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ON THE RIGIDITY OF REGULAR BICYCLE (n,k)-GONS

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Abstract. Bicycle (n,k)-gons are equilateral n-gons whose k-diagonals are equal. In this paper, the order of infinitesimal flexibility of the regular n-gon within the family of bicycle (n,k)-gons is studied. An equation characterizing first order flexible regular bicycle (n,k)-gons were computed by S. Tabachnikov in [7]. This equation was solved by R. Connelly and the author in [3]. S. Tabachnikov has also constructed nontrivial deformations of the regular bicycle (n,k)-gon for certain pairs (n,k). The main result of the paper is that if the regular bicycle (n,k)-gon is first order flexible, but is not among Tabachnikov's examples of deformable regular bicycle (n,k)-gons, then this bicycle polygon is second order flexible as well; however, it is third order rigid.

1. Introduction

Stanislav Ulam asked whether spheres are the only solids of uniform density which will float in water in equilibrium in any position (problem 19 in the "The Scottish Book" [6]). The question, known as the floating body problem makes sense in any dimension $d \geq 2$. Assuming that the density of the water is 1 and the density of the body is a given number $0 < \rho < 1$, the question is equivalent to the following geometric problem: Find all d-dimensional bodies of volume V such that for every hyperplane which cuts off a piece of volume ρV from the body, the segment connecting the centers of gravity of the body and the cut off portion is perpendicular to the hyperplane.

In the paper [7], Sergei Tabachnikov studied diverse questions related to the problem of how to determine the direction of a bicycle motion from the tire tracks of the wheels. He calls a closed smooth curve Γ a *bicycle curve*, if there is another closed curve γ, possibly with cusp singularities, such that the information that Γ and γ are the tracks of the front and rear wheels of a bicycle, respectively, is not enough to determine which way the bicycle went. A bicycle curve of length l and *perimetral density* $0 < \alpha < 1$ can be characterized by the property that the Euclidean distance between the endpoints of the arcs of length αl of the curve is constant. Bicycle curves

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are closely related to the two-dimensional floating body problem. Namely, a closed *convex* curve is a bicycle curve if and only if its convex hull D is a solution of the floatation problem with a certain density.

The floating body problem is still open for most densities ρ, but there are some known results in the plane. In 1938, H. Auerbach [1] constructed for density $\rho = 1/2$ a non-circular convex disc in the plane which can float in equilibrium in any position.

Bicycle curves with rational perimetral density are called *Zindler curves.* For certain perimetral densities, existence and non-existence theorems of non-circular Zindler curves were obtained by J. Bracho, L. Montejano and D. Oliveros in [2] (see also the references given there).

It is a natural idea to look for non-circular solutions of the 2-dimensional floatation problem among deformations of the circle. In [8], F. Wegner considered the family of those solutions of the floatation problem which have a p-fold rotational symmetry and studied whether the circle can be deformed within this class. He gave explicit formulas in terms of elliptic integrals for a one-parameter deformation of the circle within the family of bicycle curves with p-fold rotational symmetry and proved the surprising result that the curves he found solve the floatation problem for $p-2$ different densities simultaneously.

The present paper deals with a discrete version of the bicycle curve problem proposed by S. Tabachnikov [7]. A bicycle (n, k)-gon is an equilateral n-gon whose k-diagonals are equal. (A k-diagonal is a diagonal connecting a vertex to its k^{th} neighbor.) An equilateral n-gon is obviously a bicycle $(n, 1)$-gon and a bicycle (n, k)-gon is a bicycle $(n, n-k)$-gon as well, therefore, we consider only the case $2 \le k \le n/2$. We focus on the higher order infinitesimal flexibility of the regular n-gons within the family of bicycle (n, k)-gons. Concerning this question, S. Tabachnikov proved that the regular (n, k)-gon is flexible if (n, k) is in the set $\mathfrak{A} = \{(n, k) \mid (n \text{ is even and } k \text{ is odd}) \text{ or } n = 2k\}$. He also proved the following theorem.

Theorem 1. *The regular n-gon is first-order rigid as a bicycle (n, k)-gon if and only if the equation*

$$\frac{\sin(k(r+1)\pi/n)}{\sin((r+1)\pi/n)} = \frac{\sin(k(r-1)\pi/n)}{\sin((r-1)\pi/n)} \tag{1}$$

has no integer solution in r belonging to the range $2 \le r \le n/2$.

Integer solutions of equation (1) were described by R. Connelly and the author in [3]. It turned out that if the regular (n, k)-gon is first order flexible, then either $(n, k) \in \mathfrak{A}$ or

$$(n, k) \in \mathfrak{B} := \{(n, k) \mid n \text{ and } k \text{ are even and } n \text{ divides } (k+1)(n/2-k+1)\}.$$

For $(n, k) \in \mathfrak{B}$, the unique solution $2 \le r \le n/2$ of (1) is $r = n/2 - k$. Since first order rigidity implies rigidity and infinitesimal rigidity of any order, the above results give complete answer to the rigidity question of regular bicycle (n,k)-gons whenever $(n,k) \notin \mathfrak{B}$. Our main contribution to these results is the proof of the fact that if $(n,k) \in \mathfrak{B}$, then the regular n-gon is second order flexible but third order rigid in the class of bicycle (n,k)-gons.

2. Preliminaries

We shall use the notation $\phi = \pi/n$, $\eta = e^{i\phi}$, $\xi = \eta^2 = e^{2i\phi}$ throughout this paper.

