DECODING GENERALISED HYPEROCTAHEDRAL GROUPS AND ASYMPTOTIC ANALYSIS OF CORRECTIBLE ERROR PATTERNS

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ABSTRACT. We demonstrate a majority-logic decoding algorithm for decoding the generalised hyperoctahedral group $C_m \operatorname{wr} S_n$ when thought of as an error-correcting code. We also find the complexity of this decoding algorithm and compare it with that of another, more general, algorithm. Finally, we enumerate the number of error patterns exceeding the correction capability that can be successfully decoded by this algorithm, and analyse this asymptotically.

1. INTRODUCTION

Sets, or groups, of permutations may be used as error-correcting codes, with permutations in list form as the codewords, and the usual Hamming distance. This idea goes back to the 1970s, for instance to the papers of Blake [3] and Blake, Cohen and Deza [4]. Subsequently, there has been a resurgence of interest in such codes because of a potential application to "powerline communications," where electrical power cables are used to transmit data as well as electricity. For instance, the 2004 paper by Chu, Colbourn and Dukes [7] gives a description of this, and some constructions for suitable codes, while a more general survey can be found in Huczynska's 2006 paper [11]. More recently, permutation codes have been applied to "flash memory" data storage devices (see the 2010 paper of Tamo and Schwartz [12]).

In [2], the first author gives a decoding algorithm which works for arbitrary permutation groups when used as codes in this way. In [1], he considers certain families of permutation groups in more detail. In order to describe the case we are interested in in this paper, we need the following background material.

Suppose H and K are permutation groups acting on sets Γ and Δ respectively, where $|\Gamma| = m$ and $\Delta = \{1, \ldots, n\}$. The *wreath product* $G = H \operatorname{wr} K$ is constructed as follows. We consider the action of the Cartesian product

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$H^n = H \times H \times \cdots \times H$ on n disjoint copies of the set Γ, labelled by the elements of Δ. We then form the semidirect product of H^n with K, where K acts on H^n according to its action on $\Delta = \{1, \ldots, n\}$; the resulting group is $G = H \text{ wr } K$. Now, we can define an equivalence relation on $\Gamma \times \Delta$ where the equivalence classes are the copies of Γ; this equivalence relation is preserved by the action of G, and so forms a *system of imprimitivity* or *block system* for G. (See Cameron [6] for more information about permutation groups.)

One family considered in [1] were the groups $H \text{ wr } S_n$, where H is a *regular* permutation group of order m. In this paper, we consider the special case of this where H is a cyclic group of order m, so we have $G = C_m \text{ wr } S_n$ acting in its imprimitive action on n copies of $\{1, \ldots, m\}$. We call these groups *generalised hyperoctahedral groups*, as in the case $m = 2$ we have the well-known hyperoctahedral group (the automorphism group of the n-dimensional hypercube).

In Section 2 we give an alternative decoding algorithm from that given in [2], that can only be used in this case, and in Section 3 we show that this algorithm is better-performing (in terms of time and space complexity). Finally, in Section 4 we show that certain patterns of more than r errors can be successfully decoded by this algorithm, and in Section 5 we analyse the asymptotic behaviour of this.

Recall that the *minimum distance*, d, of a code is the least value of the Hamming distance over all possible pairs of codewords, and that the *correction capability* (i.e. the number of errors that can be guaranteed to be corrected), r, is given by $r = \lfloor \frac{d-1}{2} \rfloor$.

Proposition 1. *The correction capability of $G = C_m \text{ wr } S_n$ is $r = \lfloor \frac{m-1}{2} \rfloor$.*

Proof. The Hamming distance between two permutations g and h is precisely $n - |\text{Fix}(gh^{-1})|$, where $\text{Fix}(g)$ is the set of points fixed by that permutation. Using the group structure, the minimum distance of a permutation group G of degree n is therefore equal to

$$n - \max_{\substack{g \in G \\ g \neq id}} |\text{Fix}(g)|$$

(see [2] for further details). Now suppose $G = C_m \text{ wr } S_n$. Each copy of $\{1, \ldots, m\}$ forms an imprimitivity block for G, and furthermore if an element of G fixes one point in a block, it must fix all m points in that block. Consequently, the maximum number of fixed points is $(n-1)m$ (it is easy to construct elements with this many), therefore the minimum distance is $nm - (n-1)m = m$, and so the correction capability is $r = \lfloor \frac{m-1}{2} \rfloor$. $\square$

