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N-FREE EXTENSIONS OF POSETS. NOTE ON A THEOREM OF P.A.GRILLET.

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ABSTRACT. Let $S_N(P)$ be the poset obtained by adding a dummy vertex on each diagonal edge of the N's of a finite poset P. We show that $S_N(S_N(P))$ is N-free. It follows that this poset is the smallest N-free barycentric subdivision of the diagram of P, poset whose existence was proved by P.A. Grillet. This is also the poset obtained by the algorithm starting with $P_0 := P$ and consisting at step m of adding a dummy vertex on a diagonal edge of some N in P_m, proving that the result of this algorithm does not depend upon the particular choice of the diagonal edge chosen at each step. These results are linked to drawing of posets.

1. INTRODUCTION

An N is a poset made of four vertices labeled a, b, c, d such that $a < c, b < c, b < d$, b incomparable to a, a incomparable to d and d incomparable to c (see Figure 1(a)). This simple poset plays an important role in the algorithmic of posets [3]. It can be contained in a poset P in essentially two ways, this fact leading to the characterization of two basic types of posets, the *series-parallel* posets and the *chain-antichain complete* (or C.A.C) posets.

The first way is related to the comparability graph of P. An N can be contained in P as an induced poset, that is P contains four vertices on which the comparabilities are those indicated above. Finite posets with no induced N are called *series-parallel*. Indeed, since their comparability graph contains no induced P_4 (a four vertices path) they can be obtained from the one element poset by direct and complete sums (a result which goes back to Sumner [5], see also [6]). The second way is related to the (oriented) diagram of P. This is the object of this note.

In order to describe this other way, let us recall that a *covering pair* in a poset P is a pair (x, y) such that $x < y$ and there is no $z \in P$ such that $x < z < y$. The (directed)*diagram* of P is the directed graph, denoted

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by $Diag(P)$, whose vertex set is P and edges are the covering pairs of P . If (x, y) is a covering pair, we say that x *is covered by* y, or y *covers* x, a fact that we denote $x \prec_P y$, or $x \prec y$ if there is no risk of confusion, or $(x, y) \in Diag(P)$. We denote by $Inc(P)$ the set of pairs (x, y) formed of incomparable elements.

Definition 1.1. *Let a, b, c, and d four elements of P, we say that these elements form:*

(1) an N in P if $b \prec c$, $a \prec c$, $b \prec d$, and $(a, d) \in Inc(P)$;
(2) an N' in P if $b \prec c$, $a < c$, $b < d$, and $(a, d) \in Inc(P)$;
(3) an N in $Diag(P)$ if $b \prec c$, $a \prec c$, $b \prec d$, and $a \not\prec d$;

An N in P is evidently an N' in P. Provided that P is finite, an N' in P yields an N in P (indeed, if $\{a, b, c, d\}$ forms an N' as in (2), then pick a', b' such that $a \leq a' \prec c$ and $b \prec d' \leq d$. Clearly, the set $\{a', b, c, d'\}$ is an N in P). An N in P induces an N in $Diag(P)$; the converse is false: if $\{a, b, c, d\}$ is an N in $Diag(P)$ as in (3) above, then $a < d$ is possible, but then -provided that P is finite- it contains an N, eg the 4 element subset a', a, c, b, where $a \prec a' < d$. Thus, if P is finite, it contains an N under one of these three forms if it contains all. We say that P is *N-free* if it contains no N. It was proved by P.A.Grillet [1] that *a finite poset P is N-free if and only if P is chain-antichain complete (or C.A.C) that is if every maximal chain of P meets every maximal antichain of P* (the formulation N-free in terms of the N defined in (1) is due to Leclerc and Monjardet [2]).

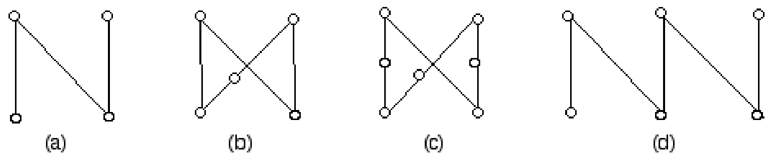

FIGURE 1. Examples of posets containing an N.