$\mathbb{R}^2$ will be identified with the complex plane $\mathbb{C}$ and we denote by $\langle x, y\rangle = \mathrm{Re}(x\bar{y})$ the standard dot product on $\mathbb{R}^2$. An n-tuple of points or vectors in the plane is just an element of $\mathbb{C}^n$. It will be convenient to identify $\mathbb{C}^n$ with the group algebra $\mathbb{C}[\mathbb{Z}_n]$ of the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. This means that we shall think of a vector $(z_1, \dots, z_n) \in \mathbb{C}^n$ both as a complex valued function on $\mathbb{Z}_n$ and as an n-periodic complex valued function on $\mathbb{Z}$. In particular, we define z_j for integer values of j by the periodicity rule $z_{j+n} = z_j$, $\forall j \in \mathbb{Z}$. There are two multiplicative operations on the group algebra $\mathbb{C}[\mathbb{Z}_n]$. The product $\mathbf{a} \cdot \mathbf{b}$ of two vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ is the pointwise product $\mathbf{a} \cdot \mathbf{b} = (a_1 b_1, \dots, a_n b_n)$. The convolution $\mathbf{a} * \mathbf{b} = (c_1, \dots, c_n)$ of them is defined by $c_j = \sum_{s=1}^n a_s b_{j-s}$. The two operations are related to one another by the discrete Fourier transform $\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$. Recall that the j^{th} coordinate of $\mathcal{F}(\mathbf{a})$ is $\mathcal{F}(\mathbf{a})_j = \sum_{s=1}^n a_s \xi^{-sj}$. According to the convolution theorem, we have $\mathcal{F}(\mathbf{a} * \mathbf{b}) = \mathcal{F}(\mathbf{a}) \cdot \mathcal{F}(\mathbf{b})$. $\mathcal{F}$ is an invertible linear transformation. It transforms the standard basis $\delta^{*1}, \dots, \delta^{*n}$ of $\mathbb{C}^n$ (δ^{*j} is the j^{th} row of the $n \times n$ unit matrix) into the basis $\chi^{-1}, \dots, \chi^{-n}$ consisting of the irreducible characters $\chi^s = (\xi^s, \xi^{2s}, \dots, \xi^{sn})$ of $\mathbb{Z}_n$. We also have $\mathcal{F}(\chi^s) = n\delta^{*s}$.

Define the *support of the discrete Fourier transform* of a function $\mathbf{a} \in \mathbb{C}[\mathbb{Z}_n]$ as the set:

$$\operatorname{supp} \mathcal{F}(\mathbf{a}) = \{ j \in \mathbb{Z}_n \mid \mathcal{F}(\mathbf{a})_j \neq 0 \}.$$

We have the following simple relations for linear combinations, products and convolutions of functions:

$$\begin{aligned} &\operatorname{supp} \mathcal{F}(\alpha\mathbf{a} + \beta\mathbf{b}) \subset \operatorname{supp} \mathcal{F}(\mathbf{a}) \cup \operatorname{supp} \mathcal{F}(\mathbf{b}), \\ &\operatorname{supp} \mathcal{F}(\mathbf{a} \cdot \mathbf{b}) \subset \operatorname{supp} \mathcal{F}(\mathbf{a}) + \operatorname{supp} \mathcal{F}(\mathbf{b}), \\ &\operatorname{supp} \mathcal{F}(\mathbf{a} * \mathbf{b}) = \operatorname{supp} \mathcal{F}(\mathbf{a}) \cap \operatorname{supp} \mathcal{F}(\mathbf{b}). \end{aligned} \tag{2}$$

In particular, the regular representation of $\mathbb{Z}_n$ on the group algebra $\mathbb{C}[\mathbb{Z}_n]$ preserves the support of the discrete Fourier transform, i.e.,

$$\operatorname{supp} \mathcal{F}(\mathbf{a} * \delta^{*k}) = \operatorname{supp} \mathcal{F}(\mathbf{a}) \quad \text{for all } k. \tag{3}$$

It will be used frequently that if $a_j = \cos(2js\phi + \phi_0)$, then

$$\operatorname{supp} \mathcal{F}(a_1, \dots, a_n) = \{\pm s\}.$$

The points $x_i = \xi^i$, $(i = 1, \dots, n)$ are the vertices of a regular n-gon. Regular n-gons are bicycle (n,k)-gons for any $2 \le k \le n/2$. When x is a vector valued function on an interval, the q^{th} derivative of the squared length $d^2 = \langle x, x\rangle$ is expressed as

$$(d^2)^{(q)} = \sum_{s=0}^{q} \binom{q}{s} \langle x^{(s)}, x^{(q-s)}\rangle.$$

In view of this, a q^{th}-order variation of the regular n-gon within the family of bicycle (n,k)-gons is given by $(q+1)$ collections of vectors $(x_1^{(s)}, \dots, x_n^{(s)}) \in \mathbb{C}^n$, $(s = 0, \dots, q)$ satisfying $x_i^{(0)} = x_i$ for all $1 \le i \le n$,

$$\sum_{s=0}^{p} \binom{p}{s} \langle x_{j+1}^{(s)} - x_j^{(s)}, x_{j+1}^{(p-s)} - x_j^{(p-s)}\rangle = 0 \quad \text{for all } 1 \le p \le q, \tag{4}$$

and

$$\sum_{s=0}^{p} \binom{p}{s} \langle x_{j+k}^{(s)} - x_j^{(s)}, x_{j+k}^{(p-s)} - x_j^{(p-s)}\rangle = c^{(p)} \quad \text{for all } 1 \le p \le q, \tag{5}$$

where the numbers $c^{(p)} \in \mathbb{R}$ are some constants.

