Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
Vol. 06(01): 108 - 118 (2019)

# Weber's problem on the Riemannian Manifolds: Some upper bounds for the minimun Weber's function

## El problema de Weber sobre variedades Riemannianas: Algunas cotas Superiores para el mínimo de la función de Weber

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*Received: Feb. 12, 2019* *Accepted: May. 19, 2019*

**DOI**: http://dx.doi.org/10.17268/sel.mat.2019.01.13

### Abstract

In this paper we obtain some upper bounds for the minimum of the Weber function on a strongly convex ball in a Riemannian manifold with positive sectional curvature; where the minimum is reached on the weighted geometric median of "m" given points in the strongly convex.

**Keywords:** The Weber problem, Weighted Geometric Median, Riemannian manifold, Strongly convex set.

### Resumen

En este artículo se obtiene algunas cotas superiores para el mínimo de la función de Weber sobre una bola fuertemente convexa en una variedad Riemanniana con curvartura seccional positiva; dicho mínimo se alcanza sobre la mediana geométrica pesada de "m" puntos dados en la bola fuertemente convexa.

**Palabras Clave**: Problema de Weber, Mediana Geométrica Pesada, Variedad Riemanniana, Conjunto fuertemente convexo.

## 1. Introduction.

Mathematically, Weber's problem in the plane consists in finding a point that minimizes the sum of the weighed distances to "m" fixed points. Weiszfeld [7] was the first to formulate an iterative method to approximate the solution to Weber's problem.

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If the fixed points are located in large regions on the surface of the earth, the approximation through a plane, and the use of the Euclidean distance, will provide unsatisfactory results; and since the sphere is used to model the planet earth, Drezner and Wesolowsky [2], Hansen [5], generalized Weber's problem to the sphere, and used the shortest arc distance to estimate the distance between two points.

Since both the plane and the sphere are regular surfaces; and since regular surfaces are a particular case of Riemannian manifolds, Fletcher [4], Aftab Khurrum [1], generalized the problem to Riemannian manifolds.

In addition, Fletcher [4] establishes conditions to guarantee the existence and uniqueness of the weighted geometric median. In this paper, upper bounds are established for the minimum of the Weber function on a strongly convex ball in a Riemannian manifolds; which is reached in the weighted geometric median.

## 2. The Weber Problem on the Plane

Consider "m" different points in the plane $a_1, a_2, \dots, a_m \in \mathbb{R}^2$, $a_i = (x_i,\ y_i), \forall\ i = 1, 2, \dots, m$ , with weights $w_i > 0\ , \forall\ i = 1, 2, \dots, m$.

Weber's problem is to find a point of $\mathbb{R}^2$ that minimizes the function $F$: $\mathbb{R}^2 \to \mathbb{R}$ , defined by:

(1) $$F(x,y) = \sum_{i=1}^{m} w_i \sqrt{(x - x_i)^2 + (y - y_i)^2}\ ;$$

ie:

(2) $$\min_{X \in \mathbb{R}^2} F(X) = \min_{X \in \mathbb{R}^2} \sum_{i=1}^{m} w_i d_i(X)\ ,$$

where $d_i(X) = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ is the Euclidean distance between the point $X = (x, y)$ and the point $a_i = (x_i,\ y_i)$. The points $a_i \in \mathbb{R}^2$ are called vértices and the function $F(x)$ is called Weber function.