2. The decoding algorithm

The decoding algorithm we give below makes use of the relatively straightforward combinatorial structure of the group $G = C_m \text{ wr } S_n$. Since the group permutes n imprimitivity blocks of size m, a permutation in G when

written in list form can be divided into n blocks of length m. The ordering of the blocks gives the action of S_n on the imprimitivity blocks, and the (cyclic) ordering of the symbols within a block gives the corresponding element of C_m. Thus each position effectively holds two pieces of information which are constant throughout that block: a block label and a cyclic shift.

Example 2. The following permutation is an element of C_5 wr S_4:

$$[7, 8, 9, 10, 6 \mid 15, 11, 12, 13, 14 \mid 20, 16, 17, 18, 19 \mid 5, 1, 2, 3, 4].$$

As can be seen, the permutation splits into four blocks of length five, containing $1, \ldots, 5$, $6, \ldots, 10$, etc.

Recall from Proposition 1 that the correction capability is $r = \lfloor \frac{m-1}{2} \rfloor$. Consequently, if we assume there to be a maximum of r errors, there will be a majority of positions in each block which contain the correct symbol. The decoding algorithm uses this fact: the majority of the block labels and the majority of the cyclic shifts will be correct, so this allows the reconstruction of the transmitted word.

If a list L has a unique most frequently-occurring element, we denote it by majority(L).

Algorithm 3. Input the received word $w = [w_1 | w_2 | \cdots | w_n]$, where $w_i = [w_{i1}, \ldots, w_{im}]$. For each i and j, we calculate q_{ij} and s_{ij} where $w_{ij} = mq_{ij} + s_{ij}$ (where $0 \le q_{ij} \le n-1$ and $0 \le s_{ij} \le m-1$), then map w_{ij} to a pair (b_{ij}, c_{ij}) as follows:

$$b_{ij} := \begin{cases} q_{ij} & \text{if } s_{ij} \neq 0 \\ q_{ij} - 1 & \text{if } s_{ij} = 0 \end{cases}$$
$$c_{ij} := s_{ij} - j \mod m$$

(where $c_{ij} \in \{0, \ldots, m-1\}$). Defining $\widehat{b}_i = \text{majority}[b_{i1}, \ldots, b_{im}]$ and $\widehat{c}_i = \text{majority}[c_{i1}, \ldots, c_{im}]$ for each i, the list $[\widehat{b}_1, \ldots, \widehat{b}_n]$ gives a permutation of $\{0, 1, \ldots, n-1\}$ corresponding to the element of S_n acting on the blocks, and the list $[\widehat{c}_1, \ldots, \widehat{c}_n]$ gives the cyclic shifts within each block. We can then reconstruct the original permutation $g := [g_1 | g_2 | \cdots | g_n]$, where $g_i = [g_{i1}, \ldots, g_{im}]$, $g_{ij} = m\widehat{b}_i + t_{ij}$ and $t_{ij} = j + \widehat{c}_i \mod m$. (Note that we assume $t_{ij} \in \{1, \ldots, m\}$.[1])

Example 4. Suppose we transmit the following element $g \in C_5$ wr S_4:

$$[7, 8, 9, 10, 6 \mid 15, 11, 12, 13, 14 \mid 20, 16, 17, 18, 19 \mid 5, 1, 2, 3, 4].$$

Then suppose we receive the following word w:

$$[17, 1, 9, 10, 6 \mid 15, 11, 12, 13, 14 \mid 20, 16, 17, 18, 19 \mid 5, 1, 2, 3, 4].$$

This clearly has errors in positions 1 and 2. Having split this into four blocks of length five, we obtain the data shown in Table 1.

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[1]This is done to reconcile two conventions, namely that permutations are of the set $\{1, \ldots, m\}$ while modular arithmetic is performed on the set $\{0, \ldots, m-1\}$.

