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FRACTIONAL ILLUMINATION OF CONVEX BODIES

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Abstract. We introduce a fractional version of the illumination problem of Gohberg, Markus, Boltyanski and Hadwiger, according to which every convex body in $\mathbb{R}^d$ is illuminated by at most 2^d directions. We say that a weighted set of points on $\mathbb{S}^{d-1}$ illuminates a convex body K if for each boundary point of K, the total weight of those directions that illuminate K at that point is at least one. We define the fractional illumination number of K as the minimum total weight of a weighted set of points on $\mathbb{S}^{d-1}$ that illuminates K. We prove that the fractional illumination number of any o-symmetric convex body is at most 2^d, and of a general convex body $\binom{2d}{d}$. As a corollary, we obtain that for any o-symmetric convex polytope with k vertices, there is a direction that illuminates at least $\lceil \frac{k}{2^d} \rceil$ vertices.

1. Definitions and Results

We work in the d–dimensional Euclidean space $\mathbb{R}^d$, denote the origin by o and the unit sphere by $\mathbb{S}^{d-1}$. The cardinality, interior, boundary and the volume of a set $X \subset \mathbb{R}^d$ are denoted by $\operatorname{card} X, \operatorname{int} X, \operatorname{bd} X$ and $\operatorname{vol} X$, respectively.

We say that a direction $u \in \mathbb{S}^{d-1}$ *illuminates* a boundary point x of the convex body K if the ray emanating from x in the direction u intersects the interior of K. A set of directions $A \subseteq \mathbb{S}^{d-1}$ illuminates K if each boundary point of K is illuminated by at least one member of A. The *illumination number* $i(K)$ of K is the minimum number of directions that illuminate K. The following was conjectured by I. Gohberg, A. S. Markus, V. G. Boltyanski and H. Hadwiger: *Every convex body in $\mathbb{R}^d$ is illuminated by at most 2^d directions (that is, $i(K) \leq 2^d$) moreover, parallelotopes are the only bodies requiring 2^d directions.* For a thorough treatment of the development of this and related problems, see [4, 11, 14].

In this note, we introduce the following fractional version of the illumination number.

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Definition 1. A *Dirac measure* on a set X is a measure of the form $\delta_x(A) = \operatorname{card}(A \cap \{x\})$ for some $x \in X$. A positive linear combination of finitely many Dirac measures is a *measure of finite support.*
Let $K \subset \mathbb{R}^d$ be a convex body (a compact convex set with non–empty interior) and μ a (non–negative) measure of finite support on $\mathbb{S}^{d-1}$. We say that μ is a *fractional illumination* of K if for every $b \in \operatorname{bd} K$

$$\mu\left(\left\{u \in \mathbb{S}^{d-1} : u \text{ illuminates } K \text{ at } b\right\}\right) \geq 1.$$

The *fractional illumination number* of K is

$$i^*(K) := \inf\left\{\mu(\mathbb{S}^{d-1}) : \mu \text{ is a fractional illumination of } K\right\}.$$

Note that if one restricts the set of measures to sums of Dirac measures (that is, if $\mu(X) = \operatorname{card}(X \cap T)$ for some $T \subset \mathbb{S}^{d-1}$) then one obtains the definition of the illumination number. Our results follow.

Theorem 2. *For every convex body $K \subset \mathbb{R}^d$*

$$i^*(K) \leq \frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)} \leq \binom{2d}{d}.$$

The second inequality is the theorem of C. A. Rogers and G. C. Shepard [13] on the volume of the difference body.

Corollary 3. *For every o-symmetric convex body $K \subset \mathbb{R}^d$, $i^*(K) \leq 2^d$. Moreover, $i^*(P) = 2^d$ if P is a parallelotope.*

As a corollary, we obtain the following:

Theorem 4. *For every o-symmetric convex polytope $P \subset \mathbb{R}^d$ with k vertices, there is a direction that illuminates at least $\left\lceil \frac{k}{2^d} \right\rceil$ vertices.*

We recall that a subset A of a convex set $K \subset \mathbb{R}^d$ is called an *antipodal set* in K if for each pair of distinct points $x, y \in A$ there is a pair of parallel hyperplanes through x and y, respectively supporting K. We denote the maximum cardinality of an antipodal set in K by $a(K)$. According to a beautiful result of L. Danzer and B. Grünbaum [5],

$$\max\{a(K) \colon K \subset \mathbb{R}^d \text{ a convex set}\} = 2^d,$$

where the maximum is attained only by K being a parallelotope and A its set of vertices. As we will see in Remark 8,

$$a(K) \leq i^*(K) \leq i(K). \tag{1.1}$$

Thus, Corollary 3 is a strengthening of the result of Danzer and Grünbaum in the case when K is o-symmetric.

The following proposition shows that there is a case of strict inequality in the first part of (1.1).

