



COLOURFUL TRANSVERSAL THEOREMS

LUIS MONTEJANO AND DEBORAH OLIVEROS

ABSTRACT. We prove colourful versions of three classical transversal theorems: the Katchalski-Lewis Theorem “T(3) implies T-k”, the “T(3) implies T” Theorem for well distributed sets, and the Goodmann-Pollack Transversal Theorem for hyperplanes.

1. INTRODUCTION

Transversal properties of families of translated copies of a compact convex set have been studied by a number of authors, with special attention to Helly-type problems. For instance, B. Grünbaum [9] conjectured the following Helly-type theorem for line transversals to a family \mathcal{F} of pairwise disjoint translates of a compact convex set K in the plane: if every subfamily of \mathcal{F} of cardinality five admits a line transversal, then the entire family admits a line transversal, a statement that was proved later by Tverberg in [17]. In [10] Grünbaum proved the following result:

Theorem. *If the members of \mathcal{F} are sufficiently far apart and every subfamily of cardinality three admits a line transversal, then the entire family admits a line transversal.*

Following the same spirit, Katchalski and Lewis [14] proved the following result:

Theorem. *There is a positive integer k , depending only on K , with the property that if every subfamily of \mathcal{F} of cardinality three admits a line transversal, then there is a line intersecting all but at most k members of \mathcal{F} .*

From a different perspective, Hadwiger proved in [11] that if \mathcal{F} is an ordered family of compact convex sets in the plane such that there is a line meeting any three sets in a manner compatible with the order, then \mathcal{F} admits a line transversal. J. E. Goodman and R. Pollack [8] extended Hadwiger’s Theorem for hyperplane transversals in the n -dimensional Euclidean space \mathbf{R}^n , with the ordering of the sets replaced by the “order type” and the

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condition that no two have a common point by the condition that the family is “separated”. The result reads as follows:

Theorem. *If every subfamily of cardinality $n + 1$ has a transversal hyperplane in a manner compatible with the “order type”, then there is a hyperplane transversal to all members of the family.*

Later, R. Wenger [20] proved Hadwiger’s Theorem without the pairwise disjointness, and R. Pollack and R. Wenger [15] were able to prove the generalization to higher dimensions without separability. It is interesting to remark that in these latter two cases the proof required topological methods.

In [3] Bárány described remarkable extensions of two classical theorems, known as the “multiplied” or “colourful” versions of the Carathéodory and Helly theorems. The colourful version of the Helly theorem was discovered by Lovász, and the colourful version of the Carathéodory theorem by Bárány in [3]. These theorems have many applications in discrete geometry. For example, they play key roles in Sarkaria’s proof of Tverberg’s Theorem [16] and in the proof of the existence of weak ϵ -nets for the family of convex sets in \mathbf{R}^n . These colourful theorems also have interesting connections with topology, as the research of Živaljević shows in [19] regarding the connection among Tverberg’s Theorem, colourful theorems and algebraic topology.

Let \mathcal{F} be a family of compact convex sets in \mathbf{R}^n . We say that \mathcal{F} is v -coloured if there is a partition of the family \mathcal{F} into v pairwise disjoint classes F_1, \dots, F_v called the chromatic classes. We say that $A \in \mathcal{F}$ has colour i , $i = 1, \dots, v$, if $A \in F_i$. Also, we say that a subset $G \subset \mathcal{F}$ is heterochromatic if no two members of G have the same colour. Furthermore, we say that \mathcal{F} is $T_s(r)$ -chromatic if every subfamily of cardinality r admits an s -flat transversal; for instance, if $s = n - 1$, $T_{n-1}(r)$ implies a transversal hyperplane for any r members of \mathcal{F} . For simplicity, $T(r)$ will denote the existence of a line transversal to any r members of \mathcal{F} .

If \mathcal{F} , an $(n + 1)$ -coloured family of convex sets in \mathbf{R}^n with the property that all members of any heterochromatic subfamily of cardinality $n + 1$, have a common point, then it is unreasonable to expect all members of \mathcal{F} to have a common point, but surprisingly, the colourful Helly Theorem states that there is a colour, say i , with the property that all i -coloured members of \mathcal{F} have a common point.

The purpose of this paper is to prove the colourful versions of three classical transversal theorems. In Section 2, we prove the following colourful version of the Goodman and Pollack extended Hadwiger Theorem using separability:

Theorem. *If \mathcal{F} is a separable $(n + 1)$ -coloured family of compact, convex sets in \mathbf{R}^n and \mathcal{F} is $T_{n-1}(n + 1)$ -chromatic in a manner compatible with the order type, then there is a colour and a hyperplane transversal to all members of the family of this colour.*

In [1] Arocha, Bracho and Montejano proved the above result in dimension 2 for intersecting planar sets.

Section 3 will be devoted to proving a colourful version of Grünbaum's classical transversal theorem;

Theorem. *If \mathcal{F} is a 3-coloured family of translated copies of a compact convex set in the plane, $T(3)$ -chromatic and well-distributed, then there is a colour and line transversal to all members of \mathcal{F} with this colour.*

Finally, in Section 4, we prove the colourful version of the Katchalsky-Lewis Theorem.

Theorem. *Given a compact convex set K in the plane, there is a constant k , depending only in K , such that if \mathcal{F} is a 3-coloured, $T(3)$ -chromatic family of translated copies of K , then there is a colour and a line transversal to all members of \mathcal{F} of this colour except for k of them.*

Throughout the rest of the paper, $\text{conv}(X)$ will denote the convex hull of X .

2. THE COLOURFUL GP TRANSVERSAL THEOREM

Let \mathcal{P} be a set of points in general position in \mathbf{R}^n , $\mathcal{P} = R \cup S$ and $R \cap S = \emptyset$. We say that a point in R is round and a point in S is square. Assume that R and S are $(n+2)$ -coloured, with the partitions

$$R = R_1 \cup \dots \cup R_{n+2} \text{ and } S = S_1 \cup \dots \cup S_{n+2}.$$

Let $\mathcal{P}_i = R_i \cup S_i$ be the set of points of colour i , $i = 1, \dots, n+2$.

Definition 2.1. *A heterochromatic Radon partition of \mathcal{P} in \mathbf{R}^n is an heterochromatic subset $\mathcal{G} \subset \mathcal{P}$ of cardinality $n+2$, with the property that the convex hull of the round points of \mathcal{G} intersects the convex hull of the square points of \mathcal{G} ; that is, $|\mathcal{G} \cap \mathcal{P}_i| = 1$ and $\text{conv}(\mathcal{G} \cap R) \cap \text{conv}(\mathcal{G} \cap S) \neq \emptyset$ for $i = 1, \dots, n+2$.*

Observation: When $n = 1$, the following situation holds: consider \mathcal{P} , a set of 3-coloured points in a line, some of them round and some square. A heterochromatic Radon triple consists of three points of \mathcal{P} , each one with a different colour and with the property that either a round point is between two square points or a square point is between two round points.