If the vertices $a_i \in \mathbb{R}^2$ are not collinear, the Weber function is strictly convex, so the problem (2) has a unique solution; Wendel and Hurter [8], proved that this solution is in the convex capsule of the vertices. Weiszfeld [7], proposed an iterative method to approximate the solution of the problem (2), by means of a sequence defined by:

(3) $$x^{k+1} = \frac{\sum_{i=1}^{m} \frac{w_i x_i}{d_i(x^k,\ y^k)}}{\sum_{i=1}^{m} \frac{w_i}{d_i(x^k,\ y^k)}} \quad , \quad y^{k+1} = \frac{\sum_{i=1}^{m} \frac{w_i y_i}{d_i(x^k,\ y^k)}}{\sum_{i=1}^{m} \frac{w_i}{d_i(x^k,\ y^k)}}\ ,$$

with starting point:

$$x^0 = \frac{\sum_{i=1}^{m} w_i x_i}{\sum_{i=1}^{m} w_i} \quad , \quad y^0 = \frac{\sum_{i=1}^{m} w_i y_i}{\sum_{i=1}^{m} w_i}\ .$$

## 3. The Weber Problem on Riemannian Manifold

Let M be a Riemannian manifold n-dimensional of class $C^\infty$ , and $d_M$ , $\nabla$ , $\mathcal{R}$ are the intrinsic metric of M, Riemannian Connection and Curvature Operator respectively; where:

$$\mathcal{R}(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z\ ,\ \ \forall\ X, Y, Z \in \Gamma(M),$$

$\Gamma(M) = \{X: M \to TM/ \quad X \in C^\infty(M)\}$ , TM is the tangent bundle of M.

**Definition 1.** Let $S \subset M$. S is called a Strongly Convex set if for each pair of points $q_1, q_2$ of the closed set $\bar{S}$ of $S$, there is a unique minimizing geodesic $\alpha: [0,1] \to S$, $\alpha(0) = q_1$, $\alpha(1) = q_2$, such that $\alpha([0,1]) \subset S$.

Also, for each $p \in M$, there is $\delta > 0$, such that the geodesic ball $B(p, \delta)$ is strongly convex [6]. Let $A \subset M$ be a strongly convex set in $M$. The set $T_x A$ of tangent vectors to $A$ at point $x$ is a convex cone in tangent space to $M$ at $x$: $T_x M$.

Given $X_x$, $Y_x \in T_x A$, then: $tX_x + tY_x \in T_x A, \forall\, t \geq 0$; and there are curves:

$$\alpha, \ \beta: [0,1] \to A, \ \ \alpha(0) = \beta(0) = x, \ \ \alpha'(0) = X_x \quad , \ \beta'(0) = Y_x.$$

Also, we consider a deformation of class $C^\infty$ of curves in $A$:

(4)
$$\tau: [0,1]x[0,t_0] \to A$$
$$(s,t) \to \tau(s,t)$$

such that

a) $\tau(s, 0) = x$.

b) $\tau(0,t) = \alpha(t) = exp_x(tX_x)$,

$\tau(1,t) = \beta(t) = exp_x(tY_x)$.

c) For each $t \in [0, t_0]$, the curve:

$$\gamma_t: [0,1] \to A \text{ , defined by } \gamma_t(s) = \tau(s,t),$$

is a parameterized geodesic with a parameter proportional to the arc length. This allows to consider the parameterized geodesic:

$$\lambda_s: [0,t_0] \to A, \ \ \forall\, s \in [0,1], \text{ such that } \lambda_s(t) = \gamma_t(s) = \tau(s,t).$$

**Theorem 1**. The vector field $J(s) = \frac{\partial \tau(s,t)}{\partial t}$, $\forall\ s \in [0,1]$, defined along the curve $\gamma_t(s) = \tau(s,t)$, is a field of Jacobi; ie:

$$\frac{D^2 J(s)}{\partial s^2} + \mathcal{R}\left(\frac{d\gamma_t(s)}{ds}, J(s)\right)\frac{d\gamma_t(s)}{ds} = 0$$