TABLE 1. Data obtained during decoding in Example 4

i	j	w_{ij}	(b_{ij}, c_{ij})	i	j	w_{ij}	(b_{ij}, c_{ij})
0	1	17	(3,1)	2	1	20	(3,4)
0	2	1	(0,4)	2	2	16	(3,4)
0	3	9	(1,1)	2	3	17	(3,4)
0	4	10	(1,1)	2	4	18	(3,4)
0	5	6	(1,1)	2	5	19	(3,4)
1	1	15	(2,4)	3	1	5	(0,4)
1	2	11	(2,4)	3	2	1	(0,4)
1	3	12	(2,4)	3	3	2	(0,4)
1	4	13	(2,4)	3	4	3	(0,4)
1	5	14	(2,4)	3	5	4	(0,4)

Taking the "majority" elements, we find the block permutation $\beta = [1, 2, 3, 0]$ and cyclic shifts of $[1, 4, 4, 4]$. We have the information needed to reconstruct the transmitted permutation: for instance with $i = 0$ and $j = 1$, we have $\widehat{b}_0 = 1$, $\widehat{c}_0 = 1$, $t_{01} = 1 + 1 \mod 5 = 2$ and so $g_{01} = 1 \times 5 + 2 = 7$. Performing these calculations for each i and j, we can recover the transmitted permutation:

$$[7, 8, 9, 10, 6 \mid 15, 11, 12, 13, 14 \mid 20, 16, 17, 18, 19 \mid 5, 1, 2, 3, 4].$$

We conclude this section by mentioning that the first author has implemented this algorithm in the computer algebra system GAP [9].

3. COMPLEXITY

We recall that Algorithm 3 has three parts: calculating the numbers (b_{ij}, c_{ij}), which involves integer arithmetic; determining the most frequently-occurring elements $\widehat{b}_i$ and $\widehat{c}_i$; and reconstructing the decoding permutation g, which involves more integer arithmetic. In order to determine the complexity of this algorithm, there are some assumptions we need to make first.

- integer arithmetic can be done via a look-up table, in constant time;
- comparing the sizes of two integers can be done in constant time;
- finding position i in a list of length k takes $O(\log k)$ time.

However, the second step is more complicated and requires the following lemma.

Lemma 5. *Let L be a list of length m with symbols chosen from $S = \{1, \ldots, k\}$. Suppose L has a unique most frequently occurring element $x \in S$. Then the time taken to determine x is $O(k + m \log k)$.*

Proof. We begin by producing an auxiliary list K of length k, initially set to $[0, 0, \ldots, 0]$. This takes k units of time. We then work through each of the positions of L: in each position, we do as follows:

- read the symbol, i, (taking one unit of time);
- find position i in K (taking $O(\log k)$ time);
- increment that entry by 1 (taking one unit of time).

This turns K into a list of the frequencies of each symbol in S in the list L. Doing this for each of the m entries of L requires a total of $O(m \log k)$ time units. We then work through K to find the position of the maximum element. This will require $O(k)$ comparisons. Combining this, we have $O(k) + O(m \log k) + O(k) = O(k + m \log k)$ as required. $\qquad\square$

We observe that the method described above is not necessarily the best possible; other methods may be faster, and which method is the best may depend on factors such as the relative sizes of m and k. However, we now use it to determine the time complexity of Algorithm 3.

Theorem 6. *The time required to perform Algorithm 3 is $O(mn \log m)$ (if $m \geq n$) or $O(n^2 + mn \log n)$ (if $m \leq n$).*

Proof. The first stage is the calculation of the numbers (b_{ij}, c_{ij}). There are mn such calculations to perform, and we have assumed that each takes a constant amount of time, requiring a total of $O(mn)$ time units.

The next stage is, in each block i, to determine the most frequently occurring block label $\widehat{b_i}$ and most frequently occurring cyclic shift $\widehat{c_i}$. This involves determining the most frequently occurring element in a list of length m with symbols chosen from a set of size n (for the block labels) and from a list of length m with symbols chosen from a set of size m (for the cyclic shifts). By Lemma 5 above, the first of these will take $O(n + m \log n)$ time units, the second $O(m + m \log m)$. As this has to be done in each of n blocks, this gives a total of $O(n^2 + mn \log n + mn + mn \log m)$.