Two q^{th}-order variations $x_s^{(p)}$ and $\hat{x}_s^{(p)}$ are called *equivalent up to congruence* if there is a smooth one-parameter family of isometries Φ_t, such that

$$\hat{x}_s^{(p)} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^p \left[\Phi_t\left(\sum_{j=0}^{q} x_s^{(j)} \frac{t^j}{j!}\right)\right]\Bigg|_{t=0} \qquad \text{for all } 0 \le p \le q.$$

A q^{th}-order variation is *trivial* if it is equivalent up to congruence to the q^{th}-order variation $x_s^{(p)} \equiv 0$ for $1 \le q \le r$ and $1 \le s \le n$. If a non-trivial first order variation $x_s^{(1)}$ exists, then there is a non-trivial q^{th}-order variation as well, it is given by $\hat{x}_s^{(p)} = 0$ for $1 \le p < q$ and $\hat{x}_s^{(q)} = x_s^{(1)}$, $1 \le s \le n$. This motivates the following definition of higher order rigidity and flexibility.

A system is *q^{th}-order rigid* if for any q^{th}-order variation of the system, the *first order variation* obtained by forgetting the higher order terms *is trivial.* A system is *q^{th}-order flexible* if it is not q^{th}-order rigid (see [4]).

There are several possibilities to factor out the action of congruences. We could, for example, pin down two consecutive vertices, say x_1 and x_2, and consider only q^{th}-order variations for which $x_1^{(p)} = x_2^{(p)} = 0$ for $1 \le p \le q$. Another possibility, which preserves the logical symmetry of the vertices, is to consider q^{th}-order variations which leave the mass center of the set of vertices at the origin and keep the angular momentum of the set of the vertices with respect to the origin to be equal to 0. Thus we may restrict our attention to q^{th}-order variations for which

$$\sum_{s=1}^{n} x_s^{(p)} = 0 \quad \text{for } 1 \le p \le q, \tag{6}$$

and

$$\sum_{s=1}^{n}\sum_{j=0}^{p}\binom{p}{j}\Omega(x_s^{(j)},x_s^{(p-j+1)})=0 \quad \text{for } 0\le p\le q, \tag{7}$$

where Ω is the standard symplectic form on $\mathbb{C}$, which assigns to $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ the signed area of the parallelogram spanned by them:

$$\Omega(z_1,z_2)=\det\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}=\operatorname{Im}(\bar z_1\cdot z_2).$$

3. Recursive formulae for higher order variations

Denote by $\Delta x_s^{(p)}$ the difference $\Delta x_s^{(p)}=x_{s+1}^{(p)}-x_s^{(p)}$. These differences must satisfy

$$\sum_{j=1}^{n}\Delta x_j^{(p)}=0. \tag{8}$$

In view of equation (6), the numbers $x_s^{(p)}$ can be expressed with the help of these differences as

$$x_s^{(p)}=\sum_{j=s}^{n+s-1}\frac{j+1}{n}\Delta x_j^{(p)}.$$

Thus, the system of equations (4), (5) and (7) can be rewritten in terms of the differences $\Delta x_s^{(p)}$.

The unit complex numbers $w_s=\eta^{2j+1}$ and iw_s form an orthonormal basis of $\mathbb{C}$ over $\mathbb{R}$. Express $\Delta x_s^{(p)}$ as a linear combination of this basis

$$\Delta x_s^{(p)}=\lambda_s^{(p)}w_s+\mu_s^{(p)}iw_s,\quad \lambda_s^{(p)},\mu_s^{(p)}\in\mathbb{R}.$$

As $\Delta x_s^{(0)}=\eta^{2s+2}-\eta^{2s}=2\sin(\phi)iw_s$, we have

$$\lambda_s^{(0)}=0 \text{ and } \mu_s^{(0)}=2\sin(\phi) \text{ for } 1\le s\le n. \tag{9}$$

Suppose that $1\le p\le q$ and we have already computed $\lambda_s^{(j)}$ and $\mu_s^{(j)}$ for $1\le s\le n$ and $0\le j<p$. Let us try to compute $\lambda_s^{(p)}$ and $\mu_s^{(p)}$.

Equation (4) is equivalent to the equation

$$\sum_{s=0}^{p}\binom{p}{s}(\lambda_j^{(s)}\lambda_j^{(p-s)}+\mu_j^{(s)}\mu_j^{(p-s)})=0, \tag{10}$$

which yields together with (9) a recursive formula for $\mu_j^{(p)}$

$$\mu_j^{(p)}=-\frac{1}{4\sin(\phi)}\sum_{s=1}^{p-1}\binom{p}{s}(\lambda_j^{(s)}\lambda_j^{(p-s)}+\mu_j^{(s)}\mu_j^{(p-s)}). \tag{11}$$

The left-hand side of (5) is equivalent to

(12)
$$\sum_{s=0}^{p}\binom{p}{s}\left\langle\sum_{l=0}^{k-1}(\lambda_{j+l}^{(s)}w_{j+l}+\mu_{j+l}^{(s)}iw_{j+l}),\sum_{l=0}^{k-1}(\lambda_{j+l}^{(p-s)}w_{j+l}+\mu_{j+l}^{(p-s)}iw_{j+l})\right\rangle.$$

Since

$$\langle w_j,w_l\rangle=\langle iw_j,iw_l\rangle=\cos(2(j-l)\phi)$$

and

$$\langle iw_j,w_l\rangle=-\langle w_j,iw_l\rangle=\sin(2(l-j)\phi),$$

equation (12) takes the form

$$\begin{aligned}\sum_{l_1,l_2=0}^{k-1}\sum_{s=0}^{p}\binom{p}{s}\Big[&(\lambda_{j+l_1}^{(s)}\lambda_{j+l_2}^{(p-s)}+\mu_{j+l_1}^{(s)}\mu_{j+l_2}^{(p-s)})\cos(2(l_1-l_2)\phi)\\&+(\lambda_{j+l_1}^{(s)}\mu_{j+l_2}^{(p-s)}-\mu_{j+l_1}^{(s)}\lambda_{j+l_2}^{(p-s)})\sin(2(l_1-l_2)\phi)\Big]=c^{(p)}.\end{aligned}\tag{13}$$