Theorem 2.2. *Let $|\mathcal{P}_i|$, the number of points of colour i , be even for every $i = 1, \dots, n+2$. Then the number of heterochromatic Radon partitions of \mathcal{P} is even.*

Proof. Let $I(\mathcal{P})$, the Radon index of \mathcal{P} , be the number of heterochromatic Radon partitions of \mathcal{P} . Note that if R and S are separated by a hyperplane, then $I(\mathcal{P}) = 0$. Hence, we may assume that there is no such hyperplane and $I(\mathcal{P}) = k$ for some $k > 0$.

Since the points of \mathcal{P} are in general position and $|\mathcal{P}|$ is finite, we may assume that there is a direction, say $d = (0, \dots, 0, 1)$ in \mathbf{R}^n , where no line in this direction is contained in a hyperplane spanned by n points of \mathcal{P} . We

shall move the points of \mathcal{P} one by one until the round points and the square points of \mathcal{P} became separated by the hyperplane

$$H = \{(x_1, x_2, \dots, x_n) \mid x_n = 0\}$$

without changing the Radon Index of \mathcal{P} modulo 2. Denote by H^+ and H^- the upper and lower closed half spaces determined by H respectively.

Suppose that a point $s_1 \in S_1$ is in H^- . Move s_1 in a vertical line with respect to time t such that s_1^t , the point at t , consistently approaches H^+ as t increases. Let $\mathcal{P}^t = (\mathcal{P} \setminus \{s_1\}) \cup \{s_1^t\}$ and suppose $s_1^0 = s_1$ and $s_1^{t_0} \in H^+$. If $I(\mathcal{P}^t) = k$ for every $t \in [0, t_0]$, then we may assume that s_1 is in H^+ .

If $I(\mathcal{P}^t)$ changes at some time, say $t = 1$, then an heterochromatic Radon partition has been created or a heterochromatic Radon partition has been destroyed. In the first case, there exists $\mathcal{A} \subset \mathcal{P} \setminus \{s_1\}$ such that $|\mathcal{A}| = n + 1$ and $(\mathcal{A} \cup \{s_1^t\}) \cap R$ and $(\mathcal{A} \cup \{s_1^t\}) \cap S$ are separated by a hyperplane for $0 \leq t < 1$ and $\mathcal{A} \cup \{s_1^1\}$ is a heterochromatic Radon partition, or in the second case, $\mathcal{A} \cup \{s_1^t\}$ is a heterochromatic Radon partition for $0 \leq t < 1$ and the sets $(\mathcal{A} \cup \{s_1^1\}) \cap R$ and $(\mathcal{A} \cup \{s_1^1\})$ are separated by a hyperplane. In either case, there is a hyperplane \tilde{H} that contains a subset \mathcal{B} of \mathcal{P}^1 that is a heterochromatic Radon partition of $\tilde{H} \cap \mathcal{P}^1$ in the $(n-1)$ -dimensional hyperplane \tilde{H} . Since the points of \mathcal{P} are in general position and the cardinality of \mathcal{B} is $n+1$, then there is a colour, say $i = 2$, such that none of the points of $\mathcal{P}_2 = \{x_{21}, x_{22}, \dots, x_{2\lambda}\}$ are in \tilde{H} . Furthermore, due to the fact that no vertical line is contained in a hyperplane determined by n points of \mathcal{P} , we have that the relative interior of the convex hull of the square points \mathcal{B} intersects the relative interior of the convex hull of the round points \mathcal{B} .

Suppose first that x_{21} is a square point. If both x_{21} and s_1^t lie on the same side of \tilde{H} for $0 \leq t < 1$, then $(\mathcal{B} \setminus \{s_1^1\}) \cup \{s_1^t, x_{21}\}$ is not a heterochromatic Radon partition, but $(\mathcal{B} \setminus \{s_1^1\}) \cup \{s_1^t, x_{21}\}$ is a heterochromatic Radon partition for $t \geq 1$, if t is sufficiently close to 1, and hence the Radon index increases by one as s_1^t crosses the hyperplane \tilde{H} . Conversely, if x_{21} and s_1^t lie on different sides of \tilde{H} , for $0 \leq t < 1$, the Radon index decreases by one as s_1^t crosses the hyperplane \tilde{H} . We repeat this process for each $x_{2i} \in \mathcal{P}_2$, $i = 1, \dots, \lambda$. Since λ is even, the parity of $I(\mathcal{P})$ does not change. Similarly, if x_{21} is a round point and both x_{21} and s_1^t lie on the same side of \tilde{H} , for $0 \leq t < 1$, then $(\mathcal{B} \setminus \{s_1^1\}) \cup \{s_1^t, x_{21}\}$ is a heterochromatic Radon partition, but $(\mathcal{B} \setminus \{s_1^1\}) \cup \{s_1^t, x_{21}\}$ is not a heterochromatic Radon partition, for $t \geq 1$, sufficiently close to 1, and hence the Radon index decreases by one as s_1^t crosses the hyperplane \tilde{H} . Conversely, if x_{21} is a round point and it is separated from s_1^t by \tilde{H} , $0 \leq t < 1$, then $(\mathcal{B} \setminus \{s_1^1\}) \cup \{x_{21}\}$ is not a heterochromatic Radon partition and hence $I(\mathcal{P}^1)$ increases by one as s_1^t crosses the hyperplane \tilde{H} . Again, the parity of $I(\mathcal{P})$ does not change after considering each $x_{2i} \in \mathcal{P}_2$.