Proof

Using the curvature operator $\mathcal{R}$, we have:

$$\mathcal{R}\left(\frac{d\gamma_t(s)}{ds}, J(s)\right)\frac{d\gamma_t(s)}{ds} = \nabla_{J(s)}\nabla_{\frac{d\gamma_t(s)}{ds}}\frac{d\gamma_t(s)}{ds} - \nabla_{\frac{d\gamma_t(s)}{ds}}\nabla_{J(s)}\frac{d\gamma_t(s)}{ds} + \nabla_{\left[\frac{d\gamma_t(s)}{ds}, J(s)\right]}\frac{d\gamma_t(s)}{ds}$$

$$= -\nabla_{\frac{d\gamma_t(s)}{ds}}\nabla_{\nabla_{\frac{d\gamma_t(s)}{dt}}}\frac{d\lambda_s(t)}{dt} = -\frac{D^2 J(s)}{\partial s^2}.$$

Therefore, $J(s)$ is a field of Jacobi. ∎

**Theorem 2.** Let $\tau(s,t)$ be a deformation given by (4), and be the geodesic:

$$\lambda_s : [0, t_0] \to A\,, \ \ \forall\, s \in [0,1], \text{ such that } \lambda_s(t) = \gamma_t(s) = \tau(s,t).$$

Then:

(5) $$\frac{d\lambda_s(0)}{dt} = (1-s)X_{x_0} + sY_{x_0}\ ,\ \forall s \in [0,1].$$

Proof

By theorem (1), the vector field $J(s) = \frac{\partial\tau(s,t)}{\partial t}$ ,$\forall\ s \in [0\,,1]$, is a field of Jacobi along the geodesic $\gamma_t(s) = \tau(s,t)$.

In particular for $t = 0$, and by (a) of the deformation (4), the Jacobi equation is reduced:

$$\frac{D^2 J(s)}{\partial s^2} = \frac{D}{ds}\frac{D}{ds}\big(J(s)\big) = \frac{D}{ds}\frac{D}{ds}\left(\frac{\partial\tau(s,0)}{\partial t}\right) = 0.$$

By the conditions:

$$\frac{\partial\tau(0,0)}{\partial t} = \frac{d\alpha(0)}{dt} = X_{x_0}\quad,\qquad \frac{\partial\tau(1,0)}{\partial t} = \frac{d\beta(0)}{dt} = Y_{x_0}\quad,$$

we have:

$$\frac{d\lambda_s(0)}{dt} = (1-s)X_{x_0} + sY_{x_0}\ ,\ \forall s \in [0,1]\ . \qquad \blacksquare$$

**Definition 2.** Given points $x_1, x_2, \dots, x_m \in M$, and positive real number $w_1, w_2, \dots, w_m$ , the function $F: M \to \mathbb{R}$ , defined by:

(6) $$F(x) = \sum_{i=1}^{m} w_i d_M(x,\ x_i)$$

is called Weber's function.

**Theorem 3.** Let $p_0 \in M$, $B(p_0, r)$ a strongly convex ball. For each $x_0 \in B(p_0, r)$, the function $F_{x_0}: B(p_0, r) \to \mathbb{R}$ defined by $F_{x_0}(x) = d_M(x_0, x)$ is a convex function.

Proof

Let $x_1\,, x_2\ \in B(p_0, r)$ be two arbitrary points. Then there are minimal geodesics

$$\alpha :\ [0,1] \to B(p_0, r),\ \ \alpha(0) = x_0\ ,\ \alpha(1) = x_1\ ,\ \ \frac{d\alpha(0)}{dt} = X_{x_0}$$

$$\beta :\ [0,1] \to B(p_0, r),\ \ \beta(0) = x_0\ ,\ \beta(1) = x_2\ ,\ \ \frac{d\beta(0)}{dt} = Y_{x_0}\ ,$$

such that:

$$l(\alpha) = \int_0^1 \left\|\frac{d\alpha(t)}{dt}\right\| dt = \ \|X_{x_0}\|,$$

$$l(\beta) = \int_0^1 \left\|\frac{d\beta(t)}{dt}\right\| dt = \ \|Y_{x_0}\|.$$

By deformation $\tau(s,t)$ given by (4), there is a minimal geodesic: $\gamma_1 :\ [0,1] \to B(p_0, r)$, such that:

$$\gamma_1(0) = \tau(0,1) = \ \alpha(1) = x_1\ \ ,$$
$$\gamma_1(1) = \tau(1,1) = \ \beta(1) = x_2\,.$$

