Contributions to Discrete Mathematics

Volume 5, Number 1, Pages 1–6
ISSN 1715-0868

A CHARACTERIZATION OF THE BASE-MATROIDS OF A GRAPHIC MATROID

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ABSTRACT. Let $M = (E, \mathcal{F})$ be a matroid on a set E, and B one of its bases. A closed set $\theta \subseteq E$ is saturated with respect to B when $|\theta \cap B| = r(\theta)$, where $r(\theta)$ is the rank of θ.

The collection of subsets I of E such that $|I \cap \theta| \leq r(\theta)$ for every closed saturated set θ turns out to be the family of independent sets of a new matroid on E, called base-matroid and denoted by M_B. In this paper we prove that a graphic matroid M, isomorphic to a cycle matroid $M(G)$, is isomorphic to M_B, for every base B of M, if and only if M is direct sum of uniform graphic matroids or, in equivalent way, if and only if G is disjoint union of cacti. Moreover we characterize simple binary matroids M isomorphic to M_B, with respect to an assigned base B.

1. Introduction

Let $M = (E, \mathcal{F})$ be a matroid on a set E, having $\mathcal{F}$ as its family of independent sets. For notations and definitions we refer to [6].

Let Ξ denote the set of all closed sets of M. Then

$$\mathcal{F} = \{S \subseteq E : |S \cap \theta| \leq r(\theta), \forall \theta \in \Xi\}.$$

A set $\theta \subseteq E$ is defined [3] saturated with respect to a base B of M if

$$|\theta \cap B| = r(\theta).$$

Thus any B-saturated closed set θ satisfies the relation $cl(\theta \cap B) = \theta$; in other words, θ coincides with the closure of its intersection with B.

If in addition θ belongs to Ξ, we have a saturated closed set. The set of all the saturated closed sets of M, with respect to a base B, is denoted by Ξ_B. A circuit is *fundamental* with respect to B when it is the fundamental circuit of an element $i \in E \setminus B$. Calling $\gamma(i)$ the unique minimal subset of

Received by the editors January 21, 2007, and in revised form September 29, 2008.
2000 *Mathematics Subject Classification.* 05B35, 90C27.
Key words and phrases. Base, uniform matroid, graphic matroid, binary matroid, cactus.
Work partially supported by MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca).

B such that $\gamma(i) \cup i \notin \mathcal{F}$, then $\gamma(i) \cup i$ is a fundamental circuit. We use the notation

$$\mathcal{F}_B = \{S \subseteq E : |S \cap \theta| \leq r(\theta), \forall \theta \in \Xi_B\}$$

and

$$M_B = (E, \mathcal{F}_B).$$

In [3] it is proved that $M = (E, \mathcal{F}_B)$ is a matroid, and in particular a transversal matroid. An application of these matroids, named base-matroids, is in the field of inverse combinatorial optimization problems; indeed many different inverse problems have been addressed in the recent literature [1, 3, 5].

Recall that a matroid M on a ground set E, whose family of independent sets is $\mathcal{F}$, is direct sum of the matroids $M_1, M_2, \dots, M_s$ on disjoint sets $E_1, E_2, \dots, E_s$ respectively, when $E_1, E_2, \dots, E_s$ is a partition of E and

$$\mathcal{F} = \{I_1 \cup \cdots \cup I_s : I_i \in \mathcal{F}(M_i), 1 \leq i \leq s\},$$

where $\mathcal{F}(M_i)$ is the family of independent sets of M_i.

A simple matroid M is binary if the symmetric difference of any two different circuits is a union of disjoint circuits. Clearly graphic matroids are examples of binary matroids.

The main aim of this paper is determining a characterization of a graphic matroid M which is isomorphic to M_B ($M \simeq M_B$), where B is any base of M. Indeed, it is proved that a matroid M, isomorphic to a cycle matroid $M(G)$, is isomorphic to M_B for every base B of M if and only if G is disjoint union of cacti or, in equivalent way, if and only if M is direct sum of uniform graphic matroids. Finally we characterize a simple binary matroid M isomorphic to M_B, with respect to an assigned base B.

2. Independent circuits

Let $\mathcal{F}$ and $\mathcal{F}_{\mathcal{B}}$ denote the collections of independent sets of M and M_B respectively. It is easy to see that

$$\mathcal{F} \subseteq \mathcal{F}_{\mathcal{B}},$$

and the inclusion is proper when a dependent set of M turns out to be independent in M_B; in this case M is not isomorphic to M_B. In other words the above relation implies that $M \simeq M_B$ if and only if

$$\mathcal{F} = \mathcal{F}_{\mathcal{B}}.$$

Lemma 2.1. *Let M be a matroid and B one of its bases. Then $M \simeq M_B$ if and only if every circuit of M is also circuit of M_B.*

Proof. If every circuit of M is also circuit of M_B, then it follows that every dependent set of M is dependent also in M_B. Then $\mathcal{F} = \mathcal{F}_{\mathcal{B}}$ and consequently $M \simeq M_B$.