We now repeat this operation for each $s_i \in H^- \cap S_1$ and replace S_1 by $S'_1 = (S_1 \setminus \{s_1\}) \cup \{s'_1\}$ $k > 5$ such that $s'_1 \subset H^+$, $S' = S'_1, S_2, \dots, S_{n+2}$,

$\mathcal{P}' = R \cup S$, $R \cup S = \emptyset$, \mathcal{P}' is a set of points in general position in \mathbf{R}^n , $I(\mathcal{P}') < I(\mathcal{P})$ and $I(\mathcal{P}') \equiv I(\mathcal{P}) \pmod{2}$. Similarly, we perform the same operation with all the points of $s_i \in H^- \cap S_j$ for every $j = 2, \dots, n+2$ and with all points $r_i \in H^+ \cap S_j$ with $j = 1, \dots, n+2$ until H strictly separates the corresponding R' from S' and then the corresponding Radon index $I(\mathcal{P}') = 0$ and $I(\mathcal{P}') \equiv I(\mathcal{P}) \pmod{2}$. Thus the number of heterochromatic Radon partitions is even. \square

Definition 2.3. Let $G = \{A_1, \dots, A_m\}$ be a family of m compact convex sets in \mathbf{R}^n and let $X = \{x_1, \dots, x_m\}$ be a set of m points in general position in \mathbf{R}^{n-1} . We say that a transversal hyperplane Γ of G meets G consistently with the order type of X if there is $y_i \in A_i \cap \Gamma$, $i = 1, \dots, m$, such that $\{y_1, \dots, y_m\}$ has the same order type as $\{x_1, \dots, x_m\}$.

In particular, every separation of the points $\{y_1, \dots, y_m\}$ by a hyperplane of Γ implies the corresponding separation of the points $\{x_1, \dots, x_m\}$ by a hyperplane of \mathbf{R}^{n-1} . For more information about the notion of order type, see [8].

By a separated family of convex sets in \mathbf{R}^n , we mean a family for which no n members have a common $(n-2)$ -flat transversal; i.e., separation in the plane means pairwise disjointness and separation in 3-dimensional space means no three members have a line transversal.

The following theorem is the coloured version of the Goodman Pollack hyperplane transversal theorem for separated families of compact convex sets.

Theorem 2.4. Let $\mathcal{F} = \{A_1, \dots, A_\lambda\}$ be a separated, $(n+1)$ -coloured family of compact convex sets in \mathbf{R}^n and let $X = \{x_1, \dots, x_\lambda\}$ be a configuration of points in \mathbf{R}^{n-1} . Suppose that every heterochromatic subfamily \mathcal{F}' of \mathcal{F} of cardinality $n+1$ has an $(n-1)$ -hyperplane transversal consistent with the order type of X . Then there exists a colour, say k , and a hyperplane transversal to all members of colour k .

For the proof of Theorem 2.4 we need the following lemma.

Lemma 2.5. Let $\mathcal{F} = \{A_1, \dots, A_{n+1}\}$ be a separated family of $n+1$ convex sets in \mathbf{R}^n , let H be a hyperplane that is disjoint from the interior of each set in \mathcal{F} . Let A_i be in H^+ and A_j be in H^- for $i = 1, \dots, r$ and $j = r+1, \dots, r+s+2$. If $\text{conv}\{\cup A_i\} \cap \text{conv}\{\cup A_j\}$ is not empty then $r+s+2 = n+2$ and H supports each set in \mathcal{F} .

Proof. Let $x \in \text{conv}\{a_i \mid A_i \subset H^+\} \cap \text{conv}\{a_i \mid A_i \subset H^-\}$. Then clearly $x \in H$. Assume x belongs to the relative interior of both $\text{conv}\{a_{i_0}, \dots, a_{i_r}\}$ and $\text{conv}\{a_{j_0}, \dots, a_{j_s}\}$ where $a_{i_k} \in A_{i_k} \subseteq H^+$, $k = 0, \dots, r$, and $a_{j_\lambda} \in A_{j_\lambda} \subseteq H^-$ for $\lambda = 0, \dots, s$. Let Γ^r be the r -plane generated by $\{a_{i_1}, \dots, a_{i_r}\}$. Due to the fact that for $k = 0, \dots, r$, $a_{i_k} \in H^+$, $x \in H$ and x belongs to the relative interior of $\text{conv}\{a_{i_0}, \dots, a_{i_r}\}$, we have that $\Gamma^r \subset H$. Similarly, Γ^s , the s -plane generated by $\{a_{j_1}, \dots, a_{j_s}\}$ satisfies that $\Gamma^s \subset H$. Then the plane

Γ , generated by Γ^r and Γ^s , is also in H , and is an $(r + s)$ -plane intersecting $r + s + 2$ of the convex sets of F . Then since F is separated, $r + s + 2$ must be $n + 1$ and hence $\{a_0, \dots, a_{n+1}\} \subset H$. \square

Proof of Theorem 2.4. We begin as in [11], using the classical contraction argument. For each $i = 1, \dots, \lambda$, fix a point $a_i \in A_i$ and let $A_i(t)$ be the contraction of A_i about a_i by a factor of t , $0 \leq t \leq 1$; $A_i(0) = A_i$ and $A_i(1) = a_i$. If $t_1 < t_2$ and the hypothesis holds for the family $\mathcal{F}(t_2) = \{A_1(t_2), \dots, A_\lambda(t_2)\}$, then clearly it holds for $\mathcal{F}(t_1) = \{A_1(t_1), \dots, A_\lambda(t_1)\}$ as well. Since the family is separated, by continuity there exists a maximum τ such that the hypothesis holds for $t \leq \tau$ and fails for $t > \tau$. If $\tau = 1$, the theorem is trivial, hence we may assume without loss of generality that $\tau = 0$; that is, the hypothesis is satisfied for the original family of convex sets A_i but not for any proper contraction $A_i(t)$. It follows (see [8]) that there is a hyperplane H and a heterochromatic subfamily \mathcal{F}' , say $\mathcal{F}' = \{B_1, \dots, B_{n+1}\}$, such that every member of \mathcal{F}' is tangent to H and satisfies the following property: if $y_i \in H \cap B_i$, $i = 1, \dots, n + 1$, then

$$\text{conv}\{y_i \mid B_i \subset H^+\} \cap \text{conv}\{y_j \mid B_j \subset H^-\} \neq \emptyset.$$

First, note that X is a finite, $(n + 1)$ -coloured subset of \mathbf{R}^{n-1} , because every point $x_j \in X$ is of colour i , $i = 1, \dots, n + 1$ if and only if the corresponding convex set A_j is of colour i .

If every convex set of \mathcal{F} of a given colour i intersects H , then the theorem is true. Suppose then that there exists a set of colour i , $C_i \in \mathcal{F}$, such that $C_i \cap H = \emptyset$, $i = 1, \dots, n + 1$. Let us consider the family consisting of the convex sets

$$\{B_1, \dots, B_{n+1}, C_1, \dots, C_{n+1}\} \subset \mathcal{F}$$

and let

$$\mathcal{P} = \{b_1, \dots, b_{n+1}, c_1, \dots, c_{n+1}\} \subset X$$

be the corresponding points in \mathbf{R}^{n-1} , according to our hypothesis. Declare a point $b_i \in \mathcal{P}$ to be a square point if the corresponding $B_i \in H^-$ and b_j to be a round point if the corresponding $B_j \in H^+$. Similarly, a point $c_i \in \mathcal{P}$ is a square point if the corresponding $C_i \in H^-$ and c_j is a round point if the corresponding $C_j \in H^+$.