For each “s” consider the geodesic:

$$\lambda_s : [0,1] \to B(p_0, r), \quad \lambda_s(t) = \tau(s,t);$$

what satisfies: $\lambda_s(0) = \tau(s,0) = x_0$ , $\lambda_s(1) = \tau(s,1) = \gamma_1(s)$.

Then, the function $F_{x_0} \circ \gamma_1 : [0,1] \to \mathbb{R}$ verifies:

$$(F_{x_0} \circ \gamma_1)(s) = F_{x_0}(\gamma_1(s)) = F_{x_0}(\lambda_s(1)) = d_M(x_0, \lambda_s(1)) =$$

$$= \int_0^1 \left\| \frac{d\lambda_s(t)}{dt} \right\| dt = \left\| \frac{d\lambda_s(0)}{dt} \right\| = \left\| (1-s)X_{x_0} + sY_{x_0} \right\|$$

$$\leq (1-s)\left\|X_{x_0}\right\| + s\left\|Y_{x_0}\right\|$$

$$= (1-s)d_M\big(x_0, \alpha(1)\big) + s d_M\big(x_0, \beta(1)\big)$$

$$= (1-s)d_M(x_0, x_1) + s d_M(x_0, x_2)$$

$$= (1-s)F_{x_0}(x_1) + sF_{x_0}(x_2), \ \forall s \in [0,1]$$

Therefore, $F_{x_0}$ is a convex function. ■

**Theorem 4.** Let $x_1, x_2, \dots, x_m \in B(p_0, r)$. Then the function $F: B(p_0, r) \to \mathbb{R}$ defined by:

(7) $$F(x) = \sum_{i=1}^{m} w_i d_M(x, x_i) \ , \ w_i > 0 , \ \forall i = 1, \dots, m$$

is convex function.

Proof

By theorem (3), the Weber’s function (6):

$$F(x) = \sum_{i=1}^{m} w_i d_M(x, x_i)$$

$$= \sum_{i=1}^{m} w_i F_{x_i}(x) ,$$

where the functions $F_{x_i}$ , $\forall i = 1, \dots, m$ , are convex functions. Also,

$w_i F_{x_i}$ , $\forall i = 1, \dots, m$ , are convex functions.

Therefore: $F(x) = \sum_{i=1}^{m} w_i d_M(x, x_i)$ is a convex function. ■

## 4. Some bound for the Weber function

In this section we introduce the weighted geometric median (Fletcher, 2009).

**Definition 3.** Let $U \subset M$ be a strongly convex set, $x_1, x_2, \dots, x_m \in U$. The Weighted Geometric Median of $F(x) = \sum_{i=1}^{m} w_i d_M(x, x_i)$, $w_i > 0, \forall i = 1, \dots, m$, $\sum_{i=1}^{m} w_i = 1$ , is:

(8) $$x^* = arg \min_{x \in U} \sum_{i=1}^{m} w_i d_M(x, x_i).$$

If $w_i = \frac{1}{m}, \forall i = 1, \dots, m$ , $x^*$ is call simply the Geometric Median [4].

Let $p_0 \in M,\ B(p_0,r) = exp_{p_0}(B(0,r))$, $B(0,r) \subset T_{p_0}M$ is the ball open in tangent space$T_{p_0}M$, where the exponential map $exp_{p_0}$ is a diffeomorphism.