The final stage is the reconstruction of g, which requires m integer arithmetic operations in each of the n blocks. As we have assumed that integer arithmetic takes constant time, this requires a total of $O(mn)$ time. So the total time required is $O(mn) + O(n^2 + mn \log n + mn + mn \log m) + O(mn)$. If $m \geq n$, this reduces to $O(mn + mn \log m) = O(mn \log mn)$, while if $m \leq n$ it reduces to $O(n^2 + mn \log n)$, as required. $\qquad\square$

We should also consider the space complexity of Algorithm 3. This time, we require a look-up table for our integer arithmetic, and there are also items that have to be stored whilst the algorithm is being performed.

Proposition 7. *The amount of storage space required by the decoding algorithm is $O(mn^2)$, and the space required to perform the algorithm is $O(mn)$.*

Proof. We need to store a look-up table, where for each of the mn symbols, for n possible divisors we record a quotient/remainder pair. This requires a total of $2mn^2 = O(mn^2)$ storage units. To perform the algorithm, we

need to store mn quotient/remainder pairs, then the two auxiliary lists (one of length m and one of length n) to find their most frequently-occurring element, and need mn units to store the reconstructed group element. This gives a total of $2mn + m + n + mn = \mathrm{O}(mn)$ units. $\qquad\square$

In [2], a more general decoding algorithm was given, which works for arbitrary permutation groups; also in [2] its complexity was analysed in a similar fashion to the above. In the case where the group is the generalised hyperoctahedral group $C_m \operatorname{wr} S_n$, bounds on the time and space complexity for the more general algorithm are given by $\mathrm{O}(m^2 n^2)$ and $O(m^3 n^3)$ respectively. Thus, from the point of view of a worst-case analysis, Theorem 6 and Proposition 7 suggest that Algorithm 3 is an improvement. (Both algorithms require $\mathrm{O}(mn)$ space to perform the algorithm.)

4. Enumerating correctible error patterns

Suppose we have transmitted a permutation $g \in C_m \operatorname{wr} S_n$ and obtained the received word w, which contains errors. The *error pattern* of w is the subset of the positions $\{1, \ldots, mn\}$ where the errors are situated. Formally, a *k-error pattern* is a subset of $\{1, \ldots, mn\}$ of size k.

We observe that Algorithm 3 will successfully decode w if there are a majority of correct elements *in each block*. Consequently, there will be error patterns of size at most nr that can successfully be corrected (where $r = \lfloor \frac{m-1}{2} \rfloor$), regardless of what the erroneous symbols are. We call an error pattern *correctible* if it contains no more than r errors in each block. In this section, we investigate how many such patterns there are.

Before we do so, we remark that for a given transmitted permutation g there are received words whose error patterns are not correctible, but which still can be decoded by Algorithm 3, depending on the nature of the erroneous symbols. For instance, consider Example 4, but suppose the received word w begins $[7, 8, 6, 15, 1 \mid \ldots]$. Three positions in that block contain errors (so the pattern is not correctible), yet Algorithm 3 would determine the correct block label and cyclic shift. On the other hand, if the received word w begins $[7, 8, 1, 1, 1 \mid \ldots]$, the error pattern is the same, but Algorithm 3 would fail. In the remainder of the paper, we are only concerned with correctible error patterns.

For positive integers k, n and r, define $\mathcal{P}_{n,r}(k)$ to be the set of all partitions of the integer k into at most n parts, and where each part has size at most r. For $\pi \in \mathcal{P}_{n,r}(k)$, we denote the number of parts of size i by $f_i(\pi)$ (so that $\sum f_i(\pi) \leq n$). We also define a quantity $c_i(\pi)$ to be

$$c_i(\pi) = \sum_{j=1}^{i-1} f_j(\pi)$$

for $i \geq 2$, with $c_1(\pi) = 0$. That is, $c_i(\pi)$ is the number of parts in π of size strictly less than i.

