Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
Vol. 04(01): 51 - 58 (2017)

Existencia de tres soluciones para el sistema hamiltoniano fraccionario.

Existence of three solution for fractional Hamiltonian system.

César Torres Ledesma* and Oliverio Pichardo Diestra†

Received, Mar. 20, 2017 *Accepted, May. 15, 2017*

DOI: http://dx.doi.org/10.17268/sel.mat.2017.01.06

Resumen

En este artículo se considera un sistema Hamiltoniano dado por

$$-\,{}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)), \quad a.e\ \ t\in[0,T] \qquad (0.1)$$
$$u(0)=u(T)=0.$$

donde $\alpha\in(1/2,1)$, $t\in[0,T]$, $u\in\mathbb{R}^n$, $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ *es una función dada y* $\nabla F(t,u)$ *es el gradiente de* F *en* u. *La novedad de este trabajo es que, usando una versión modificada del teorema del paso de montaña para funcional limitada desde abajo probamos la existencia de por lo menos tres soluciones para (0.1).*

Palabras claves. Calculo fraccionario, derivada fraccionaria, sistema Hamiltoniano fraccionario, problema de valor de contorno.

Abstract

In this paper we consider the fractional Hamiltonian system given by

$$-\,{}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)), \quad a.e\ \ t\in[0,T] \qquad (0.2)$$
$$u(0)=u(T)=0.$$

where $\alpha\in(1/2,1)$, $t\in[0,T]$, $u\in\mathbb{R}^n$, $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ *is a given function and* $\nabla F(t,u)$ *is the gradient of* F *at* u. *The novelty of this paper is that, using a modified version of mountain pass theorem for functional bounded from below we prove the existence of at least three solutions for (0.2).*

Keywords. Fractional calculus, fractional derivatives, fractional Hamiltonian system, boundary value problem

1. Introduction. Fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For

*Departamento de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n., Ciudad Universitaria, Trujillo-Perú. *etorres@unitru.edu.pe*.

†Escuela Académico Profesional de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n., Ciudad Universitaria, Trujillo-Perú. *opichardo@unitru.edu.pe* .

details and examples, one can see the monographs [17], [26], [28] and the papers [2], [3], [5], [6], [10], [19], [23], [25], [30], [33]. Moreover the existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions have been great attention in the qualitative theory of fractional differential equations, due to its mathematical interest and applications. Some recent contributions on the existence of such solutions for abstract differential equations and fractional differential equations have been made, see [1], [3], [4], [6], [14], [15], [22], [27] for details.

Recently, also equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives is an interesting and new field in fractional differential equations theory. In this topic, many results are obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory [8] (including Leray-Schauder nonlinear alternative), topological degree theory [20] (including co-incidence degree theory) and comparison method [34] (including upper and lower solutions and monotone iterative method) and so on.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become to a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [24], Rabinowitz [29] and the references listed therein.

Motivated by the above classical works, in recent paper [21], for the first time, the authors showed that the critical point theory is an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{aligned} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) &= \nabla F(t,u(t)), \text{ a.e. } t\in[0,T], \\ u(0) &= u(T) = 0. \end{aligned} \tag{1.1}$$

and obtained the existence of at least one nontrivial solution by study the critical points of the functional

$$\overline{I}(u) = \frac{1}{2}\int_0^T |{}_0D_t^{\alpha}u(t)|^2dt - \int_0^T F(t,u(t))dt. \tag{1.2}$$

After that, Torres [32] took further discussion of this type problem in scalar case by using mountain pass theorem. Bonanno, Rodríguez-López and Tersian [12] considered this type problem with impulsive effects and proved existence of three solutions by using a critical point theorem given in [13]. We note that it is not easy to use the critical point theory to study (1.1), since it is often very difficult to establish a suitable space and variational functional for the fractional boundary value problem.