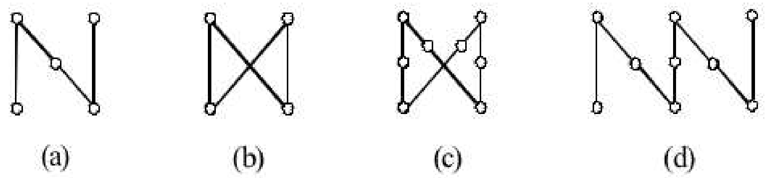

FIGURE 2. Examples of N-free posets

A *barycentric subdivision* of the diagram of a poset P consists to add finitely many vertices, possibly none, on each edge of the diagram of P.

These vertices added to those of P provides a new poset in which P is embedded. We denote by $S(P)$ the poset obtained by adding just one vertex on each edge of the diagram of P. As it is immediate to see, this poset is N-free. In his embedding theorem (Theorem 7 [1]) P.A.Grillet proves that among the N-free posets obtained as barycentric subdivisions of a finite poset P there is one, denoted $\overline{P}$, which is minimum. In this note, we provide a simple description of $\overline{P}$ and give some consequences.

If $A := \{a, b, c, d\}$ is an N in P as in (1) of Definition 1.1, we say that the pair (b, c) is the *diagonal edge* of this N. Let $N_{diag}(P)$ be the set of diagonal edges of all the N's in P and let $S_N(P)$ be the poset obtained by adding a dummy vertex on each edge in $N_{diag}(P)$.

Theorem 1.2. *Let P be a finite poset. Then $S_N(S_N(P))$ is N-free. In fact this is the smallest N-free poset $\overline{P}$ which comes from a barycentric subdivision of $Diag(P)$.*

This result translates to an algorithm which transforms a poset into an N-free poset: execute twice the algorithm consisting to add simultaneously a vertex on each N of a poset. Figure 3 shows an execution of this algorithm. Two dummy elements 6 and 7 are created during the first execution. Another two, 8 and 9, are produced during the second execution. After the second execution, the resulting poset does not contain an N. Instead

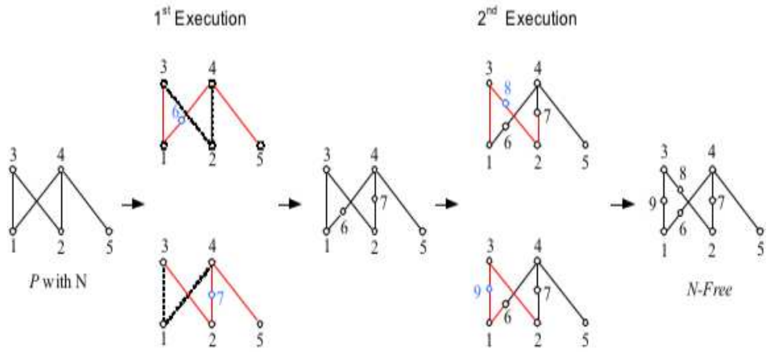

FIGURE 3. Execution of the algorithm

of adding simultaneously the dummy vertices, we may add them successively.

Theorem 1.3. *The algorithm starting with $P_0 := P$ and adding at step m a dummy vertex on a diagonal edge of some N in P_m stops on $\overline{P}$. Hence the result and the number of steps does not depends upon the particular choice of the diagonal edges choosen at each step.*

Remarks 1.4. *(1) If instead of the diagonal edges of P we consider those of $Diag(P)$, one get the same conclusion as in Theorem 1.2 and Theorem 1.3 (see Remark 2.6 below).*