Proposition 8. *For a word in $C_m \operatorname{wr} S_n$, and for $k \leq nr$, the number of k-error patterns which are correctible is given by*

$$E_{n,m,r}(k) = \sum_{\pi \in \mathcal{P}_{n,r}(k)} \prod_{i=1}^{r} \binom{n - c_i(\pi)}{f_i(\pi)} \binom{m}{i}^{f_i(\pi)}.$$

Proof. For a k-error pattern to be correctible, the errors can be spread across up to n blocks, as long as there are no more than r errors in each block. So for a given partition $\pi \in \mathcal{P}_{n,r}(k)$, for each part of π we have to choose (i) which block contains that many errors and (ii) where in that block they lie. Working through i in increasing order, for each i there are $n - c_i(\pi)$ blocks remaining, of which we choose $f_i(\pi)$ (corresponding to the $f_i(\pi)$ parts of size i). Then in each of the $f_i(\pi)$ blocks we have chosen, we choose i error positions from the m available. $\square$

Note that if $k > nr$, the set $\mathcal{P}_{n,r}(k)$ is empty, so $E_{n,m,r}(k) = 0$.

While this is a tidy combinatorial expression for the desired quantity $E_{n,m,r}(k)$, its behaviour cannot easily be seen, especially as we wish to compare it with the total number of k-error patterns $\binom{mn}{k}$. A first step would be to find a recurrence relation.

Lemma 9. *The numbers $E_{n,m,r}(k)$ satisfy the recurrence relation*

$$E_{n,m,r}(k) = \sum_{l=0}^{r} \binom{m}{l} E_{n-1,m,r}(k - l).$$

Proof. Suppose there are l errors in the nth block; there are $\binom{m}{l}$ ways of arranging these. Then there are $k - l$ errors in the remaining $n - 1$ blocks, so there are $E_{n-1,m,r}(k - l)$ ways of arranging these. Summing over all possible values of $l \leq r$, we obtain the required relation. $\square$

This recurrence relation assists us in studying the generating function for $E_{n,m,r}(k)$. Let $\mathcal{E}_{n,m,r}(x)$ denote this function, that is

$$\mathcal{E}_{n,m,r}(x) = \sum_{k \geq 0} E_{n,m,r}(k)\, x^k,$$

and define

$$F_{m,r}(x) = \sum_{l=0}^{r} \binom{m}{l} x^l.$$

Proposition 10. *The generating function $\mathcal{E}_{n,m,r}(x)$ can be rewritten as*

$$\mathcal{E}_{n,m,r}(x) = (F_{m,r}(x))^n.$$

Proof. Applying Lemma 9 and re-summing, we obtain

$$
\begin{aligned}
\mathcal{E}_{n,m,r}(x) &= \sum_{k \geq 0} E_{n,m,r}(k) x^k \\
&= \sum_{k \geq 0} \sum_{l=0}^{r} \binom{m}{l} E_{n-1,m,r}(k-l) x^{k-l} x^l \\
&= \left(\sum_{l=0}^{r} \binom{m}{l} x^l \right) \mathcal{E}_{n-1,m,r}(x).
\end{aligned}
$$

By iterating this, and with the observation that $\mathcal{E}_{0,m,r}(x) = 1$, we have

$$
\mathcal{E}_{n,m,r}(x) = \left(\sum_{l=0}^{r} \binom{m}{l} x^l \right)^n
$$

as required. $\qquad\square$

The probability that a k-error pattern is correctible is then given by

$$
p_{n,m,r}(k) = E_{n,m,r}(k) / \binom{nm}{k} ,
$$

as the number of all possible k-error patterns is $\binom{nm}{k}$, of which $E_{n,m,r}(k)$ are correctible.

From the point of view of applications it is perhaps more useful to consider the probability $P_{n,m,r}(p)$ that a received word has a correctible error pattern, under the assumption that an individual error occurs with probability p. In other words, we consider a probabilistic model in which the number k of errors is binomially distributed, $k \sim B(mn, p)$, so that the expected number of errors is given by pmn.

Proposition 11. *For a word in $C_m \operatorname{wr} S_n$, if an individual error occurs with probability p then the probability that its error pattern is correctible is given by*

$$
P_{n,m,r}(p) = (1-p)^{mn} \mathcal{E}_{n,m,r}\left(p/(1-p) \right) .
$$

Proof. An error pattern is correctible if there are at most r errors in each block. The probability that exactly l errors occur in a block of size m is given by $\binom{m}{l} p^l (1-p)^{m-l}$, so that the probability that there are at most r errors in each of n blocks is given by

$$
P_{n,m,r}(p) = \left(\sum_{l=0}^{r} \binom{m}{l} p^l (1-p)^{m-l} \right)^n ,
$$

which gives the desired expression. $\qquad\square$

Of course these two probabilistic models are not equivalent.