We can now see that $\{b_1, \dots, b_{n+1}\}$ is a heterochromatic Radon partition of \mathcal{P} , since by hypothesis it has the order type of $\mathcal{B} = \{y_1, \dots, y_{n+1}\}$ and

$$\text{conv}\{y_i \mid B_i \subset H^+\} \cap \text{conv}\{y_j \mid B_j \subset H^-\} \neq \emptyset.$$

Note that \mathcal{P} has cardinality $2n+2$ and has exactly two points of every colour. We would like to calculate $I(\mathcal{P})$, the Radon index of \mathcal{P} . Suppose that $\{d_1, \dots, d_{n+1}\} \subset \mathcal{P}$ is a heterochromatic Radon partition of \mathcal{P} and suppose

$$D_j \in \{B_1, \dots, B_{n+1}, C_1, \dots, C_{n+1}\}$$

is the convex set corresponding to the point $d_j \in X$, $j = 1, \dots, n + 1$. By hypothesis, there is an $(n - 1)$ -hyperplane H' of \mathbf{R}^n transversal to $\{D_1, \dots, D_{n+1}\}$ consistent with the order type of X . That is, there is

$a_j \in D_j$, $j = 1, \dots, n+1$ such that $\{d_1, \dots, d_{n+1}\}$ has the order type of $\{a_1, \dots, a_{n+1}\}$. Since $\{d_1, \dots, d_{n+1}\}$ is a heterochromatic Radon partition and has the order type of $\{a_1, \dots, a_{n+1}\}$, we have that

$$\text{conv}\{a_i \mid D_i \subset H^+\} \cap \text{conv}\{a_i \mid D_i \subset H^-\} \neq \emptyset.$$

By Lemma 2.5, we conclude that $\{a_1, \dots, a_{n+1}\} \subset H$, but since $a_i \in D_i \cap H$, this implies that $\{D_1, \dots, D_{n+1}\} = \{B_1, \dots, B_{n+1}\}$, because $C_i \cap H = \emptyset$, $i = 1, \dots, n+1$. So $\{d_1, \dots, d_{n+1}\} = \{b_1, \dots, b_{n+1}\}$ and $I(\mathcal{P}) = 1$, which is a contradiction to Theorem 2.2. This concludes the proof of the theorem. \square

3. THE COLOURFUL “T(3) IMPLIES T” THEOREM

In this section we will consider a family \mathcal{F} of translated copies of a compact convex set in the plane and assume that these convex sets are 3-coloured by, say green, blue and red, and suppose also that \mathcal{F} is T(3)-chromatic; that is, every three convex sets of \mathcal{F} with different colour admit a transversal line. Then our conclusion is that if the convex sets of \mathcal{F} are sufficiently far away from each other, then there is a colour, say red, such that all red convex sets of \mathcal{F} admit a transversal line.

Next we will state the formal definitions and the preliminaries for the proof of Theorem 3.6. Given a directed line \mathcal{L} in the plane, we will denote by \mathcal{L}^+ and by \mathcal{L}^- the corresponding half planes to the left and to the right of \mathcal{L} respectively. Given a set $\{\mathcal{L}_1, \dots, \mathcal{L}_k\}$, of lines orthogonal to \mathcal{L} that intersect \mathcal{L} in increasing order, we will divide the plane into $2(k+1)$ closed regions that will be denoted by $\mathcal{L}_1^+, \dots, \mathcal{L}_{k+1}^+$ and $\mathcal{L}_1^-, \dots, \mathcal{L}_{k+1}^-$ if they are to the left or to the right of \mathcal{L} respectively.

Let B be a convex set and let d be a direction in the plane. We will denote by $S_d(B)$ the supporting closed strip determined by B in the direction d and by $|S_d(B)|$ the width of this strip. For simplicity, if d is given by the unit vector $d = (0, 1)$, then we will simply denote such a strip by $S(B)$. Finally, if a directed line \mathcal{H} in the orthogonal direction d^\perp of d meets $S_d(B)$ somewhere, then the strip will be divided into two sections, $S_d^+(B)$ and $S_d^-(B)$, depending upon whether it is to the left or to the right of \mathcal{H} .

Definition 3.1. *Let A be a convex set in the plane and let $\Delta(A)$ be the convex hull of the union of all rectangles $S_d(A) \cap S_{d^\perp}(A)$ over all directions d . Then we say that two convex sets A and B are well-distributed if*

$$3\Delta(A) \cap 3\Delta(B) = \emptyset.$$

A family \mathcal{F} of convex sets in the plane is well-distributed if every two members of \mathcal{F} are well-distributed.

We say that two sets are (strictly) separated by a line \mathcal{L} if they lie in different (open) half planes determined by \mathcal{L} .

Definition 3.2. *Let A, B and C be three translated copies of a convex set in the plane. We say that the triple $\{A, B, C\}$ is in L -position if A intersects $S_d(B)$ and C intersects $S_{d^\perp}(B)$ for some direction d in the plane.*

The following two lemmas are very useful in the proof of the main theorem.

Lemma 3.3 (L-Lemma). *Let A, B and C well-distributed, translates copies of a convex set in the plane. Suppose the triple $\{A, B, C\}$ is in L -position with respect to B . Then there is no transversal line through $\{A, B, C\}$.*

Proof. It will be enough to show that for each of these three convex sets, there is a line that strictly separates it from the other two. Since A intersects $S_d(B)$, C intersects $S_{d^\perp}(B)$, and C and B are well-distributed, which implies that there is a line in direction d that strictly separates C from A and B . Similarly, there is a line in direction d^\perp that strictly separates A from C and B . Without loss of generality, suppose $d = (0, 1)$, C is above $S_d(B)$, and A is located to the right of $S_{d^\perp}(B)$. Let \mathcal{L}' be the line of slope $-|S_{d^\perp}(B)|/|S_d(B)|$ through the upper right corner of the rectangle $S_d(B) \cap S_{d^\perp}(B)$. Since A, B and C are well-distributed, \mathcal{L}' separates B from A and C and $\mathcal{L}' \cap (A \cup C) = \emptyset$. Therefore, there is a line that strictly separates B from A and C , as required. \square