A natural question is whether problem

$$\begin{aligned} -{}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) &= \nabla F(t,u(t)), \text{ a.e. } t\in[0,T], \\ u(0) &= u(T) = 0. \end{aligned} \tag{1.3}$$

is also solvable. In this case, the corresponding functional I on E^{α} given by

$$I(u) = \frac{1}{2}\int_0^T |{}_0D_t^{\alpha}u(t)|^2dt + \int_0^T F(t,u(t))dt, \tag{1.4}$$

where $E^{\alpha} = \overline{C_0^{\infty}[a,b]}^{\|\cdot\|_{\alpha}}$ and

$$\|u\|_{\alpha}^2 = \int_0^T |u(t)|^2dt + \int_0^T |{}_0D_t^{\alpha}u(t)|^2dt.$$

We note the difference between $\overline{I}$ and I: $\overline{I}$ is neither bounded from below nor from above, whereas under some condition I is bounded from below, so we can applied the least action principle to find at least one weak solution. In this paper we consider the multiplicity of weak solution for the problem (1.3). For that purpose let $\frac{1}{2} < \alpha < 1$ and F satisfies the following conditions

(F_0) $F(t,0)$ for almost every $t\in[0,T]$.

(F_1) $F(t,x)$ is measurable in t for every $x\in\mathbb{R}^n$ and continuously differentiable in x for almost every $t\in[0,T]$, and there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b\in L^1([0,T],\mathbb{R}^+)$ such that

$$|F(t,x)| \le a(|x|)b(t), \quad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x\in\mathbb{R}^n$ and for almost every $t\in[0,T]$.

(F_2) $\lim_{|x|\to\infty} F(t,x) = \infty$ uniformly in t.
(F_3) There are $d_0, d_1 \in \mathbb{R}^n$, with $d_0 \neq d_1$, such that
$(F_3)_1$ $\int_0^T \min_{|\xi - d_0|<\delta} F(t,\xi) = \int_0^T F(t,d_0)dt$ for some $\delta > 0$;
$(F_3)_2$ $\int_0^T F(t,d_0) \geq \int_0^T F(t,d_1)dt$.

The following are the main results of this paper.

Theorem 1. *Suppose that F satisfies (F_0)-(F_3), then (1.3) admits at least three weak solutions.*

The main ingredient in the proof of Theorem 1 is a version of the mountain pass theorem for functional bounded from below due to Bonanno [11]. We recall this result

Theorem 2. *Let X be a real Banach space, and let $I : X \to \mathbb{R}$ be a continuously Gâteaux-differentiable function wick satisfies (PS) and is bounded from below. Assume that*

(a) There are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\}$$

Then, I admits at least three distinct critical points.

This article is organized as follows. In Section 2 we present preliminaries with the main tools and the functional setting of the problem . In Section 3 we prove the Theorem 1.

2. Preliminaries.

2.1. Fractional operators. In this subsection we introduce some basic definitions of fractional calculus which are used further in this paper. For the proof see [17], [28] and [31].

Definition 1. *(Left and Right Riemann-Liouville fractional integral) Let u be a function defined on $[a,b]$. The left (right) Riemann-Liouville fractional integral of order $\alpha > 0$ for function u is defined by*

$$_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds, \ t\in[a,b],$$
$$_tI_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)ds, \ t\in[a,b],$$

provided in both cases that the right-hand side is pointwise defined on $[a,b]$.

Definition 2. *(Left and Right Riemman-Liouville fractional derivative) Let u be a function defined on $[a,b]$. The left and right Riemann - Liouville fractional derivatives of order $\alpha > 0$ for function u denoted by $_aD_t^\alpha u(t)$ and $_tD_b^\alpha u(t)$, respectively, are defined by*

$$_aD_t^\alpha u(t) = \frac{d^n}{dt^n}\, {_aI_t^{n-\alpha}} u(t),$$
$$_tD_b^\alpha u(t) = (-1)^n\frac{d^n}{dt^n}\, {_tI_b^{n-\alpha}} u(t),$$

where $t\in[a,b]$, $n-1\leq\alpha<n$ and $n\in\mathbb{N}$.