(2) *A poset P can be embedded into an N-free poset which does not come from a barycentric extension of its diagram, but a minimal one is not necessarily isomorphic to $\overline{P}$. The posets represented in (a) and (c) of Figure 4 are the minimal N-free barycentric extensions $\overline{A}$ of A and $\overline{B}$ of B respectively; the posets represented in (b) and $d)$ are minimal N-free extensions of A and B. There are quotients of $\overline{A}$ and $\overline{B}$. We do not know if a minimal N-free extensions of a poset P is necessarily a quotient of $\overline{P}$.*

(3) *P.A. Grillet considered infinite posets satisfying some regularity condition. We restricted ourselves to finite posets. We do not know how our results translate to the infinite.*

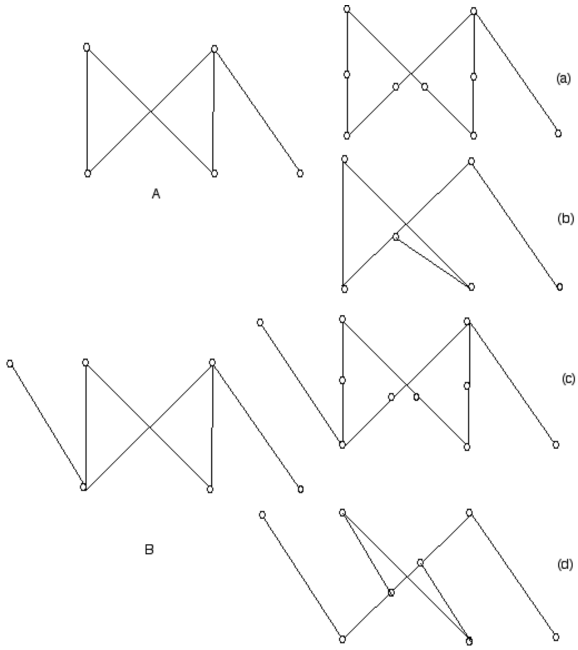

FIGURE 4. Minimal N-free extensions

The motivation for this research came from drawing of posets. A good drawing solution that works for all posets is clearly out of reach. However, if every poset can be embedded into another with a particular structure, and at the same time these particular structures can be nicely drawn, then this can lead to an interesting approximation of general ordered set drawing. In [4] was presented an approach for drawing N-free posets. The algorithm, called *LR-drawing* (LR for left-right), consists of three steps: The first step is to convert P into an N-free poset Q. The second step is to apply the LR-drawing to Q. The third and last step is to retrieve P from the drawing of Q. The first part of the algorithm requiring to look at the possible extensions of a poset into an N-free one, this suggested an other look at the barycentric extensions of a poset and lead to the present results.

2. Proofs

In this section, we consider a *finite poset* P. A basic ingredient of the proofs is the set $A(P)$ of pairs $(b,c) \in Diag(P) \setminus N_{diag}(P)$ for which there are two vertices $a,d \in P$ such that $a < c, b < d$, $(a,b),(c,d) \in Inc(P)$, and either $(a,c) \in N_{diag}(P)$ or $(b,d) \in N_{diag}(P)$. In our definition of members of $A(P)$, we could have supposed $a \prec c$ and $b \prec d$. The definition we choose is closer to the one considered in Lemma 11 of Grillet's paper.

An important feature of a barycentric subdivision is that each new element has a unique upper cover and a unique lower cover. This fact is at the root of the following lemma.

Lemma 2.1. *Let P' be a barycentric subdivision of P and $a,b,c,d \in P'$. If $a < c, b < d, (b,c) \in Diag(P')$ and $(a,b),(d,c) \in Inc(P')$ then $b,c \in P$; if, moreover, $(a,c),(b,d) \in Diag(P')$ and $a < d$ then $a,d \in P$.*