Definition 3.4. *Let \mathcal{L} be a directed line in the plane and let \mathcal{L}_1 and \mathcal{L}_2 be two different lines orthogonal to \mathcal{L} that intersect \mathcal{L} in increasing order. Let A, B and C be three convex sets in the plane. We say that the triple $\{A, B, C\}$ is in V -position with respect to the line \mathcal{L} if A is contained in the open region \mathcal{L}_1^+ , B is contained in the open region \mathcal{L}_2^- and C is contained in the open region \mathcal{L}_3^+ . Two of the sets are allowed to be tangent to the line \mathcal{L} .*

Lemma 3.5 (V-lemma). *Suppose the triple $\{A, B, C\}$ is in V -position with respect to a line \mathcal{L} . Then there is no transversal line through $\{A, B, C\}$.*

Proof. In order to prove that there is no transversal line through $\{A, B, C\}$, it will be enough to show that for each of the convex sets in $\{A, B, C\}$, there is a line that strictly separates it from the other two. Clearly, \mathcal{L}_1 separates A from B and C , and $C \cap \mathcal{L}_1 = \emptyset$. The line \mathcal{L}_2 separates A and B from C , and $A \cap \mathcal{L}_2 = \emptyset$. Furthermore, the line \mathcal{L} separates B from A and C , and one of the three convex sets does not touch the line \mathcal{L} . Hence there are three lines, sufficiently close to the lines $\mathcal{L}_1, \mathcal{L}_2$ and \mathcal{L} , with the property that they strictly separate one from the other two. \square

Throughout the proof of Theorem 3.6, we will obtain triplets that are in V -position with respect to the x -axis, but sometimes they will be V -position with respect to the y -axis. If this is the case, for simplicity, we shall say that we are using the V^\perp -Lemma.

Theorem 3.6. *Let $\mathcal{F} = R \cup G \cup B$ be a 3-coloured family of translated copies of a compact convex set in \mathbf{R}^2 , suppose that \mathcal{F} is well-distributed and $T(3)$ -chromatic. Then there is a colour and a line transversal to each convex set of \mathcal{F} with this colour.*

We repeat the classical contraction argument of Hadwiger [11], and assume in this case that there is a slope zero, positive directed line \mathcal{L} tangent to the

heterochromatic triplet $\{R_0, G_0, B_0\}$, where $R_0 \in R$, $G_0 \in G$ and $B_0 \in B$. Moreover, \mathcal{L} intersects the triplet $\{R_0, G_0, B_0\}$ in that order and R_0 and B_0 , say, are below \mathcal{L} and G_0 above \mathcal{L} . See [8] for details.

The vertical strips generated by these sets and the line \mathcal{L} divide the plane into eight closed regions that we will denote by $\mathcal{L}_1^+, \mathcal{L}_1^-, \mathcal{L}_2^+, \mathcal{L}_2^-, \mathcal{L}_3^+, \mathcal{L}_3^-$ and $\mathcal{L}_4^+, \mathcal{L}_4^-$ as shown in the following figures. Denote the corresponding strips in direction \mathcal{L}^\perp generated by R_0, G_0 and B_0 by $S(R_0), S(G_0)$ and $S(B_0)$, respectively. Finally, denote by \mathcal{H}_1 the other line tangent to G_0 parallel to \mathcal{L} , and by \mathcal{H}_2 the other line tangent to R_0 and B_0 parallel to \mathcal{L} .

Suppose the theorem is not true. Hence we may assume that there exists at least one member of each colour, say $R_1 \in R$, $B_1 \in B$, and $G_1 \in G$ such that they do not meet \mathcal{L} . In what follows, we will work with this family of six convex sets $\mathcal{F}' = \{R_0, R_1, B_0, B_1, G_0, G_1\}$, two of them red, two blue and two green, with the property that any three of them with different colours have a common transversal. Our method of proof is to analyze the possible positions of these convex sets and conclude that our initial assumption yields a contradiction. For this purpose, the next lemma is crucial.

Lemma 3.7 (I-lemma). *The orthogonal projections of the six members of \mathcal{F}' onto the line \mathcal{L} are not pairwise disjoint.*

Proof. Assume the projection of the six convex sets is pairwise disjoint. Choose a point in each convex set in \mathcal{F}' to make up a set $\mathcal{P} = \{r_0, r_1, b_0, b_1, g_0, g_1\}$. Declare, as before, that a point is square or round if the corresponding convex set is above or below the line \mathcal{L} . By Theorem 1, \mathcal{P} must contain another heterochromatic Radon partition different from $\{r_0, b_0, g_0\}$. Due to the fact that R_1, G_1 and B_1 do not intersect \mathcal{L} , any other heterochromatic Radon partition will be in V-position. The V-Lemma thus yields a contradiction to the $T(3)$ -chromatic hypothesis. Then the orthogonal projection of the six members are not pairwise disjoint. \square

Now we begin studying the possible positions for the six convex sets.

Theorem 3.8. *The forbidden positions for R_1 (see Figure 1) are:*

- (a) *the convex set R_1 does not intersect*

$$\mathcal{L}_1^- \cup S^-(R_0) \cup \mathcal{L}_2^- \cup S(G_0) \cup S(B_0) \cup \mathcal{L}_4^+,$$

- (b) *the convex set R_1 is not contained in $\mathcal{L}_3^+ \cap \text{int}(H_1^+)$, and*
(c) *the convex set R_1 is not contained in $\mathcal{L}_3^- \cap \text{int}(H_2^-)$.*

Proof.

- (a) First of all, note that R_1 does not intersect $S(G_0)$ and does not intersect $S(B_0)$, by the L-Lemma for the triple $\{R_1, G_0, B_0\}$. Next, the V-Lemma for $\{R_1, G_0, B_0\}$ implies that R_1 is not contained in $\mathcal{L}_1^- \cup S^-(R_0) \cup \mathcal{L}_2^-$ and not contained in \mathcal{L}_4^+ .
(b) The convex set R_1 is not contained in $\mathcal{L}_3^+ \cap H_1^+$ by the V^\perp -Lemma applied to the triple $\{B_0, G_0, R_1\}$.