Now, let's consider the points $x_1, x_2, \dots, x_m \in B(p_0,r)$, and be

$$v_i = exp_{p_0}^{-1}(x_i) \in B(0,r),\ \ \forall\, i = 1, \dots, m.$$

**Theorem 5.** Let M be a Riemannian manifold with sectional curvature $k(x) > 0,\ \forall\, x \in M$. Then:

(9) $$F(x) = \sum_{i=1}^{m} w_i d_M(x,\ x_i) < 2r.$$

Proof

As $k(x) > 0, \forall\, x \in M$, and by Toponogov's theorem , we have:

$$d_M(x,\ x_i) \le \|v - v_i\|,\ \ \forall\, i = 1, \dots, m$$

$$d_M(x,\ x_i) \le \|v - v_i\| < 2r,\ \ \forall\, i = 1, \dots, m$$

Also: $$w_i d_M(x,\ x_i) \le w_i\|v - v_i\| < 2rw_i,\ \ \forall\, i = 1, \dots, m$$

$$\sum_{i=1}^{m} w_i d_M(x,\ x_i) \le \sum_{i=1}^{m} w_i\|v - v_i\| < 2r\sum_{i=}^{m} w_i = 2r$$

Therefore:

$$F(x) = \sum_{i=1}^{m} w_i d_M(x,\ x_i) < 2r\ .$$ ■

**Theorem 6.** Let M be a Riemannian manifold with sectional curvature $0 < k(x) \le \Delta$ , $B(p_0,r) \subset M$ a strongly convex ball, $r < \frac{\pi}{4\sqrt{\Delta}}$ , and $x_1, x_2, \dots, x_m \in B(p_0,r)$, then exists and is unique the weighted gemetric median defined by (8). Also:

(10) $$F(x) = \sum_{i=1}^{m} w_i d_M(x,\ x_i) < \frac{\pi}{2\sqrt{\Delta}}\ .$$

Proof

By theorem (4) the function $F(x) = \sum_{i=1}^{m} w_i d_M(x,\ x_i)$ is convex; and how the sectional curvature of M is bounded, and $r < \frac{\pi}{4\sqrt{\Delta}}$ , then by Fletcher [4], exists and is unique the weighted gemetric median defined by (8).

By theorem (5):

$$F(x) = \sum_{i=1}^{m} w_i d_M(x,\ x_i) < 2r < \frac{\pi}{2\sqrt{\Delta}}\ .$$

Therefore:

$$F(x) = \sum_{i=1}^{m} w_i d_M(x,\ x_i) < \frac{\pi}{2\sqrt{\Delta}}\ .$$ ■

**Corollary 1.** Let M be a Riemannian manifold with sectional curvature $0 < k(x) \le \Delta$ , $B(p_0,r) \subset M$ a strongly convex ball, $r < \frac{\pi}{4\sqrt{\Delta}}$ , and $x_1, x_2, \dots, x_m \in B(p_0,r)$, then

(11) $$F^* = \min_{x \in B(p_0,r)} \sum_{i=1}^{m} w_i d_M(x,\ x_i) < \frac{\pi}{2\sqrt{\Delta}}\ .$$

Proof

By theorem (6), exists and is unique the weighted geometric median $x^* \in B(p_0, r)$; and be $F^* = F(x^*)$. Then:

$$F^* = F(x^*) \leq F(x) < \frac{\pi}{2\sqrt{\Delta}}, \ \forall\, x \in B(p_0, r). \qquad ■$$

Let $x_1, x_2, \dots, x_m \in B(p_0, r)$, $v_i = exp_{p_0}^{-1}(x_i) \in B(0, r)$, $\forall\, i = 1, \dots, m$,

$$v = exp_{p_0}^{-1}(x) \in B(0, r), \ \forall\, x \in B(p_0, r).$$

This allows to consider the function $f: B(0, r) \subset T_{p_0}M \to \mathbb{R}$, defined by:

$$f(v) = \sum_{i=1}^{m} w_i \|v - v_i\|.$$

**Theorem 7.** Let M be a Riemannian manifold with sectional curvature $k(x) > 0$, $\forall\, x \in M$. Then:

(12) $$F(x) \leq f(v), \qquad \forall\, x \in B(p_0, r).$$

Proof

As $k(x) > 0$, $\forall\, x \in M$, and by Toponogov theorem we have:

$$d_M(x, x_i) \leq \|v - v_i\|, \ \forall\, i = 1, \dots, m.$$

Also: $$w_i d_M(x, x_i) \leq w_i \|v - v_i\|, \quad w_i > 0, \ \forall\, i = 1, \dots, m.$$