Proof. If b or c is not in P then (b,c) is a new edge, hence either b, or c, is a dummy vertex. If b is a dummy vertex, we have $c < d$, whereas if c is a dummy vertex we have $a < b$, contradicting our hypothesis. If $a \notin P$ then (a,c) is a new edge and, since $(a,c) \in Diag(P')$, a is a dummy vertex on some edge $(a',c) \in Diag(P)$; from $a < d$, we get $c < d$, a contradiction. Applying this to the dual poset P^{dual} we get $d \in P$. □

Lemma 2.2. *Let $\{a,b,c,d\}$ four elements of P such that $(a,c),(b,d) \in Diag(P)$, $(b,c) \in Diag(P) \setminus N_{diag}(P)$.*

(1) $a < d$ and if $(a,d) \notin Diag(P)$ then $(a,c),(b,d) \in N_{diag}(P)$;

(2) If $(a,c) \in N_{diag}(P)$ then

(a) $(x,b) \in Inc(P)$ for every $x \in P$ such that $(a,x) \in$Diag (P) and $\{a,c,x,y\}$ witnesses the fact that $(a,c) \in N_{diag}(P)$ for some $y \in P$;

(b) $(a,d) \in N_{diag}(P)$ iff $(a,d) \in Diag(P)$.

(3) $(a,c) \in N_{diag}(P)$ if and only if there is some $x \in P$ such that $(a,x) \in Diag(P)$ and $(x,b) \in Inc(P)$.

Proof. (1) If $a \not< d$ then $\{a,b,c,d\}$ is an N in P hence $(b,c) \in N_{diag}(P)$ contradicting the fact that $(b,c) \in Diag(P) \setminus N_{diag}(P)$. Let $x \in P$ such that $a \prec x \leq d$. Then $\{x,a,c,b\}$ is an N in P hence $(a,c) \in N_{diag}(P)$. With this argument applied to P^{dual} we get $(b,d) \in N_{diag}(P)$.

(2) Suppose $(a,c) \in N_{diag}(P)$. Let us prove (a). Let x,y such that $(a,x),(y,c) \in Diag(P)$ such that $\{x,a,c,y\}$ witnesses that $(a,c) \in N_{diag}(P)$. If $(x,b) \notin Inc(P)$ then $b < x$. Let $b' \in P$ such that $b \prec b' \leq x$. Then $\{y,c,b',b\}$ is an N in P thus $(b,c) \in N_{diag}(P)$ contradicting our hypothesis. Let us prove (b). Suppose $(a,d) \in Diag(P)$. Let x,y as above. Since $(x,b) \in Inc(P)$, $\{x,a,d,b\}$ is an N in P, hence $(a,d) \in N_{diag}(P)$. The converse is obvious.

(3) follows immediately from $(2-a)$. □

Lemma 2.3. $N_{diag}(S_N(P)) = A(P)$

Proof. Set $P' := S_N(P)$.

(a) $N_{diag}(P') \subseteq A(P)$. Let $(b,c) \in N_{diag}(P')$.

Claim 1 $(b,c) \in Diag(P) \setminus N_{diag}(P)$. Moreover, if $A := \{a,b,c,d\}$ is an N in P' with $a \prec_{P'} c$ and $b \prec_{P'} d$ then a or d are in $P' \setminus P$.

Proof of Claim 1 According to Lemma 2.1 we have $b, c \in P$. Since $(b,c) \in Diag(P')$, it follows $(b,c) \in Diag(P) \setminus N_{diag}(P)$. Since $b, c \in P$, if a and d are in P then $\{a,b,c,d\}$ is an N in P and thus (b,c) has been subdivided, hence $(b,c) \notin Diag(P')$ a contradiction. □

Let A as above.

Case 1. $a \in P' \setminus P$. In this case a is a dummy vertex on some edge $(a',c) \in N_{diag}(P)$. Since $(b,d) \in Diag(P')$ there is some $d' \in P$ such that $b \prec_P d'$ and $d \leq d'$ ($d' = d$ if $d \in P$, otherwise $(b,d) \in Diag(P')$ in which case d is a dummy vertex on (b,d')). Thus $A' := \{a',b,c,d'\}$ witnesses the fact that $(b,c) \in A(P)$.