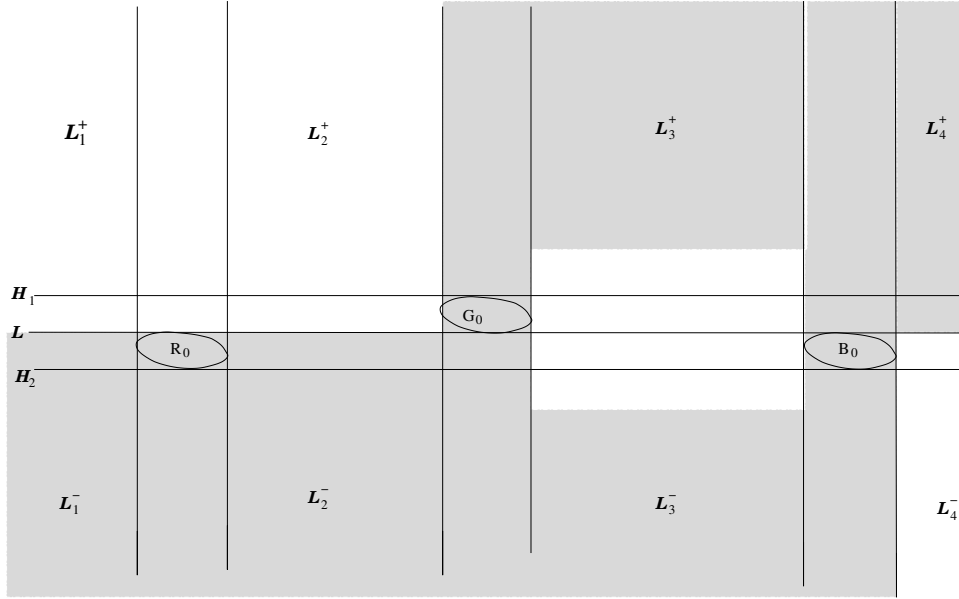


FIGURE 1

(c) Completely analogous to (b).

□

Theorem 3.9. *The forbidden positions for B_1 (see Figure 2) are:*

(a) *the convex set B_1 does not intersect*

$$\mathcal{L}_4^- \cup S^-(B_0) \cup \mathcal{L}_3^- \cup S(G_0) \cup S(R_0) \cup \mathcal{L}_1^+,$$

(b) *the convex set B_1 is not contained in $\mathcal{L}_2^+ \cap \text{int}(H_1^+)$, and*

(c) *the convex set B_1 is not contained in $\mathcal{L}_2^- \cap \text{int}(H_2^-)$.*

Proof. The proof is completely analogous to the proof of Theorem 3.8, using the obvious symmetry between the situations of R_0 and B_0 . □

Theorem 3.10. *The forbidden positions for G_1 (see Figure 3) are:*

(a) *the convex set G_1 does not intersect*

$$S(R_0) \cup \mathcal{L}_2^+ \cup S^+(G_0) \cup \mathcal{L}_3^+ \cup S(B_0),$$

(b) *the convex G_1 is not contained in $(\mathcal{L}_2^- \cup S^-(G_0) \cup \mathcal{L}_3^-) \cap \text{int}(H_2^-)$, and*

(c) *the convex set G_1 does not intersect $S(G_0)$.*

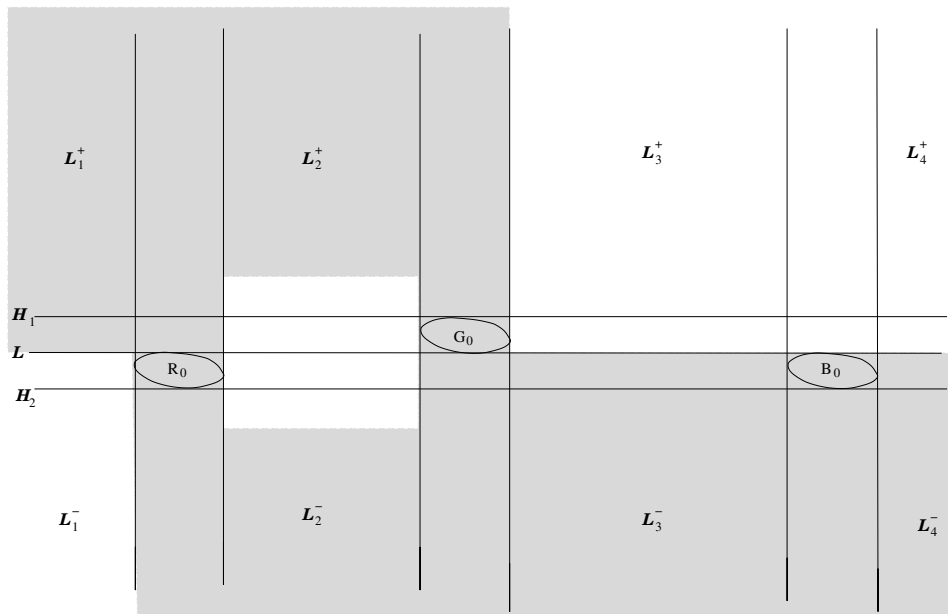


FIGURE 2

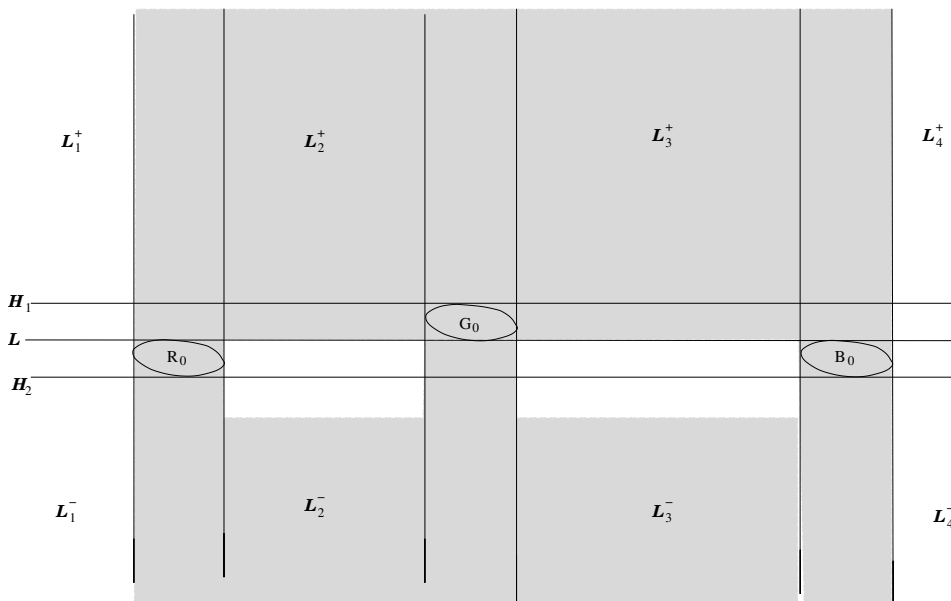


FIGURE 3

Proof.