Then $$F(x) = \sum_{i=1}^{m} w_i d_M(x, x_i) \leq \sum_{i=1}^{m} w_i \|v - v_i\| = f(v).$$

Therefore:

$$F(x) \leq f(v), \ \forall\, x \in B(p_0, r). \qquad ■$$

**Theorem 8.** Let M be a Riemannian manifold with sectional curvature $0 < k(x) \leq \Delta$, $B(p_0, r) \subset M$ a strongly convex ball, $r < \frac{\pi}{4\sqrt{\Delta}}$, and be a $x_1, x_2, \dots, x_m \in B(p_0, r)$,

$$\sum_{i=1}^{m} w_i = 1, \qquad F^* = \min_{x \in B(p_0, r)} (F(x)), \qquad f^* = \min_{v \in B(0, r)} (f(v)).$$

Then:

(12) $$F^* \leq f^* < \frac{\pi}{2\sqrt{\Delta}}.$$

Proof

Let $x^* \in B(p_0, r)$ such that: $F^* = F(x^*)$, and $v^* \in B(0, r)$ such that: $f^* = f(v^*)$. We have

**Case 1.** Let $v^* = exp_{p_0}^{-1}(x^*)$, then:

$$F^* = F(x^*) \leq f(v^*) = f^* < \frac{\pi}{2\sqrt{\Delta}}.$$

**Case 2.** Let $v_{x^*} = exp_{p_0}^{-1}(x^*)$, such that $v_{x^*} \neq v^*$, then $f(v^*) < f(v_{x^*})$.

As $F(x) \leq f(v)$, y $v = \varphi(x)$, where $\varphi = exp_{p_0}^{-1}$, we have:

$$F(x) \leq f(\varphi(x)) = G(x), \text{ where } G = f \circ \varphi: B(p_0, r) \to \mathbb{R}$$

then:

$$\min_{x \in B(p_0,r)} (F(x)) \leq \min_{x \in B(p_0,r)} (G(x))$$
$$= \min_{x \in B(p_0,r)} (f \circ \varphi(x))$$
$$= f^* = \min_{x \in B(p_0,r)} (f(v)) < \frac{\pi}{2\sqrt{\Delta}}.$$

Therefore:

$$F^* \leq f^* < \frac{\pi}{2\sqrt{\Delta}}.$$ ■

**4. Example**

In this section we present results on the unit sphere.

Let $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 / \ \ x^2 + y^2 + z^2 = 1\}$ the unit sphere, whose sectional curvature $k(x) = 1, \ \forall\, x \in S^2$; therefore the bounded above for the sectional curvature is $\Delta = 1$.

Now, consider the ball strongly convex $B(p_0, r)$, where $p_0 = (0, 0, 1)$, and $r = \frac{\pi}{4}$, Drezner and Wesolowsky [2], and using spherical Coordinates:

$$x = \cos(\varphi)\, cos(\theta), \quad y = \cos(\varphi)\, sin(\theta), \ \ z = \sin(\varphi), \quad \frac{\pi}{4} < \varphi < \frac{\pi}{2}\ ,\ 0 < \theta < 2\pi\ ,$$

a PC with Intel Core i3 microprocessor, Windows 7 operating system, 2.20 GHz, 4GB of RAM,

we have the following cases:

**Case 1.** We consider $m = 5$ points chosen randomly

| Coordinates and weights | | | |
|---|---|---|---|
| | $\theta_i$ | $\varphi_i$ | $w_i$ |
| 1 | 5.1191 | 0.8620 | 0.2000 |
| 2 | 5.6913 | 1.0041 | 0.2000 |
| 3 | 0.7979 | 1.2149 | 0.2000 |
| 4 | 5.7389 | 1.5374 | 0.2000 |
| 5 | 3.9732 | 1.5432 | 0.2000 |

Table 1. Spherical coordinates of the points and their weights.