5. Asymptotic analysis

5.1. Asymptotics of $p_{n,m,r}(k)$ for $C_m \operatorname{wr} S_n$ as $n \to \infty$.
Let us focus on the asymptotics of $p_{n,m,r}(k)$ when the error frequency k/mn is fixed. Here, we will discuss the case where $n \to \infty$ as m is fixed.

We first give an expression for $E_{n,m,r}(k)$ which is amenable to asymptotic treatment.

Lemma 12.
$$E_{n,m,r}(k) = \frac{1}{2\pi i} \oint (F_{m,r}(z))^n \frac{dz}{z^{k+1}} \ ,$$
where the contour of integration is a counterclockwise circle about the origin.

Proof. This follows directly from Proposition 10 and the Cauchy Integral Formula. □

An asymptotic analysis of the integral in Lemma 12 is obtained from a saddle-point approximation. (See Flajolet and Sedgewick [8] for background material on this technique.) For n and k large, the behaviour of the integral is determined by the exponential of
$$n \log F_{m,r}(z) - k \log z \ .$$
There is a unique positive saddle ζ given by
$$0 = \frac{d}{d\zeta} \left[\log F_{m,r}(\zeta) - \frac{k}{n} \log \zeta \right] \ ,$$
and the asymptotics are obtained by approximating the integrand around the saddle by a Gaussian. This is the content of Theorem VIII.8 of [8], which we use to obtain the following result.

Proposition 13. *Let $\lambda = k/n$ be a fixed positive number with $0 < \lambda < r$, let ζ be the unique positive root of the equation*
$$\zeta \frac{F'_{m,r}(\zeta)}{F_{m,r}(\zeta)} = \lambda \ ,$$
and let
$$\xi = \frac{d^2}{d\zeta^2} \left[\log F_{m,r}(\zeta) - \frac{k}{n} \log \zeta \right] \ .$$
Then, with $k = \lambda n$ an integer, one has, as $n \to \infty$,
$$E_{n,m,r}(k) = \frac{F_{m,r}(\zeta)^n}{\zeta^{k+1}\sqrt{2\pi n \xi}}(1 + o(1)) \ .$$
In addition, a full expansion in descending powers of n exists. These estimates hold uniformly for λ in any compact interval of $[0, r]$.

Proof. One easily checks that the conditions of Theorem VIII.8 in [8] are satisfied. □

Fixing m and the fraction of errors $k/mn = \lambda/m$, this allows us to control the asymptotics of $p_{n,m,r}(k)$ for large n.
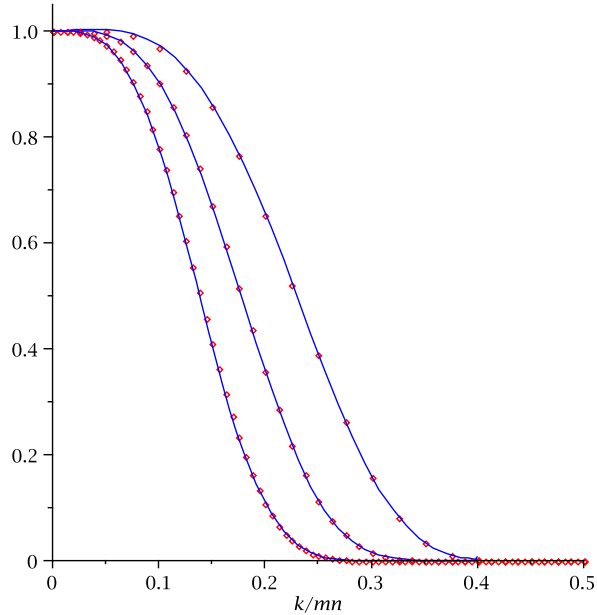
FIGURE 1. The probability $p_{n,m,r}(k)$ that a k-error pattern is correctible versus the error frequency k/mn for $C_m \operatorname{wr} S_n$, when $m = 5$ (and thus $r = \lfloor \frac{5-1}{2} \rfloor = 2$). Shown is a comparison of the exact values (shown as points) and the asymptotic result (shown as curves) from Proposition 13 for $n = 8$, 16, and 32 (from right to left).