- (a) The convex set G_1 does not intersect $S(R_0)$, and does not intersect $S(B_0)$ by the L-Lemma for the triple $\{G_1, R_0, B_0\}$. The V-Lemma for $\{R_0, G_1, B_0\}$ implies that G_1 is not contained in $\mathcal{L}_2^+ \cup S^+(G_0) \cup \mathcal{L}_3^+$.
- (b) The V-Lemma for $\{R_0, G_1, B_0\}$, but now using the horizontal line H_1 as axis, implies that G_1 is not contained in $(\mathcal{L}_2^- \cup S^-(G_0) \cup \mathcal{L}_3^-) \cap \text{int}(H_2^-)$.
- (c) Follows from (a) and (b), keeping in mind that G_0 and G_1 are well-distributed. □

In the following cases, we shall prove that the vertical strips generated by the convex set $\{R_0, R_1, B_0, B_1, G_0, G_1\}$ are pairwise disjoint, contradicting the I-lemma.

Lemma 3.11.

$$(3.1) \quad S(R_1) \cap S(B_1) = \emptyset.$$

Proof. Suppose $S(R_1) \cap S(B_1) \neq \emptyset$. By the L-lemma for $\{G_0, B_1, R_1\}$, G_0 does not intersect the horizontal strip generated by B_1 and by R_1 . Moreover, by the V^\perp -Lemma for the triple $\{R_1, G_0, B_1\}$, we know that if the horizontal strip of G_0 separates R_1 and B_1 , then $S(G_0) \cap (S(R_1) \cup S(B_1)) \neq \emptyset$, which yields a contradiction to Theorems 3.8 and 3.9. Thus $(R_1 \cup B_1)$ is contained either in the interior of H_1^+ or in the interior of \mathcal{L}^- . But this is again a contradiction to Theorems 3.8 and 3.9. □

Lemma 3.12.

$$(3.2) \quad S(G_1) \cap S(B_1) = \emptyset.$$

Proof. Assume $S(G_1) \cap S(B_1) \neq \emptyset$. By the L-lemma for $\{R_0, G_1, B_1\}$ with respect to G_1 and R_1 , G_1 and R_1 do not intersect the horizontal strip R_0 . Moreover, by the V^\perp -lemma for the triple $\{G_1, R_0, B_1\}$, we know that if the horizontal strip $S(R_0)$ separates G_1 from B_1 then $S(R_0) \cap (S(G_1) \cup S(B_1)) \neq \emptyset$, which yields a contradiction to Theorems 3.9 and 3.10. Thus $(G_1 \cup B_1)$ is contained either in the interior of \mathcal{L}^+ or in the interior of H_2^- . Then Theorems 3.9 and 3.10 imply that the only possible positions for $(G_1 \cup B_1)$ are: *i*) $(G_1 \cup B_1)$ contained in the interior of \mathcal{L}_4^+ *ii*) $(G_1 \cup B_1)$ contained in the interior of $(\mathcal{L}_1^- \cap H_2^-)$, *iii*) $(G_1 \cup B_1)$ contained in the interior of $(\mathcal{L}_2^- \cap H_2^-)$.

i) Assume that $(G_1 \cup B_1)$ is contained in the interior of \mathcal{L}_4^+ . By the V-lemma for $\{R_1, B_0, G_1\}$, the convex set R_1 is not contained in the interior of $(\mathcal{L}_1^+ \cup S^+(R_0) \cup \mathcal{L}_2^+)$. Moreover, by the V-lemma for $\{G_0, R_1, B_1\}$, the convex set R_1 is not contained in interior of \mathcal{L}_3^- , consequently Theorem 3.8 implies that R_1 must be in the interior of \mathcal{L}_4^- . By the V^\perp -lemma for

triple $\{R_1, B_0, G_1\}$, R_1 must intersect H_2 , and for the triple $\{R_1, G_0, B_1\}$, B_1 must intersect H_1 . Clearly $S(R_1) \cap S(G_1) = \emptyset$, otherwise the L-lemma would yield a contradiction for $\{G_1, R_1, B_0\}$. Next, note that R_1 is not to the right of $S(G_1)$ by the V-lemma for $\{B_0, G_1, R_1\}$. Then $S(R_1)$ is to the left of $S(G_1)$. However since $S(B_1) \cap S(G_1) \neq \emptyset$ and $S(B_1) \cap S(R_1) = \emptyset$ by (3.1), then $S(R_1)$ is to the left of $S(B_1)$ as well, which contradicts the V-lemma for $\{G_0, R_1, B_1\}$.

ii) Suppose now that $(G_1 \cup B_1)$ is contained in the interior of $(\mathcal{L}_1^- \cap H_2^-)$. By the V-lemma for $\{R_1, B_0, G_1\}$, the convex set R_1 is not contained in the interior of $\mathcal{L}_2^+ \cup S^+(R_0) \cup \mathcal{L}_3^+$. Moreover, by the V-lemma for $\{B_1, G_0, R_1\}$, the convex set R_1 is not contained in the interior of $\mathcal{L}_3^- \cup S^+(B_0) \cup \mathcal{L}_4^-$. Consequently Theorem 3.8 implies that R_1 must be in the interior of \mathcal{L}_1^+ , but this is a contradiction to the V^\perp -lemma for triple $\{G_1, B_0, R_1\}$.

iii) If $(G_1 \cup B_1)$ is contained in the interior of $(\mathcal{L}_2^- \cap H_2^-)$ and since G_1 and B_1 are well-distributed, one of them is completely contained in

$$\mathcal{L}_2^- \cap \text{int}(H_2^-),$$

which is a contradiction either to Theorem 3.9 or to Theorem 3.10. \square

Lemma 3.13.

$$(3.3) \quad S(G_1) \cap S(R_1) = \emptyset.$$

Proof. The proof is completely analogous to the proof of (3.2), using the obvious symmetry between the situations of R_0 and B_0 . \square

Lemma 3.14.