Proposition 5. *Let P be a regular pentagon on the plane. Then $a(P) = 2$ while $i^*(P) \geq \frac{5}{2}$.*

The second inequality in (1.1) may be strict as well. Recall that for a smooth convex body in $\mathbb{R}^d$, we have $i(K) = d + 1$ (cf. [11]).

Proposition 6. *For every convex body $K \subset \mathbb{R}^d$, we have that $i^*(K) \geq 2$. If K is smooth then $i^*(K) = 2$.*

Finally, we formulate a weaker version of the Gohberg–Markus–Boltyanski–Hadwiger Conjecture:

Conjecture 7. *For every convex body $K \subset \mathbb{R}^d$*

$$i^*(K) \leq 2^d,$$

and equality is attained only if K is a parallelotope.

The validity of this conjecture was unknown even in the o-symmetric case. Corollary 3 confirms Conjecture 7 in the case when K is o-symmetric. In summary, we study i^*, because it is a quantity between the quantity a, the maximum of which in a given dimension is well-understood, and the quantity i, the maximum of which is only conjectured.

Fractional illumination is a special case of the more general notion of fractional transversals (for the definition, see Section 2). This concept first appeared in papers by Z. Füredi [6], by L. Lovász [8], and by C. Berge and M. Simonovits [2]. For details on (fractional) transversals cf. [1, 7, 9, 10] and [12].

2. Fractional Transversals

We recall some definitions from combinatorics. A *set system* on a base set X is a family $\mathcal{F}$ of some non–empty subsets of X. A *transversal* of $\mathcal{F}$ is a subset $T \subset X$ with the property that $T \cap F \neq \varnothing$ for any $F \in \mathcal{F}$. The *transversal number* $\tau(\mathcal{F})$ of $\mathcal{F}$ is the minimum cardinality of a transversal of $\mathcal{F}$. A *fractional transversal* of $\mathcal{F}$ is a measure μ of finite support on X with the property that $\mu(F) \geq 1$ for any $F \in \mathcal{F}$. The *fractional transversal number* $\tau^*(\mathcal{F})$ of $\mathcal{F}$ is

$$\tau^*(\mathcal{F}) := \inf \{\mu(X) : \mu \text{ is a fractional transversal of } \mathcal{F}\}.$$

The dual notion of transversals is matchings. A *matching* of $\mathcal{F}$ is a subset $\mathcal{M} \subset \mathcal{F}$ with the property $\operatorname{card}(F \in \mathcal{M} : x \in F) \leq 1$ for any $x \in X$. The *matching number* $\nu(\mathcal{F})$ of $\mathcal{F}$ is the maximum cardinality of a matching of $\mathcal{F}$. A *fractional matching* of $\mathcal{F}$ is a measure μ of finite support on $\mathcal{F}$ with the property that $\mu(\{F \in \mathcal{F} : x \in F\}) \leq 1$ for any $x \in X$. The *fractional matching number* $\nu^*(\mathcal{F})$ of $\mathcal{F}$ is

$$\nu^*(\mathcal{F}) := \sup \{\mu(\mathcal{F}) : \mu \text{ is a fractional matching of } \mathcal{F}\}.$$

Clearly, for any set system, we have

$$\nu(\mathcal{F}) \leq \nu^*(\mathcal{F}) \leq \tau^*(\mathcal{F}) \leq \tau(\mathcal{F}). \tag{2.1}$$

We note that $\nu^*(\mathcal{F}) = \tau^*(\mathcal{F})$ for any set sytem, a fact we are not using.

3. Proofs

Proof of Theorem 2. First, we re-phrase our problem in terms of fractional transversals. We fix a convex body K. For a point $b \in \mathrm{bd}(K)$, let

$$F_b := \left\{ u \in \mathbb{S}^{d-1} : u \text{ illuminates } K \text{ at } b \right\}.$$

We consider the following set system with base set $\mathbb{S}^{d-1}$.

$$\mathcal{F} := \{F_b : b \in \mathrm{bd}(K)\}.$$

Now, clearly,

$$i(K) = \tau(\mathcal{F}) \quad \text{and} \quad i^*(K) = \tau^*(\mathcal{F}). \tag{3.1}$$

For a point $b \in \mathrm{bd}(K)$, let

$$G_b := \mathrm{int}(K) - b.$$

We consider the following set system with base set $\mathbb{R}^d \setminus \{o\}$:

$$\mathcal{G} := \{G_b : b \in \mathrm{bd}(K)\}.$$

We note that $\cup\mathcal{G} = (K-K) \setminus \{o\}$. Let $\pi : \mathbb{R}^d \setminus \{o\} \longrightarrow \mathbb{S}^{d-1}$ be the central projection onto the sphere. Now, if we have a measure μ on $\mathbb{R}^d \setminus \{o\}$ then we obtain a measure $\pi_*(\mu)$ on $\mathbb{S}^{d-1}$ by setting $\pi_*(\mu)(A) := \mu(\pi^{-1}(A))$ for a set A in $\mathbb{S}^{d-1}$ for which $\pi^{-1}(A)$ is measurable. Clearly, if μ is a fractional transversal of $\mathcal{G}$ then $\pi_*(\mu)$ is a fractional transversal of $\mathcal{F}$. We note that if T is a transversal of $\mathcal{G}$ then $\pi(T)$ is a transversal of $\mathcal{F}$.