Example 14. Figure 1 shows the probability $p_{n,m,r}(k)$ that a k-error pattern is correctible for $C_m \operatorname{wr} S_n$, for three different values of n and when $m = 5$ (and thus $r = \lfloor \frac{5-1}{2} \rfloor = 2$). To reduce the error in the asymptotic formula for small values of k, we replace both the numerator $E_{n,m,r}(k)$ and denominator $\binom{nm}{k}$ of $p_{n,m,r}(k)$ by the respective leading terms of the asymptotic expansion given by Proposition 13 (for the denominator we use $r = m$). One expects heuristically that first-order corrections to the leading asymptotics will largely cancel each other. Numerically this seems to be confirmed, as even for moderate values of n the agreement between the asymptotic result and the exact values is remarkably good.

5.2. **Asymptotics of $P_{n,m,r}(p)$ for $C_m \operatorname{wr} S_n$ as $m \to \infty$.** Let us now consider the asymptotics of $P_{n,m,r}(p)$, given that the probability p of a single error is fixed. Here, we will discuss the case of $m \to \infty$ as n is fixed.

To deal with the truncated binomial sum $F_{m,r}(x)$, we will use the following integral formulation.

Lemma 15.
$$F_{m,r}(x) = \frac{1}{2\pi i} \oint (1 + sx)^m \frac{ds}{s^{r+1}(1-s)} \, ,$$
where the contour of integration is a clockwise circle about the origin of radius less than one.

Proof. Expand the integrand as a power series in s (which is absolutely convergent for $|s| < 1$) and integrate term-by-term. $\square$

As in the previous subsection, an asymptotic analysis of the integral in Lemma 15 is obtained from a saddle-point approximation. For m and r large, the behaviour of the integral is determined by the exponential of
$$m \log(1 + sx) - r \log s \, .$$
There is a unique saddle σ given by
$$0 = \frac{d}{d\sigma} \left[\log(1 + \sigma x) - \frac{r}{m} \log \sigma \right] \, ,$$
namely,
$$\sigma = \frac{1}{x} \frac{r}{m-r} \, .$$
Importantly, the saddle collides with the amplitude critical point $s = 1$ when $x = r/(m - r)$. This changes the asymptotic behaviour significantly, and we need a *uniform* asymptotic expansion to take this into consideration. The standard procedure here is to re-parameterise the contour by a quadratic, i.e.,
$$\log(1 + sx) - \frac{r}{m} \log s = -\frac{t^2}{2} - \gamma t + \delta \, ,$$
where γ and δ are determined by matching the location of the saddle point $s = \sigma$ with $t = -\gamma$, and the location of the critical point $s = 1$ with $t = 0$.

In the following result, erfc denotes the *complementary error function*, which is defined as
$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt.$$
We shall also need
$$\rho(\beta, x) = \sqrt{\log(1 + x) - \beta \log x - h(\beta)}$$
where
$$h(\beta) = -\beta \log \beta - (1 - \beta) \log(1 - \beta) \, ,$$
and
$$A(\beta, x) = \frac{1}{\sqrt{\beta(1-\beta)} \left(1 - \dfrac{\beta}{x(1-\beta)} \right)} - \frac{1}{\sqrt{2}\rho(\beta, x)} \, .$$
Note that the radicand in $\rho(\beta, x)$ has a quadratic zero at $x = \beta/(1 - \beta)$, and that the correct sign has to be chosen to make $\rho(\beta, x)$ an analytic function near that point.

Proposition 16. *Let $\beta = r/m$ be a fixed positive number with $0 < \beta < 1$. Then, with $r = \beta m$ an integer, we have, as $m \to \infty$,*

$$F_{m,r}(x) = (1+x)^m \left[\frac{1}{2} \operatorname{erfc}(\sqrt{m}\rho(\beta,x)) + \frac{A(\beta,x)}{\sqrt{2m\pi}} e^{-m\rho(\beta,x)^2} \right] (1 + o(1)) .$$

In addition, a full expansion in descending powers of m exists. These estimates hold uniformly for β in any compact interval of $[0,1]$ and x in any compact domain of $\mathbb{C} \setminus \mathbb{R}^-$.