Rearranging (13) we obtain the following system of linear equations for the $\lambda_j^{(p)}$'s

$$\sum_{l_1,l_2=0}^{k-1}4\sin(\phi)\lambda_{j+l_1}^{(p)}\sin(2(l_1-l_2)\phi)=c^{(p)}-R_j^{(p)},\tag{14}$$

where

$$\begin{aligned}R_j^{(p)}=\sum_{l_1,l_2=0}^{k-1}\Big[&4\sin(\phi)\mu_{j+l_1}^{(p)}\cos(2(l_1-l_2)\phi)\\&+\sum_{s=1}^{p-1}\binom{p}{s}\big[(\lambda_{j+l_1}^{(s)}\lambda_{j+l_2}^{(p-s)}+\mu_{j+l_1}^{(s)}\mu_{j+l_2}^{(p-s)})\cos(2(l_1-l_2)\phi)\\&+(\lambda_{j+l_1}^{(s)}\mu_{j+l_2}^{(p-s)}-\mu_{j+l_1}^{(s)}\lambda_{j+l_2}^{(p-s)})\sin(2(l_1-l_2)\phi)\big]\Big].\end{aligned}\tag{15}$$

Summation with respect to l_2 on the left hand side of (14) can be brought to a closed form, which yields

$$\sum_{l=0}^{k-1}\lambda_{j+l}^{(p)}4\sin(k\phi)\sin((2l-k+1)\phi)=c^{(p)}-R_j^{(p)}.\tag{16}$$

Introducing the vectors $\Lambda^{(p)}=(\lambda_1^{(p)},\dots,\lambda_n^{(p)})$, $\mathbf{R}^{(p)}=(R_1^{(p)},\dots,R_n^{(p)})$ and $\mathbf{b}=(b_1,\dots,b_n)$, where

$$b_s=\begin{cases}0 & \text{if } 1\le s\le n-k,\\ 4\sin(k\phi)\sin((1-k-2s)\phi) & \text{if } n-k+1\le s\le n,\end{cases}$$

the system of equations (16) for $j = 1, \dots, n$ can be written in a compact form

$$\Lambda^{(p)} * \mathbf{b} = c^{(p)}\chi^n - \mathbf{R}^{(p)}. \tag{17}$$

Applying the discrete Fourier transform to equation (17) we get

$$\mathcal{F}(\Lambda^{(p)}) \cdot \mathcal{F}(\mathbf{b}) = nc^{(p)}\delta^{*n} - \mathcal{F}(\mathbf{R}^{(p)}). \tag{18}$$

When the j^{th} coordinate $\mathcal{F}(\mathbf{b})_j$ of $\mathcal{F}(\mathbf{b})$ is not zero, the j^{th} coordinate of $\mathcal{F}(\Lambda^{(p)})$ is uniquely given by

$$\mathcal{F}(\Lambda^{(p)})_j = \mathcal{F}(c^{(p)}\chi^n - \mathbf{R}^{(p)})_j / \mathcal{F}(\mathbf{b})_j. \tag{19}$$

However, whenever $\mathcal{F}(\mathbf{b})_j = 0$, $\mathcal{F}(c^{(p)}\chi^n - \mathbf{R}^{(p)})_j$ must be 0 as well, otherwise equation (18) has no solutions.

To write our formulas in a concise form, we introduce the notation

$$\sigma_j = \sum_{s=0}^{k-1} \eta^{(k-1-2s)j} = \begin{cases} k & \text{if } j = 0, \\ \dfrac{\sin(kj\phi)}{\sin(j\phi)} & \text{if } 1 \le j \le n-1. \end{cases}$$

It is important to note that we have $\sigma_{n+j} = (-1)^{k-1}\sigma_j$, thus, unlike most of the sequences appearing in this paper, the sequence (σ_j) may not be periodic in n.

The discrete Fourier transform of $\mathbf{b}$ can be computed explicitly:

$$\mathcal{F}(\mathbf{b}) = \sum_{j=1}^{n} -2i\sin(k\phi)(\sigma_{j+1} - \sigma_{j-1})\eta^{j(k-1)}\delta^{*j}. \tag{20}$$

This formula shows that $\mathcal{F}(\mathbf{b})_n = 0$. Consequently, (18) can have a solution only if

$$c^{(p)} = \frac{1}{n}\mathcal{F}(\mathbf{R}^{(p)})_n = \frac{1}{n}\sum_{j=1}^{n} R_j^{(p)}. \tag{21}$$

The coefficients $\mp 2i\sin(k\phi)(\sigma_2 - k)\eta^{\pm(k-1)}$ of δ^{*1} and $\delta^{*(n-1)}$ in (20) are never equal to 0 since σ_2 is the sum of k complex numbers of unit length, consequently $|\sigma_2| < k$.

4. First order flexibility

When we apply the above general method to compute first order variations of the regular n-gon in the family of bicycle (n,k)-gons, equations (11), (15) and (21) give $\mu_j^{(1)} \equiv 0$ for all $1 \le j \le n$, $\mathbf{R}^{(1)} = \mathbf{0}$ and $c^{(1)} = 0$.

Thus, (18) reduces to

$$\mathcal{F}(\Lambda^{(1)}) \cdot \mathcal{F}(\mathbf{b}) = 0. \tag{22}$$

If $\mathcal{F}(\mathbf{b})_n$ is the only vanishing coefficient of $\mathcal{F}(\mathbf{b})$, then the general solution of (22) is

$$\Lambda^{(1)} = A\chi^n, \quad (A \in \mathbb{R}).$$

These are exactly those solutions which correspond to the trivial first order variations. Consequently, a non-trivial first order variation exists if and only if $\mathcal{F}(\mathbf{b})$ has at least two vanishing coefficients. In view of the explicit form of $\mathcal{F}(\mathbf{b})$ presented in (20), this implies Theorem 1. This proof of Theorem 1 is essentially the same proof that was given by Tabachnikov in [7].