Remark 1. *According to definition 1 and definition 2, if α becomes an integer $n\in\mathbb{N}$ we recover the usual definitions, namely*

$$_aI_t^n u(t) = \frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1}u(s)ds, \ t\in[a,b], \ n\in\mathbb{N},$$
$$_tI_b^n u(t) = \frac{1}{\Gamma(n)}\int_t^b (s-t)^{n-1}u(s)ds, \ t\in[a,b], \ n\in\mathbb{N},$$

and

$$_aD_t^n u(t) = u^{(n)}(t), \ t\in[a,b],$$
$$_tD_b^\alpha u(t) = (-1)^n u^{(n)}(t), \ t\in[a,b].$$

Remark 2. *If $u \in C[a,b]$, it is obvious that the Riemann-Liouville fractional integral of order $\alpha > 0$ is bounded in $[a,b]$. On the other hand, following [17], it is knows that the Riemann-Liouville fractional* *derivative of order* $\alpha \in [n-1, n)$ *exists a.e. on* $[a.b]$ *if* $u \in AC^n[a,b]$*, where* $AC^n[a,b]$ *is the space of functions* u *such that* $u \in C^{n-1}([a,b])$ *and* $u^{(n-1)}$ *is absolutely continuous on* $[a,b]$.

Now we enounce some properties of the Riemann-Liouville fractional integral and derivative operators.

Theorem 3.

$$
\begin{aligned}
{}_aI_t^{\alpha}({}_aI_t^{\beta}u(t)) &= {}_aI_t^{\alpha+\beta}u(t) \ \ \textit{and} \\
{}_tI_b^{\alpha}({}_tI_b^{\beta}u(t)) &= {}_tI_b^{\alpha+\beta}u(t), \ \ \forall \alpha, \beta > 0,
\end{aligned}
$$

in any point $t \in [a,b]$ *for continuous function* u *and for almost every point in* $[a,b]$ *if the function* $u \in L^1[a,b]$.

Theorem 4. *(Left inverse) Let* $u \in L^1[a,b]$ *and* $\alpha > 0$*,*

$$
\begin{aligned}
{}_aD_t^{\alpha}({}_aI_t^{\alpha}u(t)) &= u(t), \ \textit{a.e.}\ t \in [a,b] \ \ \textit{and} \\
{}_tD_b^{\alpha}({}_tI_b^{\alpha}u(t)) &= u(t), \ \textit{a.e.}\ t \in [a,b].
\end{aligned}
$$

Theorem 5. *For* $n-1 \leq \alpha < n$*, if the left and right Riemann-Liouville fractional derivatives* ${}_aD_t^{\alpha}u(t)$ *and* ${}_tD_b^{\alpha}u(t)$*, of the function* u *are integral on* $[a,b]$*, then*

$$
\begin{aligned}
{}_aI_t^{\alpha}({}_aD_t^{\alpha}u(t)) &= u(t) - \sum_{k=}^{n} [{}_aI_t^{k-\alpha}u(t)]_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}, \\
{}_tI_b^{\alpha}({}_tD_b^{\alpha}u(t)) &= u(t) - \sum_{k=1}^{n} [{}_tI_n^{k-\alpha}u(t)]_{t=b} \frac{(-1)^{n-k}(b-t)^{\alpha-k}}{\Gamma(\alpha-k+1)},
\end{aligned}
$$

for $t \in [a,b]$.

Theorem 6. *(Integration by parts)*

$$
\int_a^b [{}_aI_t^{\alpha}u(t)]v(t)dt = \int_a^b u(t)\,{}_tI_b^{\alpha}v(t)dt, \ \alpha > 0, \tag{2.1}
$$

provided that $u \in L^p[a,b]$*,* $v \in L^q[a,b]$ *and*

$$
p \geq 1, \ q \geq 1 \ \ \textit{and} \ \ \frac{1}{p} + \frac{1}{q} < 1 + \alpha \ \ \textit{or} \ \ p \neq 1, \ q \neq 1 \ \ \textit{and} \ \ \frac{1}{p} + \frac{1}{q} = 1 + \alpha.
$$

$$
\int_a^b [{}_aD_t^{\alpha}u(t)]v(t)dt = \int_a^b u(t)\,{}_tD_b^{\alpha}v(t)dt, \ \ 0 < \alpha \leq 1, \tag{2.2}
$$

provided the boundary conditions

$$
\begin{aligned}
u(a) = u(b) = 0, \ u' \in L^{\infty}[a,b], \ v \in L^1[a,b] \ \ \textit{or} \\
v(a) = v(b) = 0, \ v' \in L^{\infty}[a,b], \ u \in L^1[a,b],
\end{aligned}
$$

are fulfilled.