Conversely, if $M \simeq M_B$, from the condition $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{B}}$ it follows $\mathcal{F} = \mathcal{F}_{\mathcal{B}}$. Then it is not possible that there exists a dependent subset of M which turns out to be independent in M_B. □

We first consider the case of a circuit of M, dependent in M_B.

Proposition 2.2. *Assume that a circuit C of M satisfies the inequality $|C \cap \theta| > r(\theta)$ for a suitable closed set θ of M saturated with respect to a base B. Then $\theta = cl(C)$.*

Proof. There are two cases to consider depending on the condition that C is not contained or contained in θ.

If C is not contained in θ, then $C \cap \theta$ is a proper subset of C; then it is independent in M and consequently independent also in M_B. Thus $|C \cap \theta| \leq r(\theta)$, a contradiction.

In the second case, we have $|C \cap \theta| = |C|$; then $r(C) \leq r(\theta)$. As $r(C) = |C| - 1$, we obtain the following double inequality $|C| - 1 \leq r(\theta) < |C|$. Then $r(\theta) = |C| - 1$ and therefore $\theta = cl(C)$. □

Definition 2.3. *A circuit C of M is said to be independent with respect to B, or B-*independent*, if*

$$|cl(C) \cap B| < |C| - 1.$$

*Moreover C is dependent with respect to B, or B-*dependent*, if it is not independent with respect to B; that is,*

$$|cl(C) \cap B| = |C| - 1.$$

Thus $cl(C)$ is saturated with respect to B.

Notice that if a circuit C is B-dependent, then $C \notin \mathcal{F}_B$. In other words C is dependent in M_B; in particular it is a circuit of M_B. On the contrary, if C is B-independent, then C is independent in M_B and consequently M is not isomorphic to M_B.

Recall ([2]) that a circuit C of a matroid M has a *chord* e if there are two circuits C_1 and C_2 such that $C_1 \cap C_2 = \{e\}$ and $C = C_1 \triangle C_2$. In this case we say that C is the sum of C_1 and C_2 and also that $C \cup \{e\}$ is split into C_1 and C_2.

When a chord belongs to a base B, we say that it is a B-chord.

Lemma 2.4. *A circuit of M, fundamental with respect to B, is B-dependent and does not contain B-chords.*

Proof. Let C be a circuit of M fundamental with respect to B. If $|C| = m+1$, then $|C \cap B| = m$ and C is B-dependent. If C contains a B-chord e, then $cl(C)$ contains $m+1$ elements which belong to B. This implies the impossible relation $r(cl(C)) = m + 1$. □

Proposition 2.5. *Let M be a uniform matroid of rank n. Then for every base B of M it is $M \simeq M_B$.*

Proof. Let C be a circuit of M, that is a $(n+1)$-subset of $E(M)$. It follows that $|C \cap E| > r(E)$, so that C is dependent also in M_B. It is in particular a circuit because every proper subset of C is independent in M and consequently in M_B. The result follows from Lemma 2.1. □

3. Graphic matroids

In this section we consider the problem of characterizing graphic matroids M isomorphic to M_B for every base B of M. Let $G = (V, E)$ be a graph without loops and parallel edges, having V and E as the sets of vertices and edges respectively.

Recall that two cycles of a graph are said *intersecting* when the intersection of their edge sets is not empty.

Lemma 3.1. *A cycle matroid $M(G)$, having rank n, is uniform if and only if G is either an n-tree or an $(n+1)$-cycle.*

Proof. Let us assume that M is uniform. If m is the number of edges of G, then either $m = n$ or $m > n$. In the first case $M(G)$ does not contain dependent sets; then G does not contain cycles and G is an n-tree. If $m > n$, the condition that M is uniform implies that every $(n+1)$-subset forms a minimal dependent set, that is a $(n+1)$-cycle of G. Let C be a $(n+1)$-cycle and $e = (u, v)$ a possible edge of $E \setminus C$. Then u and v can not belong to C because otherwise we obtain a chord of C and then a cycle having length lesser than $n+1$. Thus at least one of the vertices u and v does not belong to C; this implies that there exists a spanning tree having cardinality greater than n, a contradiction.

Conversely, if G is either an n-tree or an $(n+1)$-cycle, then in both the cases $M(G)$ has rank n. In the first case it is a free matroid, while in the second case it is the uniform matroid $U_{n,n+1}$. □

Lemma 3.2. *Let G be a graph having two intersecting cycles; then G contains two cycles C and H such that $C \triangle H$ is one cycle and $C \cap H$ is a path.*

Proof. Let C and Q two intersecting cycles; $C \triangle Q$ is a set of disjoint cycles. Let D one of these cycles, where $D = C' \cup H'$, $C' \subseteq C$ and $H' \subseteq Q$ are paths and H' is vertex disjoint from C', but on the end vertices.

The subgraph $C \triangle D$ coincides with $(C \setminus C') \cup H'$. In other words, it is obtained from C by replacing the path C' by the path H'. Then $C \triangle D$ is one cycle and $C \cap D = C'$. □

Proposition 3.3. *Let G be a graph having two intersecting cycles. Then $M(G)$ contains a base B in relation to which M_B is not isomorphic to M.*

Proof. Let G be a graph having two intersecting cycles, say C and H. By Lemma 3.2 we may assume that $C \triangle H$ is one cycle, say D, and $C \cap H = P$ is a path of length ≥ 1. Assume that $D = C' \cup H'$ where C' , H' are paths contained in C and H, respectively.