$$(3.4) \quad S(R_0) \cap S(R_1) = \emptyset.$$

Proof. Clearly by the well-distributed property, if R_1 is above H_1 then B_1 is not in \mathcal{L}_1^- , otherwise the V^\perp -lemma for $\{R_1, G_0, B_1\}$ would yield a contradiction. Furthermore B_1 is not in \mathcal{L}_2^- either, by the V-Lemma for $\{R_1, G_0, B_1\}$. Then B_1 is, according to Theorem 3.9, in the admissible positions but totally to the right of R_0 and meeting H_1 , by the V-lemma for $\{R_1, G_1, B_1\}$. Assuming the former, we note that G_1 is not in \mathcal{L}_1^+ by the V-lemma applied to the triplet $\{G_1, R_0, B_1\}$. Similarly G_1 is not in \mathcal{L}_1^- and not in \mathcal{L}_4^+ by the V-lemma for $\{G_1, R_1, B_0\}$. Then G_1 is totally to the right of R_0 below \mathcal{L} and meeting H_2 , by the V^\perp -lemma for $\{G_1, R_0, B_1\}$. Recall that $S(G_1)$ and $S(B_1)$ are disjoint, by (3.2). Suppose then that $S(B_1)$ is to the right of $S(G_1)$, then the V-lemma yields a contradiction for $\{R_1, G_1, B_1\}$. Then $S(B_1)$ is to the left of $S(G_1)$, and the V-lemma for $\{R_0, B_1, G_1\}$ yields a contradiction. \square

Lemma 3.15.

$$(3.5) \quad S(B_0) \cap S(B_1) = \emptyset.$$

Proof. The proof is completely analogous to the proof of (3.4), using the obvious symmetry between the situations of R_0 and B_0 . \square

Proof of Theorem 3.6. Observe that Theorems 3.8, 3.9 and 3.10, together with (3.1), (3.2), (3.3), (3.4) and (3.5), imply that the six vertical strips

$$\{S(R_0), S(R_1), S(G_0), S(G_1), S(B_0), S(B_1)\}$$

are pairwise disjoint, but this is a contradiction to the I-lemma. \square

4. THE COLOURFUL “T(3) IMPLIES T-K” THEOREM

The purpose of this section is to prove a colourful version of the Katchalski-Lewis Theorem (B).

Theorem 4.1. *Let K be a compact convex set in \mathbf{R}^2 . Then there is a positive integer k with the property that if \mathcal{F} is any 3-coloured family of pairwise disjoint translated copies of K , which is $T(3)$ -chromatic, then there is a colour and a line transversal to every convex set of \mathcal{F} of this colour, except possibly for k of them.*

Before giving the proof of the theorem, we need the following lemma.

Lemma 4.2. *Let K be a compact convex set in the plane. Then there exists a positive integer k , depending only on K , with the following property: if \mathcal{F} and F' are two families of pairwise disjoint translated copies of K , $|F'| \leq 5$ and $|\mathcal{F}| \geq k$, then there is $K^0 \in \mathcal{F}$ such that every member of F' is well distributed of K^0 .*

Proof. Let us first prove the lemma when $F' = \{K_0\}$. Recall from the last section that two sets K^0 and K_0 are well-distributed if

$$3\Delta(K_0) \cap 3\Delta(K^0) = \emptyset.$$

Let $\Lambda(K_0)$ be the union of all translated copies of $3\Delta(K_0)$ that intersect $3\Delta(K_0)$. Note that if K^0 is a translated copy of K and K^0 is not contained in $\Lambda(K_0)$, then K^0 is well-separated from K_0 . Note that the area of $\Lambda(K_0)$ depends only on the convex body K . Then, the lemma follows in this case for

$$k > \frac{\text{area of } \Lambda(K_0)}{\text{area of } K},$$

and in general

$$k > 5 \frac{\text{area of } \Lambda(K_0)}{\text{area of } K}.$$

\square

Proof of Theorem 4.1. Let k be as in the lemma for the convex set K . Note that there are more than k translated copies of K of each colour, otherwise the theorem follows trivially. We shall first prove that there are $R_0 \in R$, $B_0 \in B$, and $G_0 \in G$, with the property that the triple $\{R_0, B_0, G_0\}$ is well-distributed. Let R_0 be any element of R and since there are more than k blue convex sets, by the lemma, there is $B_0 \in B$ such that B_0 is well-distributed of R_0 . Analogously, since there are more than k blue convex

sets, by the lemma, there is $G_0 \in G$, such that the triple $\{R_0, B_0, G_0\}$ is well-distributed.

Note that trivially, the family $\mathcal{F} = R \cup G \cup B$ has the property that every heterochromatic, well-distributed triple $\{R_0, B_0, G_0\}$ of \mathcal{F} has a transversal line. Now, choosing a point in every member of \mathcal{F} , we may as before apply the classical contraction argument and homothetically reduce every member of \mathcal{F} until the very last moment, $0 < \tau \leq 1$, in which the reduced family \mathcal{F}_τ has the above transversal-line property. This classic argument shows that there is a line \mathcal{L} tangent to the triplet $\{R_0^\tau, G_0^\tau, B_0^\tau\}$ where, R_0^τ, G_0^τ and B_0^τ are the τ -reduced copies of some $R_0 \in R$, $B_0 \in B$ and $G_0 \in G$, and where $\{R_0, B_0, G_0\}$ is a well-distributed triple.

Without loss of generality, we may assume that \mathcal{L} intersects them in that order, has slope zero and R_0^τ and B_0^τ , say, are below \mathcal{L} and G_0^τ above \mathcal{L} . For the sake of simplicity, we will rename R_0^τ, B_0^τ and G_0^τ and the whole family \mathcal{F}_τ simply as R_0, B_0, G_0 and $\mathcal{F} = R \cup G \cup B$. We would like to show that there is a colour such that \mathcal{L} is a transversal line to all members of this colour except possibly k of them. Assume then that the former is false. Consider now all red members of \mathcal{F} in R which do not intersect \mathcal{L} . If the number of these red convex sets is less than k , the theorem is proved; otherwise, by the lemma, there is $R_1 \in R$ which does not intersect \mathcal{L} and is such that the family $\{R_0, R_1, B_0, G_0\}$ is well-distributed. By the same argument, there are $G_1 \in G$ and $B_1 \in B$ which do not intersect \mathcal{L} and are such that the family $\{R_0, R_1, B_0, B_1, G_0, G_1\}$ is well-distributed. Clearly, this subfamily has the following properties:

- (1) it is 3-coloured family of pairwise disjoint translated copies of K and is $T(3)$ -chromatic and well-distributed;
- (2) it has two members of each colour;
- (3) there is a line \mathcal{L} with slope zero, tangent to the triplet $\{R_0, G_0, B_0\}$, that intersects them in that order and R_0 and B_0 , say, below \mathcal{L} and G_0 above \mathcal{L} , and
- (4) the sets R_1, B_1 , and G_1 do not intersect \mathcal{L} .

Under exactly these conditions, we proved in Theorem 3.6 that such a family can not exist. \square

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INSTITUTO DE MATEMÁTICAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
E-mail address: `luis@math.unam.mx`
E-mail address: `dolivero@math.unam.mx`