2.1.1. Fractional Derivative Space. In order to establish a variational structure which enables us to reduce the existence of solutions of BVP (1.1) to the one of finding critical points of corresponding functional, it is necessary to construct appropriate function spaces. For this setting we take some results from [21].

Let us recall that for any fixed $t \in [0,T]$ and $1 \leq p < \infty$,

$$
\begin{aligned}
\|u\|_{L^p[0,t]} &= \left(\int_0^t |u(s)|^p ds \right)^{1/p}, \\
\|u\|_{L^p} &= \left(\int_0^T |u(s)|^p ds \right)^{1/p} \ \ \text{and} \\
\|u\|_{\infty} &= \max_{t \in [0,T]} |u(t)|.
\end{aligned}
$$

Definition 3. *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative spaces $E_0^{\alpha,p}$ is defined by*

$$E_0^{\alpha,p} = \{u \in L^p[0,T]/ \ {}_0D_t^\alpha u \in L^p[0,T] \text{ and } u(0) = u(T) = 0\} = \overline{C_0^\infty[0,T]}^{\|\cdot\|_{\alpha,p}}.$$

where $\|.\|_{\alpha,p}$ is defined by

$$\|u\|_{\alpha,p}^p = \int_0^T |u(t)|^p dt + \int_0^T |{}_0D_t^\alpha u(t)|^p dt. \tag{2.3}$$

Proposition 1. *[21] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 1. *[31] Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. For any $u \in L^p[0,T]$ we have*

$$\|{}_0I_\xi^\alpha u\|_{L^p[0,t]} \leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|u\|_{L^p[0,t]}, \text{ for } \xi \in [0,t], \ t \in [0,T]. \tag{2.4}$$

Proposition 2. *[21] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > 1/p$ we have*

$${}_0I_t^\alpha({}_0D_t^\alpha u(t)) = u(t).$$

Moreover, $E_0^{\alpha,p} \in C[0,T]$.

Proposition 3. *[21] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > 1/p$ we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0D_t^\alpha u\|_{L^p}. \tag{2.5}$$

If $\alpha > 1/p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\|{}_0D_t^\alpha u\|_{L^p}. \tag{2.6}$$

According to (2.5), we can consider in $E_0^{\alpha,p}$ the following norm

$$\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u\|_{L^p}, \tag{2.7}$$

and (2.7) is equivalent to (2.3).

Proposition 4. *[21] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and $\{u_k\} \rightharpoonup u$ in $E_0^{\alpha,p}$. Then $u_k \to u$ in $C[0,T]$, i.e.*

$$\|u_k - u\|_\infty \to 0, \ k \to \infty.$$

We denote by $E^\alpha = E_0^{\alpha,2}$, this is a Hilbert space with respect to the norm $\|u\|_\alpha = \|u\|_{\alpha,2}$ given by (2.7).

3. Main result.

We recall the notion of solution for (1.3).

Definition 4. *$u \in E^\alpha$ be a weak solution of (1.3) if*

$$\int_0^T \left({}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t) + \langle \nabla F(t,u(t)), v(t)\rangle\right) dt = 0, \ \text{ for any } v \in E^\alpha. \tag{3.1}$$

Under (F_1) the corresponding functional I on E^α given by

$$I(u) = \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 dt + \int_0^T F(t,u(t))dt, \tag{3.2}$$

is continuously differentiable and weakly lower semicontinuous on E^α (see [21]), and

$$I'(u)v = \int_0^T {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t)dt + \int_0^T (\nabla F(t,u(t)), v(t))dt, \ \ \forall v \in E^\alpha.$$

Therefore critical points of I are weak solutions of (1.3).
Proof of Theorem 1 Our aim is to apply Theorem 2. To this end, let X be the fractional space E^α and let I be the functional defined in (3.2). We are going to prove that I is bounded from below and that it satisfies the (PS) condition. In fact: By (F_2), there are a constants K_0 and C_1 such that

$$F(t,x) \geq -K_0 + C_1,$$

Therefore

$$\begin{aligned} I(u) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 dt + \int_0^T F(t,u(t))dt \\ &\geq \frac{1}{2}\|u\|_\alpha^2 - K_0T + C_1T, \end{aligned} \tag{3.3}$$

hence $I(u) \to \infty$ as $\|u\|_\alpha \to \infty$ and I is bounded from below.
To show that I satisfies the Palais - Smale condition, let $\{u_k\} \in E^\alpha$ such that