Proof. This result follows from equation (9.4.22) in Section 9.4 of Bleistein and Handelsman [5] (with $r = 0$). $\qquad\square$

As an alternative to using Lemma 15, one could have written $F_{m,r}(x)$ in terms of an incomplete Beta function and used results of Temme [13].

The main result now follows immediately from Propositions 11 and 16.

Corollary 17. *As $m \to \infty$, we have*

$$P_{n,m,r}(p) = \left[\frac{1}{2} \operatorname{erfc}\left(\sqrt{m}\rho\left(\tfrac{r}{m}, \tfrac{p}{1-p}\right) \right) + \frac{A\left(\tfrac{r}{m}, \tfrac{p}{1-p}\right)}{\sqrt{2m\pi}} e^{-m\rho\left(\tfrac{r}{m}, \tfrac{p}{1-p}\right)^2} \right]^n (1 + o(1)) .$$
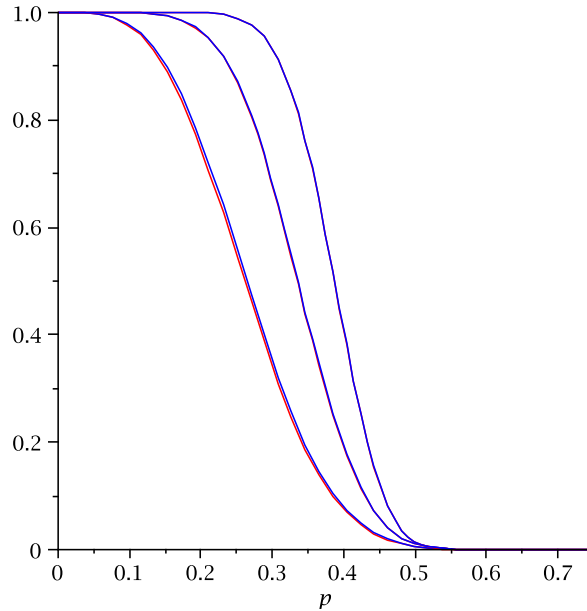

Figure 2. Shown are six curves: for $m = 8$, 16 and 32 (from left to right), the respective curves for $P_{n,m,r}(p)$ and for the asymptotic result from Proposition 17 are barely distinguishable.

Example 18. As we consider $r = \lfloor \frac{m-1}{2} \rfloor$, asymptotically $\beta = 1/2$, but even for small values of m, such as $m = 8$ and $\beta = 3/8$, the expression in Proposition 16 provides a surprisingly accurate approximation, as can be seen from Figure 2.

6. Conclusion

In this note we have introduced a new algorithm for decoding the generalised hyperoctahedral group $C_m \operatorname{wr} S_n$. If $n \ll m$, the performance of the algorithm is better from both the complexity perspective and from the number of correctible error patterns, when compared to the case $n \gg m$.

In particular, for large m the complexity of the algorithm is $O(m \log m)$, whereas it is $O(n^2)$ for large n. As is evident from Proposition 11, the number of correctible error patterns is a monotonically decreasing function of n (this behaviour is demonstrated in Figure 1). On the other hand, using the properties of the complementary error function one can deduce from Corollary 17 that for $p < r/m$ the number of correctible error patterns increases monotonically as a function of m for m sufficiently large (this behaviour is demonstrated in Figure 2).

The results of the asymptotic analysis in Section 5 give reasonable approximations even for moderately small values of m and n. Extending Proposition 16, it is possible to give refined asymptotic estimates which are uniform in m and n.

As another direction, one could consider replacing the group $C_m \operatorname{wr} S_n$ with another wreath product, and modifying the algorithm to suit. First, one could replace the symmetric group S_n with another group K; however, this would give a much smaller number of codewords, and also the decoding algorithm would need to include a "membership-testing" algorithm (see Holt *et al.* [10]) to check whether the decoded permutation was an element of K. Second, one could replace the cyclic group with another group H; however, this would require a more sophisticated decoding process.

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