Case 2. $d \in P' \setminus P$. This case reduces to Case (1) above by considering the dual poset P^{dual}. From Claim 1 there is no other case. The proof of (a) is complete.

(b) $A(P) \subseteq N_{diag}(P')$. Let $(b,c) \in A(P)$. Let $\{a,b,c,d\}$, with $(a,c), (b,d) \in Diag(P)$, witnessing it. If $(a,c) \in N_{diag}(P)$, let u be a dummy vertex on (a,c) then $\{u,c,b,d'\}$, where $d' := d$ if $(b,d) \notin N_{diag}(P)$ and d' is a dummy vertex on (b,d) otherwise, is an N in P' hence $(b,c) \in N_{diag}(P')$. If $(b,d) \in N_{diag}(P)$, apply the above case to P^{dual}. □

Lemma 2.4. $A(S_N(P)) = \emptyset$

Proof. Suppose the contrary. Set $P' := S_N(P)$ and let $(b,c) \in A(P')$. Let $A := \{a,b,c,d\}$, with $(a,c), (b,d) \in Diag(P')$, witnessing the fact that $(b,c) \in A(P')$. According to (1) of Lemma 2.2 applied to P', we have $a < d$. Thus from Lemma 2.1, we have $a,b,c,d \in P$.

Case 1. $(a,c) \in N_{diag}(P')$. According to (3) of Lemma 2.2 applied to P' there is some $x \in P'$ such that $(a,x) \in Diag(P')$ and $(x,b) \in Inc(P')$.

Next, $x \in P' \setminus P$. Indeed, $\{x,a,c,b\}$ is an N in P'. Thus, if $x \in P$, this is an N in P and $(a,c) \in N_{diag}(P)$, hence a dummy vertex is added on (a,c) in P' contradicting $(a,c) \in Diag(P')$. Finally, we consider two subcases:

Subcase 1.1. $(a,d) \in Diag(P')$. In this case, $(x,d) \in Inc(P)$ and, since $x \notin P$, $(a,x) \in Diag(P') \setminus Diag(P)$. Hence, there is $x' \in P$ such that x is a dummy vertex of $(a,x') \in N_{diag}(P)$. Let $A' := \{x',a,c,b\}$. We have $(a,c), (b,c) \in Diag(P) \setminus N_{diag}(P)$ and $(a,x') \in N_{diag}(P)$. Thus $(a,c) \in A(P)$. According to (1) of Lemma 2.2 $(b,x') \in Diag(P)$. Next, according to (3) of Lemma 2.2, there is some $v \in P$ such that $(v,x') \in Diag(P)$ and $(v,c) \in Inc(P)$. It follows that $\{v,x',b,c\}$ is an N in P hence $(b,x') \in N_{diag}(P)$. If b' is a dummy vertex on (b,x') then $\{b',b,c,a\}$ is an N in P' hence $(b,c) \in N_{diag}(P')$ contradicting $(b,c) \in A(P')$. Thus this subcase leads to a contradiction.

Subcase 2.2. $(a, d) \notin Diag(P')$. In this case, we may suppose $x < d$. In fact $(x, d) \in Diag(P')$. Indeed, if $(x, d) \notin Diag(P')$ then there is $d' \in P$ such that $x <_{P'} d' \prec_P d$. But, then $\{d', d, c, b\}$ is an N in P, thus $(b, d) \in N_{diag}(P)$ proving that $(b, d) \notin Diag(P')$ a contradiction. It follows that x is a dummy vertex added on (a, d) and that $(a, d) \in N_{diag}(P)$. Since $(a, d) \in N_{diag}(P)$, $(b, d) \in Diag(P) \setminus N_{diag}(P)$ and $(b, c) \in Diag(P)$, $(b, d) \in A(P)$. Since $(a, c) \in Diag(P)$ it follows from $(2 - b)$ of Lemma 2.2 that $(a, c) \in N_{diag}(P)$ contradicting $(a, c) \in Diag(P')$. This subcase leads to a contradiction too.

