pmatrix} = m_i \text{ and } \phi \begin{pmatrix} c \\ d \end{pmatrix} = n_j,$$

we have

$$(10) \quad \begin{aligned} ((m_i - n_j)) &= (a+b+1) - (a-b+1) - (c+d+1) - (c-d+1) \\ &= 2b - 2c - 2. \end{aligned}$$

Clearly (10) and (8.c) imply (8.f). Since the last column

$$\begin{pmatrix} a_r \\ b_r \end{pmatrix}$$

corresponds to the smallest part of the n -color partition, we see that (8.b) and (8) imply (8.e). To see the inverse implication, we see that

$$\begin{aligned} \phi^{-1} : m_i &= \begin{pmatrix} (m+i-2)/2 \\ (m-i)/2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \\ \phi^{-1} : n_j &= \begin{pmatrix} (n+j-2)/2 \\ (n-j)/2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} a &= \frac{m+i-2}{2}, \\ b &= \frac{m-i}{2}, \\ c &= \frac{n+j-2}{2}, \\ d &= \frac{n-j}{2}. \end{aligned}$$

So

$$(11) \quad a - b = i - 1,$$

$$(12) \quad c - d = j - 1,$$

$$(13) \quad b - c = \frac{1}{2}((m_i - n_j)) + 1.$$

Clearly (11) and (12) imply (8.a). (8.f) and (13) imply (8.c). (8.b) is implied by (9) and (8.e). This completes the proof of Theorem 8. \square

To illustrate the bijection we have constructed, we give an example for $\nu = 17$ shown in Table 1.

Frobenius partitions enumerated by $E_1(\nu)$	Images under ϕ
$\begin{pmatrix} 10 \\ 6 \end{pmatrix}$	17_5
$\begin{pmatrix} 12 \\ 4 \end{pmatrix}$	17_9
$\begin{pmatrix} 14 \\ 2 \end{pmatrix}$	17_{13}
$\begin{pmatrix} 16 \\ 0 \end{pmatrix}$	17_{17}
$\begin{pmatrix} 8 & 2 \\ 5 & 0 \end{pmatrix}$	$14_4 + 3_3$
$\begin{pmatrix} 8 & 3 \\ 4 & 0 \end{pmatrix}$	$13_5 + 4_4$
$\begin{pmatrix} 10 & 2 \\ 3 & 0 \end{pmatrix}$	$14_8 + 3_3$

TABLE 1

Thus $E_1(17) = A_1(17) = 7$.

The proofs of Theorems 9–11 are almost similar to the proof of Theorem 8 and hence are omitted .

Proof of Theorem 12. As in the proof of Theorem 8, here also we establish a one-one correspondence between the F -partitions enumerated by $E_5(\nu)$ and the $(n + 2)$ -color partitions enumerated by $A_5(\nu)$. We do it by mapping each column

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

of F -partition to a single part m_i of an $(n + 2)$ -color partition enumerated by $A_5(\nu)$. The mapping ϕ is given by

$$(14) \quad \phi : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a + b + 1)_{(b-a+3)},$$

and the inverse mapping ϕ^{-1} is given by

$$(15) \quad \phi^{-1} : m_i \rightarrow \begin{pmatrix} (m - i + 2)/2 \\ (m + i - 4)/2 \end{pmatrix}.$$

Clearly $(a + b + 1)$ and $(b - a + 3)$ have same parity, (14) and (12.b) imply (12.d). Now suppose that we have two adjacent columns

$$\begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \begin{pmatrix} c \\ d \end{pmatrix}$$

in an F -partition enumerated by $E_5(\nu)$ with

$$\phi \begin{pmatrix} a \\ b \end{pmatrix} = m_i \text{ and } \phi \begin{pmatrix} c \\ d \end{pmatrix} = n_j.$$

Then since $(a + b + 1)_{(b-a+3)} = m_i$ and $(c + d + 1)_{(d-c+3)} = n_j$, we have

$$(16) \quad \begin{aligned} ((m_i - n_j)) &= m - n - i - j \\ &= (a + b + 1) - (b - a + 3) - (c + d + 1) - (d - c + 3) \\ &= 2(a - d) - 6 \end{aligned}$$