Suppose from now on that $(n,k) \in \mathfrak{B}$. Then $r = n/2 - k$ is the unique integer solution of (1) belonging to the interval $[2, n/2]$, and $\Lambda^{(1)}$ solves (22) if and only if it has the form

$$\Lambda^{(1)} = \mathcal{F}^{-1}(A\delta^{*n} + B\delta^{*r} + C\delta^{*(n-r)}) = \frac{1}{n}(A\chi^n + B\chi^r + C\chi^{(n-r)}), \tag{23}$$

where $A, B, C \in \mathbb{C}$. Since $\Lambda^{(1)} \in \mathbb{R}^n$, A must be real and $B = \overline{C}$. If $B = |B|e^{i\phi_0}$, then the speed vectors v_j have the following explicit form

$$x_j^{(1)} = \left[i\omega x_j + (x_0^{(1)} - i\omega)\right] + \frac{2|B|}{n}\left(\sum_{s=0}^{j-1}\cos(2sr\phi + \phi_0)w_s\right). \tag{24}$$

where $\omega = -A/(2n\sin(\phi))$. We can reduce the number of parameters by the additional conditions on the invariance of the mass center and the angular momentum. In order to satisfy condition (6) on the invariance of the mass center, $x_0^{(1)}$ must be equal to

$$x_0^{(1)} = i\omega + \frac{|B|}{n}\left(\frac{e^{i\phi_0}}{\xi^{r+1} - 1} + \frac{e^{-i\phi_0}}{\xi^{1-r} - 1}\right)\eta \tag{25}$$

and then we also have

$$x_j^{(1)} = i\omega\xi^j + \frac{|B|}{n}\left(\frac{e^{i\phi_0}\xi^{j(r+1)}}{\xi^{r+1} - 1} + \frac{e^{-i\phi_0}\xi^{j(1-r)}}{\xi^{1-r} - 1}\right)\eta. \tag{26}$$

The angular momentum condition (7) for $q = 1$ gives

$$\sum_{j=1}^{n}\Omega(x_j^{(0)}, x_j^{(1)}) = n\omega = 0.$$

The variation is trivial if and only if $B = 0$. Since we are interested in non-trivial variations, we shall assume $B \neq 0$.

We call two q^{th}-order variations $x_s^{(p)}$ and $\hat{x}_s^{(p)}$ *equivalent up to regular reparameterization* if there is a smooth map $h : (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined on a neighborhood of 0, such that $h(0) = 0$, $h'(0) \neq 0$ and

$$\hat{x}_s^{(p)} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^p \left[\left(\sum_{j=0}^{q} x_s^{(j)}\frac{h(t)^j}{j!}\right)\right]\Bigg|_{t=0} \qquad \text{for all } 0 \leq p \leq q.$$

If $B \neq 0$, then a regular reparameterization of the variation can rescale the multiplier $|B|/n$ in the formula for $x_j^{(1)}$ to any non-zero number, so we may assume without loss of generality that

$$x_j^{(1)} = \left(\frac{e^{i\phi_0}\xi^{j(r+1)}}{\xi^{r+1}-1} + \frac{e^{-i\phi_0}\xi^{j(1-r)}}{\xi^{1-r}-1}\right)\frac{\eta}{2} \text{ and } \Delta x_j^{(1)} = \cos(2jr\phi+\phi_0)w_j. \tag{27}$$

5. Second order flexibility

Theorem 2. *If $(n,k)\in\mathfrak{B}$, the non-trivial first order variation* (27) *can be extended to a second order variation.*

Proof. Denote by $\mathbf{M}^{(p)}$ the vector $(\mu_1^{(p)},\dots,\mu_n^{(p)})$. Using the recursive formulae of section 3 and the relations in (2), we obtain the following relations for the support of the discrete Fourier transform of $\Lambda^{(p)}$, $\mathbf{M}^{(p)}$ and $\mathbf{R}^{(p)}$.

$$\begin{aligned} &\operatorname{supp}\mathcal{F}(\Lambda^{(0)}) = \emptyset, && \operatorname{supp}\mathcal{F}(\mathbf{M}^{(0)}) = \{0\},\\ &\operatorname{supp}\mathcal{F}(\Lambda^{(1)}) = \{\pm r\}, && \operatorname{supp}\mathcal{F}(\mathbf{M}^{(1)}) = \emptyset, \end{aligned}$$

$$\begin{aligned} \operatorname{supp}\mathcal{F}(\mathbf{M}^{(2)}) &\subset \operatorname{supp}\mathcal{F}(\Lambda^{(1)}) + \operatorname{supp}\mathcal{F}(\Lambda^{(1)}) = \{0,\pm 2r\},\\ \operatorname{supp}\mathcal{F}(\mathbf{R}^{(2)}) &\subset \operatorname{supp}\mathcal{F}(\mathbf{M}^{(2)}) \cup \big(\operatorname{supp}\mathcal{F}(\Lambda^{(1)}) + \operatorname{supp}\mathcal{F}(\Lambda^{(1)})\big)\\ &= \{0,\pm 2r\}. \end{aligned}$$

As $\{\pm r\}$ is not in the support of $\mathcal{F}(\mathbf{R}^{(2)})$, equation (18) can be solved for $\Lambda^{(2)}$. Any solution $\Lambda^{(2)}$ yields a second order extension of the non-trivial first order variation (27). To see this we have to check that condition (8) is fulfilled for $p=2$. Indeed, $(n-1)\notin\operatorname{supp}\mathcal{F}(\Lambda^{(2)})$ and $(n-1)\notin\operatorname{supp}\mathcal{F}(\mathbf{M}^{(2)})$ imply

$$\sum_{j=1}^{n}\Delta x_j^{(2)} = \sum_{j=1}^{n}(\lambda_j^{(2)}w_j + \mu_j^{(2)}iw_j) = \big(\mathcal{F}(\Lambda^{(2)})_{n-1} + i\mathcal{F}(\mathbf{M}^{(2)})_{n-1}\big)\eta = 0.$$

□

Although the above proof avoids the explicit computation of the second order variation, we shall need the explicit formulas for the proof of third order rigidity, so we work out the details.