Case 2. $(b, d) \in N_{diag}(P')$. This case reduces to the previous one by considering the dual poset P^{dual}. Hence, it leads to a contradiction.

Consequently $A(P') = \emptyset$. The proof is complete. □

Proof of Theorem 1.2. Set $P' := S_N(P)$ and $P'' := S_N(P')$. We prove first that P'' is N-free. This amounts to prove that $N_{diag}(P'')$ is empty. This immediately follows from Lemma 2.3 and Lemma 2.4. Indeed, we have $N_{diag}(P'') := N_{diag}(S_N(P')) = A(P') = A(S_N(P)) = \emptyset$. Next, we prove that P'' is minimum. Let Q be a barycentric subdivision of $Diag(P)$ which is N-free. Then, clearly, Q include $S_N(P)$. Since Q is also a barycentric subdivision of $Diag(P')$, Q includes also $S_N(P')$. Thus P'' is the smallest N-free poset obtained as a barycentric subdivision of $Diag(P)$. It coincides with the poset $\overline{P}$ constructed by P.A.Grillet. □

Lemma 2.5. *Let P' with $P \subseteq P' \subseteq S_N(S_N(P))$; then $N_{diag}(P') \subseteq N_{diag}(P) \cup A(P)$.*

Proof. Let $(b, c) \in N_{diag}(P')$. Suppose $(b, c) \notin N_{diag}(P) \cup A(P)$. Let $Q := S_N(S_N(P))$. We claim that $(b, c) \in N_{diag}(Q)$. Let $A := \{a, b, c, d\}$ be an N of P' witnessing the fact that $(b, c) \in N_{diag}(P')$. Since, from Lemma 2.3 $(b, c) \notin N_{diag}(P) \cup N_{diag}(S_N(P))$, $(b, c) \in Diag(Q)$ thus $A' := \{a', b, c, d'\}$ where $a \leq a' \prec_Q c$ and $b \prec_Q d'$ is an N in Q proving our claim. Next, with $(b, c) \in N_{diag}(Q)$ and $Q := S_N(S_N(P))$, we get from Lemma 2.3 that $(b, c) \in A(S_N(P))$. Since, from Lemma 2.4, $A(S_N(P)) = \emptyset$, we get a contradiction. This proves the lemma. □

Proof of Theorem 1.3 An immediate induction using Lemma 2.5 shows that each P_m is a subset of $Q := S_N(S_N(P))$. Since Q is the least N-free subset of $S(P)$ containing P the algorithm stops on Q. The number of steps is the size of $N_{diag}(P) \cup A(P)$. □

Remarks 2.6. *(1) If $A := \{a, b, c, d\}$ is an N in $Diag(P)$ as in (3) of Definition 1.1, we say that the pair (b, c) is the* diagonal edge *of this N. Let $N_{diag}(Diag(P))$ be the set of diagonal edges of all the N's in $Diag(P)$ and let $S_N(Diag(P))$ be the poset obtained by adding a dummy vertex on each edge in $N_{diag}(Diag(P))$. Clearly, $N_{diag}(Diag(P)) \subseteq N_{diag}(P) \cup A(P)$. Thus, with the same proof as for Theorem 1.3, we obtain that the algorithm consisting to add at step m a dummy vertex on an edge* *of some N in* $Diag(P_m)$ *ends on* $\overline{P}$. *Similarly, with Lemma 2.5 we get that* $S_N(Diag(S_N(Diag(P)))) = \overline{P}$;

(2) *The fact that the algorithm given in Theorem 1.3 stops is obvious: at each step,* P_m *is a subset of* $S(P)$. *The fact that the number of steps in independent of the choosen edges is more significant. This suggests a deepest investigation. We just note that if* P_m *contains just one N then* P_{m+1} *is N-free (we leave the proof to the reader).*

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