Clearly (16) and (12.c) imply (12.e). Now if $a_r = 0$, then

$$\phi \begin{pmatrix} a_r \\ b_r \end{pmatrix} = (b_r + 1)_{b_r+3}$$

which is of the form i_{i+2} , and if a_r is a nonzero even then

$$\phi \begin{pmatrix} a_r \\ b_r \end{pmatrix} = (a_r + b_r + 1)_{b_r-a_r+3}.$$

In this case we consider a “phantom” column

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

as the last column. Since

$$\phi \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0_2,$$

we see that (12.f) holds and the parts $(a_r + b_r + 1)_{(b_r - a_r + 3)}$ and 0_2 satisfy (12.e). It is worthwhile to mention here that the “phantom” column is dropped from the full Frobenius symbol.

To see the reverse implication, we consider the inverse images of two consecutive parts m_i, n_j of $(n + 2)$ -color partition enumerated by $A_5(\nu)$, viz.,

$$\begin{aligned} \phi^{-1} : m_i &= \begin{pmatrix} (m - i + 2)/2 \\ (m + i - 4)/2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \\ \phi^{-1} : n_j &= \begin{pmatrix} (n - j + 2)/2 \\ (n + j - 4)/2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}, \end{aligned}$$

that is,

$$\begin{aligned} a &= \frac{m - i + 2}{2}, \\ b &= \frac{m + i - 4}{2}, \\ c &= \frac{n - j + 2}{2}, \\ d &= \frac{n + j - 4}{2}. \end{aligned}$$

And so,

$$(17) \quad b - a = i - 3,$$

$$(18) \quad d - c = j - 3,$$

$$(19) \quad a - d = \frac{1}{2}((m_i - n_j)) + 3.$$

Clearly (19) and (12.e) imply (12.c), (17) and (18) imply (12.b). (12.f) implies that there is a column of the form

$$\begin{pmatrix} 0 \\ i - 1 \end{pmatrix}$$

which has to be the last column in the F -partition, and i_{i+2} must be the smallest part of its partition, since if $i_{i+2} > n_j$, then

$$((i_{i+2} - n_j)) = -2 - n - j < 0.$$

Also 0_2 is allowed to be a part in an $(n+2)$ -color partition enumerated by $A_5(\nu)$. 0_2 corresponds to a “phantom” columns

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

which is dropped from the corresponding F -partition. This in view of (12.e) implies that a_r is nonzero and even. Otherwise, if i_{i+2} ($i \neq 0$) is the last part in the $(n+2)$ -color partition, then using (9), we see that it corresponds to a column

$$\begin{pmatrix} 0 \\ i-1 \end{pmatrix}$$

which implies $a_r = 0$. This completes the proof of the Theorem 12. \square

To illustrate the bijection we have constructed, we close this section with the example for $\nu = 11$, shown in Table 2.

Frobenius partitions enumerated by $E_1(\nu)$	Images under ϕ
$\begin{pmatrix} 0 \\ 10 \end{pmatrix}$	11_{13}
$\begin{pmatrix} 4 & 0 \\ 6 & -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$	$11_5 + 0_2$
$\begin{pmatrix} 2 & 0 \\ 8 & -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$	$11_9 + 0_2$
$\begin{pmatrix} 5 & 0 \\ 4 & 0 \end{pmatrix}$	$10_2 + 1_3$
$\begin{pmatrix} 3 & 0 \\ 6 & 0 \end{pmatrix}$	$10_6 + 1_3$
$\begin{pmatrix} 4 & 0 \\ 4 & 1 \end{pmatrix}$	$9_3 + 2_4$

TABLE 2

3. CONCLUSION

Theorems 3 through 12 lead to the following five 3-way combinatorial identities:

$$(20) \quad A_k(\nu) = B_k(\nu) = E_k(\nu), \quad 1 \leq k \leq 5,$$

while the identities $A_k(\nu) = B_k(\nu)$ are Theorems 3 through 7 given above and were found in [7], the other identities viz , $A_k(\nu) = E_k(\nu)$ and $B_k(\nu) = E_k(\nu)$ are new.

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