Substituting into (11) yields

$$\mu_j^{(2)} = -\frac{\cos^2(2jr\phi+\phi_0)}{2\sin(\phi)}.$$

The components of $\mathbf{R}^{(2)}$ are computed by (15) as follows

$$\begin{aligned} R_j^{(2)} = 2\sum_{l_1,l_2=0}^{k-1}\cos(2(l_1-l_2)\phi)\big[&-\cos^2(2(j+l_1)r\phi+\phi_0)\\ &+\cos(2(j+l_1)r\phi+\phi_0)\cos(2(j+l_2)r\phi+\phi_0)\big]. \end{aligned} \tag{28}$$

All the products of three cosines on the right hand side can be transformed into the sum of four cosines using the identity

$$\cos(\alpha)\cos(\beta)\cos(\gamma) = \frac{1}{4}(\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma) \\ +\cos(\alpha-\beta+\gamma)+\cos(-\alpha+\beta+\gamma)).$$

This transformation leads to

$$\begin{aligned} R_j^{(2)} = \frac{1}{2}\sum_{l_1,l_2=0}^{k-1} \big[&-\cos((4r+2)\phi l_1 - 2\phi l_2 + (4jr\phi+2\phi_0)) \\ &-\cos((4r-2)\phi l_1 + 2\phi l_2 + (4jr\phi+2\phi_0)) \\ &-2\cos(2\phi l_1 - 2\phi l_2) \\ &+\cos((2r+2)\phi l_1 + (2r-2)\phi l_2 + (4jr\phi+2\phi_0)) \\ &+\cos((2r-2)\phi l_1 + (2r+2)\phi l_2 + (4jr\phi+2\phi_0)) \\ &+\cos((2r+2)\phi l_1 - (2r+2)\phi l_2) \\ &+\cos((2-2r)\phi l_1 + (2r-2)\phi l_2)\big]. \end{aligned} \tag{29}$$

All the cosines on the right hand side have the form $\cos(\alpha l_1+\beta l_2+\gamma)$. Thus, with the help of the identity

$$\sum_{l_1,l_2=0}^{k-1}\cos(\alpha l_1+\beta l_2+\gamma) = \frac{\sin(k\alpha/2)}{\sin(\alpha/2)}\frac{\sin(k\beta/2)}{\sin(\beta/2)}\cos\left(\gamma+(k-1)\frac{\alpha+\beta}{2}\right), \tag{30}$$

$R_j^{(2)}$ can be brought to the closed form

$$R_j^{(2)} = \left[1-\sigma_1^2\right] + \left[1-\left(\frac{\sigma_{2r+1}+\sigma_{2r-1}}{2}\right)\sigma_1\right]\cos((4j+2k-2)r\phi+2\phi_0).$$

By (21), we have

$$c^{(2)} = 1-\sigma_1^2.$$

Remark. Since $c^{(2)} = 1-\sigma_1^2 < 0$, our computation yields that for any infinitesimal variation of the regular bicycle (n,k)-gon with fixed side lengths, the first derivative of the squared length of the k-diagonals must be 0, and its second derivative must be negative. This result is in accordance with G. Lükő's theorem (Theorem II in [5]) claiming that the arithmetical mean of the lengths of the k-diagonals of an n-gon with unit length of sides is maximized by the regular n-gon.

The discrete Fourier transform of $c^{(2)}\chi^n - \mathbf{R}^{(2)}$ has only one pair of non-zero coefficients at the $\pm(2r)^{\text{th}}$ place and these coefficients are equal to

$$\mathcal{F}(c^{(2)}\chi^n - \mathbf{R}^{(2)})_{\pm 2r} = -\frac{n}{2}\left[1-\frac{(\sigma_{2r+1}+\sigma_{2r-1})\sigma_1}{2}\right]e^{\pm((2k-2)r\phi+2\phi_0)i}.$$

As $\mathcal{F}(\mathbf{b})$ has three vanishing coefficients, $\mathcal{F}(\mathbf{b})_0$ and $\mathcal{F}(\mathbf{b})_{\pm r}$, $\Lambda^{(2)}$ satisfies (18) if and only if it has the form

$$\lambda_j^{(2)} = A^{(2)} + B^{(2)}\cos(2rj + \phi_1) + C^{(2)}\cos(4rj + 2\phi_0 - \pi/2), \tag{31}$$

where $A^{(2)}$, $B^{(2)}$ and ϕ_1 are arbitrary real numbers,

$$C^{(2)} = \frac{2 - (\sigma_{2r+1} + \sigma_{2r-1})\sigma_1}{4\sin(k\phi)(\sigma_{2r+1} - \sigma_{2r-1})}. \tag{32}$$

6. Third order rigidity

Our goal in this section is to prove the following theorem

Theorem 3. *For any $(n,k) \in \mathfrak{B}$, the regular n-gon is third order rigid in the family of bicycle (n,k)-gons.*

Proof. We shall prove the theorem by showing that (18) for $p = 3$ cannot be solved due to the fact that $\mathcal{F}(\mathbf{b})_{\pm r} = 0$ but $\mathcal{F}(\mathbf{R}^{(3)})_{\pm r} \neq 0$. An explicit formula for $\mathbf{R}^{(3)}$ seems to be large, however, as we are interested only in the $\pm r^{\text{th}}$ coefficient of its discrete Fourier transform, we are allowed to ignore those terms that have no contribution to this Fourier coefficient. When two expressions are equal modulo terms having no impact on the $\pm r^{\text{th}}$ coefficient of $\mathcal{F}(\mathbf{R}^{(3)})$, we write the $\equiv$ sign between them.