Using the Weiszfeld algorithm in the sphere was obtained:

| Results | |
|---|---|
| Optimal Value: $F^*$ | 0.3222718 |
| Geometric Median: $x^*$ | ( 0.0436559, -0.0280022, 0.9986541) |

Table 2. The Geometric Median and the Optimal value on the Sphere.

Using the Weiszfeld algorithm in the tangent space $T_{p_0}S^2$ was obtained:

| Results | |
|---|---|
| Optimal Value: $f^*$ | 0.3365514 |
| Geometric Median: $v^*$ | ( 0.0445683, -0.0286065, 1.0000000) |

Table 3. The Geometric Median and the Optimal value on the Tangent space $T_{p_0}S^2$.

Table 2 and Table 3 show that: $F^* < f^*$.

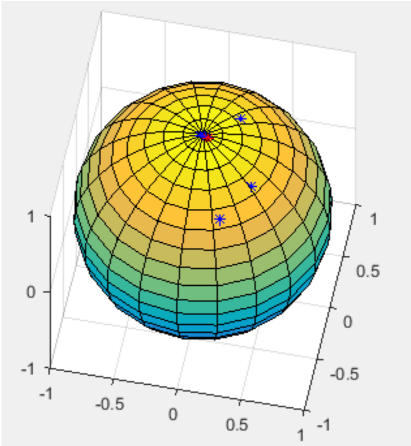


Figure 1. The red point is the geometric median for 5 points.

**Case 2.** We consider $m = 10$ points chosen randomly

| Coordinates and weights | | | |
|---|---|---|---|
| | $\theta_i$ | $\varphi_i$ | $w_i$ |
| 1 | 4.4362 | 1.1300 | 0.1000 |
| 2 | 0.2000 | 1.0851 | 0.1000 |
| 3 | 1.7400 | 1.3866 | 0.1000 |
| 4 | 0.2901 | 1.4099 | 0.1000 |
| 5 | 0.6103 | 0.9322 | 0.1000 |
| 6 | 5.1739 | 1.1701 | 0.1000 |
| 7 | 4.3657 | 1.1354 | 0.1000 |
| 8 | 1.9924 | 1.2930 | 0.1000 |
| 9 | 5.9704 | 1.3425 | 0.1000 |
| 10 | 0.2164 | 1.3781 | 0.1000 |

Table 4. Spherical coordinates of the points and their weights.

Using the Weiszfeld algorithm in the sphere was obtained:

| Results | |
|---|---|
| Optimal Value: $F^*$ | 0.3029679 |
| Geometric Median: $x^*$ | ( 0.1638406, 0.0266638, 0.9861264) |

Table 5. The Geometric Median and the Optimal value on the Sphere

Using the Weiszfeld algorithm in the tangent space $T_{p_0}S^2$ was obtained:

| Results | |
|---|---|
| Optimal Value: $f^*$ | 0.3134207 |
| Geometric Median: $v^*$ | ( 0.1633452, 0.0262530, 1.0000000) |

Table 6. The Geometric Median and the Optimal value on the Tangent space $T_{p_0}S^2$.

Table 5 and Table 6 show that: $F^* < f^*$.