We fix an $\varepsilon > 0$. We define a fractional transversal of $\mathcal{G}$ as follows. Let X be a finite subset of $K - K$ such that

$$(1-\varepsilon)\frac{\mathrm{vol}(K)}{\mathrm{vol}(K-K)} \leq \frac{\mathrm{card}((\mathrm{int}(K) - b) \cap X)}{\mathrm{card}\, X}$$

for every $b \in \mathrm{bd}\, K$. We may construct X as the intersection of $K - K$ with a sufficiently fine grid.

Now, let μ be the following measure on $\mathbb{R}^d$:

$$\mu(A) := \frac{\mathrm{card}(A \cap X)\, \mathrm{vol}(K-K)}{(1-\varepsilon)\, \mathrm{card}(X)\, \mathrm{vol}(K)}$$

for any $A \subset \mathbb{R}^d$. Clearly, μ is a transversal of $\mathcal{G}$. By observing that

$$\mu(\mathbb{R}^d \setminus \{o\}) = \frac{\mathrm{vol}(K-K)}{(1-\varepsilon)\, \mathrm{vol}(K)},$$

we finish the proof of the Theorem. □

Remark 8. The matching number of $\mathcal{F}$ in the proof is the maximum cardinality of a set $A \subset \mathrm{bd}(K)$ with the property that no two of its points are illuminated by the same direction. It is not difficult to see that $A \subset \mathrm{bd}(K)$ is such a set if, and only if, A is an antipodal set in K. Thus, $a(K) = \nu(\mathcal{F})$.

Proof of Corollary 3. To prove the second assertion, we note that if K is a parallelotope then $\nu(\mathcal{F}) = a(K) = 2^d$. □

Proof of Theorem 4. Suppose the contrary, that is that each point of $\mathbb{S}^{d-1}$ belongs to at most r members of

$$\mathcal{F}' := \{F_v : v \text{ is a vertex of } K\}.$$

where $r < \frac{k}{2^d}$. Let t be such that $\frac{2^d}{k} < t < \frac{1}{r}$ and let μ be the following measure on $\mathcal{F}$:

$$\mu(F) = t \text{ for all } F \in \mathcal{F}', \text{ and } \mu(F) = 0 \text{ if } F \notin \mathcal{F}'.$$

Since $rt < 1$, μ is a fractional matching of $\mathcal{F}$, on the other hand

$$\nu^*(\mathcal{F}) \geq \mu(\mathcal{F}) = \operatorname{card}(\mathcal{F}') \cdot t = k \cdot t > 2^d$$

contradicting (2.1) and Corollary 3.

□

Proof of Proposition 5. Let the vertices of P be $\{v_1, v_2, \ldots, v_5\}$ in a cyclic order. Then v_i and v_{i+2} form an antipodal pair in P, where indices are taken modulo 5. However, v_i and v_{i+1} are *not* antipodal. It follows that $a(P) = 2$.

Next, let

$$\mathcal{F}' := \{F_{v_i} : i = 1, \ldots, 5\}.$$

Since among any three members of $\{v_1, v_2, \ldots, v_5\}$ there are two that are antipodal, it follows that no three members of $\mathcal{F}'$ intersect. Hence, the measure μ on $\mathcal{F}$ defined as

$$\mu(F) = \frac{1}{2} \text{ for all } F \in \mathcal{F}', \text{ and } \mu(F) = 0 \text{ if } F \notin \mathcal{F}'$$

is a fractional matching of $\mathcal{F}$. Thus,

$$i^*(P) = \tau^*(\mathcal{F}) \geq \nu^*(\mathcal{F}) \geq \mu(\mathcal{F}) = \frac{5}{2}.$$

□

Proof of Proposition 6. The first assertion follows from the fact that $\nu(\mathcal{F}) \geq 2$, which is a consequence of the existence of a pair of antipodal points on the boundary of K. To prove the second asertion, we notice that F_b (defined in the proof of Theorem 2) is an open hemisphere for each $b \in \operatorname{bd}(K)$. Clearly, for any $\varepsilon > 0$ there is a measure μ of finite support on $\mathbb{S}^{d-1}$ such that $\mu(\mathbb{S}^{d-1}) \leq 2(1+\varepsilon)$ and $\mu(H) \geq 1$ for any open hemisphere H of $\mathbb{S}^{d-1}$. □

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