We have the following expression for $\mu_j^{(3)}$

$$\begin{aligned}
\mu_j^{(3)} &= -\frac{3\lambda_j^{(1)}\lambda_j^{(2)}}{2\sin(\phi)} \\
&= -\frac{3\cos(2jr\phi + \phi_0)\begin{pmatrix} A^{(2)} + B^{(2)}\cos(2jr\phi + \phi_1) \\ \qquad + C^{(2)}\cos(4jr\phi + 2\phi_0 - \frac{\pi}{2})\end{pmatrix}}{2\sin(\phi)} \\
&\equiv -\frac{3(A^{(2)}\cos(2jr\phi + \phi_0) + \frac{C^{(2)}}{2}\cos(2jr\phi + \phi_0 - \frac{\pi}{2}))}{2\sin(\phi)}.
\end{aligned}$$

When $p = 3$ the recursive formula for $\mathbf{R}^{(p)}$ takes the form

$$\begin{aligned}
R_j^{(3)} = \sum_{l_1,l_2=0}^{k-1} \Big[&(4\sin(\phi)\mu_{j+l_1}^{(3)} + 6\lambda_{j+l_1}^{(1)}\lambda_{j+l_2}^{(2)})\cos(2(l_1 - l_2)\phi) \\
&+ 6\lambda_{j+l_1}^{(1)}\mu_{j+l_2}^{(2)}\sin(2(l_1 - l_2)\phi)\Big].
\end{aligned} \tag{33}$$

Expanding the second and third terms we obtain

$$\begin{aligned}
\lambda_{j+l_1}^{(1)}\lambda_{j+l_2}^{(2)} \equiv{}& A^{(2)}\cos(2(j + l_1)r\phi + \phi_0) \\
&+ \frac{C^{(2)}}{2}\cos((2j + 4l_2 - 2l_1)r + \phi_0 - \pi/2)
\end{aligned} \tag{34}$$

and

$$\begin{aligned}\lambda^{(1)}_{j+l_1}\mu^{(2)}_{j+l_2} \equiv &-\frac{\cos(2(j+l_1)r\phi+\phi_0)}{4\sin(\phi)}\\ &-\frac{\cos((2j+4l_2-2l_1)r\phi+\phi_0)}{8\sin(\phi)}.\end{aligned} \tag{35}$$

Substituting (34) and (35) back into (33) we get

(36)

$$\begin{aligned}R^{(3)}_j \equiv \sum_{l_1,l_2=0}^{k-1} \Big[&3C^{(2)}\cos((2j+4l_2-2l_1)r\phi+\phi_0-\pi/2)\cos(2(l_1-l_2)\phi)\\ &-3C^{(2)}\cos(2(j+l_1)r\phi+\phi_0-\pi/2)\cos(2(l_1-l_2)\phi)\\ &-\frac{3\cos(2(j+l_1)r\phi+\phi_0)}{2\sin(\phi)}\sin(2(l_1-l_2)\phi)\\ &-\frac{3\cos((2j+4l_2-2l_1)r\phi+\phi_0)}{4\sin(\phi)}\sin(2(l_1-l_2)\phi)\Big].\end{aligned}$$

Products of sines and cosines can be transformed into linear combinations of cosines using the identities

$$\cos x\cos y = \frac{\big(\cos(x-y)-\cos(x+y)\big)}{2}$$

and

$$\cos x\sin y = \frac{\big(\cos(x+y-\pi/2)-\cos(x-y+\pi/2)\big)}{2}.$$

The application of this transformation to (36) yields

(37)

$$\begin{aligned}R^{(3)}_j \equiv \sum_{l_1,l_2=0}^{k-1} \Big[&3\frac{C^{(2)}}{2}\cos((2-2r)\phi l_1+(4r-2)\phi l_2+(2jr+\phi_0-\tfrac{\pi}{2}))\\ &+3\frac{C^{(2)}}{2}\cos((-2-2r)\phi l_1+(4r+2)\phi l_2+(2jr+\phi_0-\tfrac{\pi}{2}))\\ &-3\frac{C^{(2)}}{2}\cos((2r+2)\phi l_1-2\phi l_2+(2jr+\phi_0-\tfrac{\pi}{2}))\\ &-3\frac{C^{(2)}}{2}\cos((2r-2)\phi l_1+2\phi l_2+(2jr+\phi_0-\tfrac{\pi}{2}))\\ &-\frac{3}{4\sin(\phi)}\cos((2r+2)\phi l_1-2\phi l_2+(2jr+\phi_0-\tfrac{\pi}{2}))\\ &+\frac{3}{4\sin(\phi)}\cos((2r-2)\phi l_1+2\phi l_2+(2jr+\phi_0-\tfrac{\pi}{2}))\\ &-\frac{3}{8\sin(\phi)}\cos((2-2r)\phi l_1+(4r-2)\phi l_2+(2jr+\phi_0-\tfrac{\pi}{2}))\\ &+\frac{3}{8\sin(\phi)}\cos((-2-2r)\phi l_1+(4r+2)\phi l_2+(2jr+\phi_0-\tfrac{\pi}{2}))\Big].\end{aligned}$$

Summation over l_1 and l_2 can be brought to a closed form using identity (30) and gives