$$|I(u_k)| \leq M, \quad \lim_{k\to\infty} I'(u_k) = 0. \tag{3.4}$$

By (3.3), $\{u_k\}$ is bounded in E^α. Since E^α is reflexive space, going if necessary to a subsequence, we may assume that $u_k \rightharpoonup u$ in E^α, thus we have

$$\begin{aligned} \langle I'(u_k) - I'(u), u_k - u\rangle &= \langle I'(u_k), u_k - u\rangle - \langle I'(u), u_k - u\rangle \\ &\leq \|I'(u_k)\|\|u_k - u\|_\alpha - \langle I'(u), u_k - u\rangle \to 0. \end{aligned} \tag{3.5}$$

as $k \to \infty$. Moreover according (2.6) and Proposition 4, we get that u_k is bounded in $C[0,T]$ and

$$\lim_{k\to\infty} \|u_k - u\|_\infty = 0.$$

Hence we have

$$\int_0^T \langle \nabla F(t,u_k(t)) - \nabla F(t,u(t)), u_k(t) - u(t)\rangle dt \to 0, \quad k \to \infty.$$

Moreover, an easy computation show that

$$\begin{aligned} \langle I'(u_k) - I'(u), u_k - u\rangle &= \|u_k - u\|_\alpha^2 \\ &\quad + \int_0^T \langle \nabla F(t,u_k(t)) - \nabla F(t,u(t)), u_k(t) - u(t)\rangle dt. \end{aligned}$$

So $\|u_k - u\|_\alpha \to 0$ as $k \to \infty$. That is $\{u_k\}$ converges strongly to u in E^α.
Now, let

$$k = \frac{T^{\alpha - \frac{1}{2}}}{\Gamma(\alpha)(2\alpha - 1)^{1/2}}$$

given by 2.6. Put $r = \frac{\delta}{k}$ where δ is given by (F_3), $u_0(t) = d_0$ and $u_1(t) = d_1$ for all $t \in [0,T]$. Clearly, $u_0, u_1 \in X$,

$$I(u_0) = \int_0^T F(t,d_0)dt \quad \text{and} \quad I(u_1) = \int_0^T F(t,d_1)dt.$$

Moreover, fix $u \in X$ such that $\|u - u_0\|_\alpha = r$. Taking (2.6) into account, one has

$$\|u - u_0\|_\infty \leq k\|u - u_0\|_\alpha = kr = \delta.$$

Therefore,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_\alpha^2 + \int_0^T F(t,u(t))dt \\ &\geq \int_0^T F(t,u(t))dt \geq \int_0^T \min_{|\xi - d_0| < \delta} F(t,\xi)dt \end{aligned}$$

for all $u \in X$ such that $\|u - u_0\|_\alpha = r$; that is,

$$\inf_{\|u-u_0\|_\alpha = r} I(u) \geq \int_0^T \min_{|\xi - d_0| \leq \delta} F(t, \xi) dt.$$

From $(F_3)_1$ and $(F_3)_2$, and owing to our setting, one has

$$\inf_{\|u-u_0\|=r} I(u) \geq I(u_0) \geq I(u_1).$$

Hence, Theorem 2 ensures the conclusion.

REFERENCES

[1] Agarwal R., De Andrade B. and Cuevas C., *On type of periodicity and ergodicity to a class of fractional order differential equations*, Adv. Difference Equ. ID 179750, 25 pages(2010).

[2] Agarwal R, Benchohra M. and Hamani S., *Boundary value problems for fractional differential equations*, Georg. Math. J., **16**, 3, 401-411(2009).

[3] Agarwal R., Belmekki M. and Benchohra M., *A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative*, Adv. Difference Equ. **9**, 47 pages(2009).

[4] Agarwal R, Dos Santos J. and Cuevas C., *Analytic resolvent operator and existence results for fractional integro-differential equations*, J. Abs. Diff. Equ. **2**, 2, 26-47(2012).

[5] Agrawal O., Tenreiro Machado J. and Sabatier J., *Fractional derivatives and their application: Nonlinear dynamics*, Springer-Verlag, Berlin, 2004.