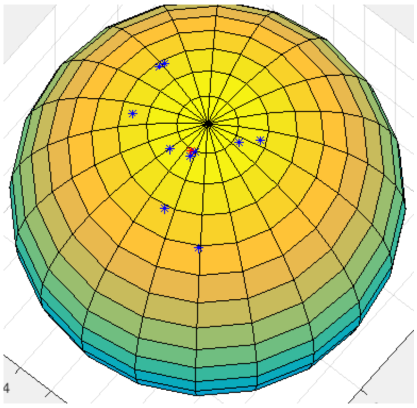

Figure 2. The red point is the geometric median for 10 points.

**Case 3.** We consider $m = 20$ points chosen randomly

| Coordinates and weights | | | |
|---|---|---|---|
| | $\theta_i$ | $\varphi_i$ | $w_i$ |
| 1 | 1.3053 | 1.2526 | 0.0500 |
| 2 | 1.8928 | 0.9913 | 0.0500 |
| 3 | 2.9589 | 1.2589 | 0.0500 |
| 4 | 1.4482 | 1.3440 | 0.0500 |
| 5 | 5.3049 | 0.9596 | 0.0500 |
| 6 | 1.2237 | 0.8776 | 0.0500 |
| 7 | 1.4195 | 1.0184 | 0.0500 |
| 8 | 1.0726 | 1.0358 | 0.0500 |
| 9 | 1.4305 | 1.1185 | 0.0500 |
| 10 | 2.7376 | 1.1843 | 0.0500 |
| 11 | 1.9547 | 0.8526 | 0.0500 |
| 12 | 5.8018 | 0.9916 | 0.0500 |
| 13 | 2.7031 | 1.4145 | 0.0500 |
| 14 | 1.1612 | 0.8083 | 0.0500 |
| 15 | 5.6855 | 1.5149 | 0.0500 |
| 16 | 6.1559 | 1.3590 | 0.0500 |
| 17 | 2.7575 | 1.1692 | 0.0500 |
| 18 | 0.6982 | 1.2398 | 0.0500 |
| 19 | 1.6215 | 0.9718 | 0.0500 |
| 20 | 2.5681 | 1.1458 | 0.0500 |

Table 7. Spherical coordinates of the points and their weights.

Using the Weiszfeld algorithm in the sphere was obtained:

| Results | |
|---|---|
| Optimal Value: $F^*$ | 0.3630473 |
| Geometric Median: $x^*$ | ( 0.0300179, 0.2946810, 0.9551241) |

Table 8. The Geometric Median and the Optimal value on the Sphere.

Using the Weiszfeld algorithm in the tangent space $T_{p_0}S^2$ was obtained:

| Results | |
|---|---|
| Optimal Value: $f^*$ | 0.3653496 |
| Geometric Median: $v^*$ | ( 0.0280437, 0.2905243, 1.0000000) |

Table 9. The Geometric Median and the Optimal value on the Tangent space $T_{p_0}S^2$.

Table 8 and Table 9 show that: $F^* < f^*$.

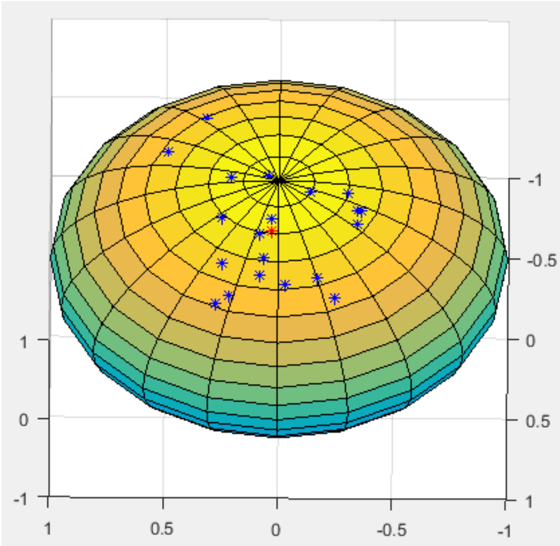


Figure 3. The red point is the geometric median for 20 points.

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