(38)
$$\begin{aligned} R_j^{(3)} \equiv \Big[&3\frac{C^{(2)}}{2}(\sigma_{r-1}\sigma_{2r-1}+\sigma_{r+1}\sigma_{2r+1}-\sigma_{r+1}\sigma_1-\sigma_{r-1}\sigma_1) \\ &+\frac{3}{4\sin(\phi)}(\sigma_{r-1}-\sigma_{r+1})\sigma_1-\frac{3}{8\sin(\phi)}(\sigma_{r-1}\sigma_{2r-1}+\sigma_{r+1}\sigma_{2r+1})\Big] \\ &\times\cos((2j+k-1)r\phi+\phi_0-\frac{\pi}{2}). \end{aligned}$$

The conclusion of this computation is that $\mathcal{F}(\mathbf{R}^{(3)})_{\pm r}$ vanishes if and only if the coefficient of $\cos((2j+k-1)r\phi+\phi_0-\frac{\pi}{2})$ in (38) is 0. Since $\sigma_{r-1}=\sigma_{r+1}=\pm 1$, as $(n,k)\in\mathfrak{B}$, this condition reduces to the equation

$$3\frac{C^{(2)}}{2}(\sigma_{2r-1}+\sigma_{2r+1}-2\sigma_1)+\frac{3}{8\sin(\phi)}(\sigma_{2r+1}-\sigma_{2r-1})=0. \tag{39}$$

To prove that (39) never holds, we simplify its left hand side.

Lemma 1. *If $(n,k)\in\mathfrak{B}$, $r=n/2-k$ as usual, then*

$$C^{(2)}(\sigma_{2r-1}+\sigma_{2r+1}-2\sigma_1)+\frac{\sigma_{2r+1}-\sigma_{2r-1}}{4\sin(\phi)}=\frac{\sin((k+1)\phi)\sin(1-k)\phi}{\cos(k\phi)\sin(2\phi)}. \tag{40}$$

Proof. The identity of the Lemma was obtained and can be verified with the computer algebra software MuPAD. We describe the main steps of the computation which give this identity. These steps can be executed by other computer algebra softwares like Maple, Mathematica, etc. as well and should produce the same result. In principle it is also possible to go through this computation by hand, though it must be a tedious work.

Expressing $C^{(2)}$ with the help of (32) we obtain the following expression for the left hand side of (40)

$$\frac{2-(\sigma_{2r+1}+\sigma_{2r-1})\sigma_1}{4\sin(k\phi)(\sigma_{2r+1}-\sigma_{2r-1})}(\sigma_{2r-1}+\sigma_{2r+1}-2\sigma_1)+\frac{\sigma_{2r+1}-\sigma_{2r-1}}{4\sin(\phi)}. \tag{41}$$

Introduce the new parameters $x=k\phi$ and $y=\phi$. Then using the fact that $n=2k+2r$ and that n divides $(k+1)(r+1)$, σ_1 and $\sigma_{2r\pm1}$ can be expressed as follows

$$\sigma_1=\frac{\sin(x)}{\sin(y)},\quad \sigma_{2r+1}=\frac{\sin(2y-x)}{\sin(2x-y)},\quad \sigma_{2r-1}=\frac{\sin(x+2y)}{\sin(2x+y)}. \tag{42}$$

Substituting (42) into (41) we can write the left hand side of (40) as a rational function of trigonometric polynomials of x and y. With the `expand` function of MuPAD, this can be written as a rational function of the variables $\cos(x)$, $\sin(x)$, $\cos(y)$ and $\sin(y)$. If we apply the `factor` function to the result of the previous step, both the numerator and the denominator will be decomposed into the product of irreducible polynomials of $\cos(x)$, $\sin(x)$,

$\cos(y)$ and $\sin(y)$. Removing the common factors of the numerator and the denominator manually or using the `simplify` function, we obtain

$$\frac{\cos^2(y) - \sin^2(y) - \cos^2(x) + \sin^2(x)}{2\cos(x)\cos y \sin(y)},$$

which is exactly the right hand side of (40). $\square$

It is clear that the right hand side of (40) cannot be 0. Thus, equation (39) never holds, consequently, $\mathcal{F}(\mathbf{R}^{(3)})_{\pm r} \neq 0$, as we wanted to show. $\square$

7. Concluding remarks

The present paper determines the order of infinitesimal flexibility of any regular bicycle polygon. However, since there are examples of third order rigid linkages which are flexible (see [4]), the following question is still open.

Question . *Is the regular n-gon flexible in the family of bicycle (n,k)-gons for $(n,k) \in \mathfrak{B}$?*

References

[1] H. Auerbach, *Sur un problème de M. Ulam concernant l'équilibre des corps flottants*, Studia Math. **7** (1938), 121–142.

[2] J. Bracho, L. Montejano, and D. Oliveros, *Carousels, Zindler curves and the floating body problem*, Periodica Math. Hung. **49** (2004), no. 2, 9–23.

[3] R. Connelly and B. Csikós, *Classification of first order flexible regular bicycle polygons*, Studia Math. Hung., to appear.

[4] R. Connelly and H. Servatius, *Higher-order rigidity – What is the proper definition*, Discrete Comput. Geom. **11** (1994), 193–200.

[5] G. Lükő, *On the mean length of the chords of a closed curve*, Israel J. Math. **4** (1966), 23–32.

[6] R.D. Mauldin (ed.), *The Scottish Book: Mathematics from the Scottish Café*, Birkhäuser, Boston, 1981.

[7] S. Tabachnikov, *Tire track geometry: Variations on a theme*, Israel J. Math. **151** (2006), 1–28.

[8] F. Wegner, *Floating bodies of equilibrium, explicit solution*, v2 8 Sep 2006, arXiv:physics/0603160, pp. 1–25.

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