[6] Anh A. and Mcvinish R., *Fractional differential equations driven by Lévy noise*, J. Appl. Math. and Stoch. Anal. **16**, 2, 97-119(2003)

[7] Atanackovic T. and Stankovic B., *On a class of differential equations with left and right fractional derivatives*, ZAMM., **87**, 537-539(2007).

[8] Bai Z. and Lü H., *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl.., **311**, 495-505(2005).

[9] Baleanu D. and Trujillo J., *On exact solutions of a class of fractional Euler-Lagrange equations*, Nonlinear Dyn., **52**, 331-335(2008).

[10] Benchohra M., Henderson J., Ntouyas S. and Ouahab A., *Existence results for fractional order functional differential equations with infinite delay*, J. of Math. Anal. Appl., **338**, 2, 1340-1350(2008).

[11] Bonanno G., *A characterization of the mountain pass geometry for functionals bounded from below*, DIE, **25**, No 11-12, 1135-1142 (2012.)

[12] Bonanno G., Rodríguez-López R. and Tersian S. *Existence of solutions to boundary value problem for impulsive fractional differential equations*, Fract. Calc. Appl. Anal., **17**, No 3, 717-744 (2014).

[13] Bonanno G. and Riccobono G. *Multiplicity results for Sturm-Liouville boundary value problems*, Appl. Math. Comput., **210**, 294-297 (2009).

[14] Cuevas C., N'Guérékata G. and Sepulveda A., *Pseudo almost automorphic solutions to fractional differential and integro-differential equations*, CAA, **16**, 1, 131-152(2012).

[15] El-Sayed A. *Fractional order evolution equations*, J. Frac. Cal., **7**, 89-100(1995).

[16] Ervin V. and Roop J., *Variational formulation for the stationary fractional advection dispersion equation*, Numer. Meth. Part. Diff. Eqs, **22**, 58-76(2006).

[17] Kilbas A., Srivastava H. and Trujillo J., *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, vol 204, Amsterdam, 2006.

[18] Klimek M., *Existence and uniqueness result for a certain equation of motion in fractional mechanics*, Bull. Polish Acad. Sci. Tech. Sci., **58**, Nº 4, 573-581(2010).

[19] Hilfer R., *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

[20] Jang W., *The existence of solutions for boundary value problems of fractional differential equations at resonance*, Nonlinear Anal., **74**, 1987-1994(2011).

[21] Jiao F. and Zhou Y., *Existence results for fractional boundary value problem via critical point theory*, Intern. Journal of Bif. and Cahos, **22**, N 4, 1-17(2012).

[22] Lakshmikantham V., *Theory of fractional functional differential equations*, Nonl. Anal., **69**, 3337-3343(2008).

[23] Lakshmikantham V. and Vatsala A., *Basic theory of fractional differential equations*, Nonl. Anal., **69**, 8, 2677-2682(2008).

[24] Mawhin J. and Willen M., *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences 74, Springer, Berlin, 1989.

[25] Metsler R. and Klafter J., *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. A, **37**, 161-208(2004).

[26] Miller K. and Ross B., *An introduction to the fractional calculus and fractional differential equations*, Wiley and Sons, New York, 1993.

[27] N'Guérékata G., *A Cauchy problem for some fractional abstract differential equation with nonlocal conditions*, Non. Anal., **70**, 1873-1876(2009).

[28] Podlubny I., *Fractional differential equations*, Academic Press, New York, 1999.

[29] Rabinowitz P., *Minimax method in critical point theory with applications to differential equations*, CBMS Amer. Math. Soc., No **65**, 1986.

[30] Sabatier J., Agrawal O. and Tenreiro Machado J., *Advances in fractional calculus. Theoretical developments and applications in physics and engineering*, Springer-Verlag, Berlin, 2007.

[31] Samko S., Kilbas A. and Marichev O. *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach, New York, 1993.

[32] Torres C. *Mountain pass solution for fractional boundary value problem*, Journal of Fractional Calculus and Applications, **1** (5), 1-10 (2014) .

[33] West B., Bologna M. and Grigolini P., *Physics of fractal operators*, Springer-Verlag, Berlin, 2003.

[34] Zhang S., *Existence of a solution for the fractional differential equation with nonlinear boundary conditions*, Comput. Math. Appl., **61**, 1202-1208(2011).