Let B a spanning tree of $C \cup H$ obtained by taking all the edges of C but an edge e of P and all the edges of H but e and another edge, say f, of $H \setminus P$. We may extend B to a spanning tree of G , which we still denote B. Then we may see that H is not B-fundamental because contains two edges which do not belong to B. Then

$$|cl(H) \cap B| = |H| - 2$$

and H is B-independent. This implies that $M(G)$ is not isomorphic to M_B, with respect to the base B. □

Recall that a connected graph G is called a *cactus* when any edge belongs to at most one cycle. In other words G is a cactus if and only if it is connected and its possible cycles are edge-disjoint.

Corollary 3.4. *If the cycle matroid $M(G)$ is isomorphic to M_B for every spanning tree B of G, then G is a graph whose components are cacti.*

Proof. From Proposition 3.3 it follows that G has not intersecting cycles; in other words the components of G are cacti. □

Theorem 3.5. *A cycle matroid $M(G)$ is isomorphic to the base-matroid M_B, for every base B of M, if and only if G is a disjoint union of cacti.*

Proof. If a cycle matroid $M(G)$ is isomorphic to the base-matroid M_B, for every base B, then, by Proposition 3.3, G does not contain intersecting cycles and by Corollary 3.4 the components of G are cacti.

Conversely, if the components of G are cacti, then G has not intersecting cycles. If there exists a base B in relation to which M_B is not isomorphic to M, then, by Lemma 2.1, there exists a cycle Q of G, which turns out to be independent in M_B. Clearly by Lemma 2.4 Q is not fundamental with respect to B. Denote by f an element of $Q \setminus B$; then the fundamental cycle $F(f)$, obtained by adding f to B, and Q are distinct and intersecting, a contradiction. □

Theorem 3.6. *For every base B of a graphic matroid M, $M \simeq M_B$ if and only if M is direct sum of uniform graphic-matroids.*

Proof. Let $M = \oplus M_i$ be direct sum of uniform graphic matroids and B a base of M. The $B = \oplus B_i$, where B_i is a base of M_i. By Proposition 2.5 $M_i \simeq M_{i_{B_i}}$ and therefore $M \simeq M_B$.

Now assume that M is isomorphic to the cycle-matroid $M(G)$ and moreover that $M \simeq M_B$ in relation to a base B of M, that is a spanning tree of G. Then by Theorem 1 G is union of disjoint cacti and therefore does not contain intersecting cycles. This implies that $E(G)$ can be partitioned into edge-disjoint cycles, say $C_1, C_2, \ldots, C_r$, $r \geq 0$, and edge-disjoint trees, say $T_1, T_2, \ldots, T_s$, $s \geq 0$. Then M is direct sum of the matroids on $C_1, C_2, \ldots, C_r$ and $T_1, T_2, \ldots, T_s$, which turn out to be all uniform.

Thus $M(G)$ is direct sum of uniform graphic matroids. □

Now we generalize the result of the previous theorem to the case of a simple binary matroid.

Theorem 3.7. *Let M be a simple binary matroid on E and B a base of M. Then $M \simeq M_B$ if and only if either all the circuits of M are fundamental or* *every circuit not fundamental with respect to B contains at least one chord which belongs to B.*

Proof. If $M \cong M_B$, then by Lemma 2.1 every circuit of M is also a circuit of M_B; in other words every circuit of M has to be B-dependent. Let C be a n-circuit, not fundamental with respect to B. Because it is B-dependent, then $|cl(C) \cap B| = n - 1$.

From the condition that C is not fundamental it follows there exists at least an element, say a, which belongs to $(cl(C) \setminus C) \cap B$. Because M is binary, from the proof of Lemma 2.1 of [2], it follows that every element of $cl(C) \setminus C$ is a chord; then the element a is a B-chord.

Conversely, assume that every possible circuit, not fundamental with respect to B, contains at least one B-chord. Our aim is to prove that it is B-dependent; by Lemma 2.1 this implies that $M \cong M_B$. Let C be an n-circuit, not B-fundamental, having a B-chord, say c_1. Let H_1, H_2 be two circuits in which $C \cup c_1$ is splitted. If H_1 and H_2 are both B-fundamental, then $(C \cap B) \cup c_1$ is an independent set of cardinality $n - 1$ whose closure coincides with $cl(C)$. Then C is B-dependent.

Now, assume that at least one of the above circuits, say H_2, is not B-fundamental. Then it contains at least one chord c_2 which belongs to B, such that $H_2 \cup c_2$ can be decomposed into two distinct circuits intersecting in c_2. By repeating the above procedure, we arrive to obtain that C can be decomposed into a number, say s, of fundamental circuits. Thus C contains s elements which do not belong to B and $s-1$ chords which belong to B. If T is the set of similar chords, then $|cl(C) \cap B| = |(C \cap B) \cup T| = n - s + s - 1 = n - 1$ and C is still B-dependent. □

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