Straightening Identities in the Onsager Algebra

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Abstract - The purpose of this work is to formulate and prove some straightening identities in the Onsager algebra. Our identities allow one to rewrite specific products of basis elements as linear combinations of products which are in a different order. Such identities could be helpful in understanding the representation theory of the Onsager algebra.

Keywords : Lie algebras; Onsager algebra

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1 Introduction

The Onsager Algebra is named after the Nobel Prize winning physical chemist and theoretical physicist, Lars Onsager. He is most known for the analytic description of the two dimensional Ising model in a zero magnetic field, [9]. This model was used to describe ferromagnetism in statistical mechanics. It was also able to model critical behaviors of many different physical systems.

More appropriate mathematics in quantum mechanics replaced the Onsager algebra and it was unused until the formation of the Dolan-Grady relations, [3]. These relations utilized a linear combination of variables with non-linear properties which complemented the Hamiltonian in quantum mechanics, [6]. In 1978, Perk proved that the Onsager algebra and Dolan-Grady relations were identical, [10]. El-Chaar summarized much of what was known about the Onsager algebra and through a series of proofs demonstrated its relationships to other well known mathematics, including the isomorphism between the Onsager algebra and the Dolan-Grady relations, [4].

The Poincaré-Birkhoff-Witt (PBW) Theorem is a well-known result, which can be found in standard references such as [1]. It tells us that given a particular order on the basis of a Lie algebra L, we can reorder any product into a linear combination of products in that order, but it does not tell us what exact linear combination we will get after reordering, meaning straightening the product. Straightening identities, which are explicit formulas detailing the results of such reordering, are needed to understand the representation theory of the Lie algebra in positive characteristic. Such identities for a simple complex Lie algebra $\mathfrak{g}$ were formulated and proved by Kostant in [7]. In 1978, Howard Garland extended these identities in a limited way to the loop algebras, $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, [5]. The Onsager algebra is a subalgebra of the loop algebra for $\mathfrak{g} = \mathfrak{sl}_2$. In 2011, the first author generalized Garland's identities to the general map algebra case

$\mathfrak{g} \otimes A$, where A is any associative commutative complex algebra with a unit, [2]. The essence of this work is to formulate and prove such straightening identities for certain products in the universal enveloping algebra of the Onsager algebra.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers, $\mathbb{Z}$ be the set of integers and $\mathbb{N}$ be the set of positive integers. We recall the definition of a Lie algebra over $\mathbb{C}$ given in [1].

Definition 2.1 *A vector space, L, over $\mathbb{C}$, with an operation $[.,.]$ (called the Lie bracket), is a complex Lie Algebra if the following axioms are satisfied for all $x, y, z \in L$ and all $c \in \mathbb{C}$:*

$$\begin{aligned} [x+y, z] &= [x,z] + [y,z] \\ [cx, y] &= c[x,y] \\ [x, y+z] &= [x,y] + [x,z] \\ [x, cy] &= c[x,y] \\ [x,x] &= 0 \\ [x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= 0 \end{aligned}$$

The first four axioms above imply that the Lie bracket is bilinear and the last axiom is called the Jacobi Identity.

Let $\mathbb{C}[t^{\pm 1}] = \mathbb{C}[t, t^{-1}]$ be the set of complex Laurent polynomials in one variable. Then, given a complex Lie algebra L, the loop algebra of L is the Lie algebra $L \otimes \mathbb{C}[t^{\pm 1}]$ with Lie bracket given by bilinearly extending the bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

where $x, y \in L$ and $a, b \in \mathbb{C}[t^{\pm 1}]$.

Let $\{e, f, h\}$ be the standard basis for the Lie algebra $\mathfrak{sl}_2$, consisting of all 2×2 matrices with trace zero and Lie bracket given by $[A, B] = AB - BA$ (the commutator bracket). So

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that $[e, f] = h$.

Definition 2.2 *The Onsager algebra of $\mathfrak{sl}_2$, $O(\mathfrak{sl}_2)$, is the Lie subalgebra of $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}]$ which is stabilized by the automorphism $\Gamma : \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}] \to \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}]$ given by*

$$\begin{aligned} e \otimes t^k &\mapsto -f \otimes t^{-k} \\ f \otimes t^k &\mapsto -e \otimes t^{-k} \\ h \otimes t^k &\mapsto -h \otimes t^{-k} \end{aligned}$$

where $k \in \mathbb{Z}$.

Given $j \in \mathbb{Z}$ and $k \in \mathbb{N}$ define

$$x_j := e \otimes t^j - f \otimes t^{-j}$$
$$w_k := h \otimes \left(t^k - t^{-k}\right).$$

Proposition 2.3 (El-Char, [4]) *A basis for $O(\mathfrak{sl}_2)$ is given by the set*

$$\{x_j, w_k \mid \; j \in \mathbb{Z}, \; k \in \mathbb{N}\}.$$

2.1 The universal enveloping algebra

Given a Lie algebra, L, let $U(L)$ be its universal enveloping algebra.

Notice that $[x_n, x_m] = w_{m-n}$, $[w_l, x_m] = 2x_{m+l} - 2x_{m-l}$, and $[w_l, w_k] = 0$. Thus x_m, x_n, and w_l do not commute in $U(O(\mathfrak{sl}_2))$, but w_l and w_k do commute. In this work our preferred order on the basis for $O(\mathfrak{sl}_2)$ will be to first order the x_m in descending order of the subscript and then put any w_l at the end.

Definition 2.4 *Given $u \in U(O(\mathfrak{sl}_2))$, define the kth divided power by*

$$u^{(k)} := \frac{u^k}{k!} \in U(O(\mathfrak{sl}_2)).$$

3 Straightening Identities

Definition 3.1 *Define the tangent numbers, T_n, as follows:*

$$\tan x = \sum_{n=1}^{\infty} \frac{T_n x^{2n-1}}{(2n-1)!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \dots.$$

In other words, the T_n are numerators of the numerical coefficients in the Maclaurin series for the tangent function. So $T_1 = 1$, $T_2 = 2$, $T_3 = 16$, $T_4 = 272$, etc.

Before stating and proving our straightening identities theorem we state and prove the following necessary lemma, which, to our knowledge, has not been proven previously.

Lemma 3.2 *For all $b, r \in \mathbb{N}$ and $c, d \in \{0, 1\}$ not both 0,*

$$\sum_{i=0}^{b-1} 2T_{i+1}T_{b-i} \binom{r}{2i+c} \binom{r-2i-c}{2(b-i-1)+d} = (c+d)T_{b+1} \binom{r}{2b-2+c+d}.$$

Proof. If $2b - 2 + c + d > r$, notice that both the right hand side binomial coefficient and the second binomial coefficient on the left hand side are degenerate, meaning that they have the form $\binom{n}{k}$ where $k > n$, and hence that they are equal to 0 by definition. So assume that $r \geq 2b-2+c+d$. Then none of the binomial coefficients are degenerate and we have:

$$\sum_{i=0}^{b-1} 2T_{i+1}T_{b-i}\binom{r}{2i+c}\binom{r-2i-c}{2(b-i-1)+d}$$

$$= \binom{r}{2b-2+c+d}\sum_{i=0}^{b-1} 2T_{i+1}T_{b-i}\binom{2b-2+c+d}{2i+c}.$$

So the Lemma is equivalent to,

$$\sum_{i=0}^{b-1} 2T_{i+1}T_{b-i}\binom{2b-2+c+d}{2i+c} = (c+d)T_{b+1}. \tag{1}$$

If $c = 1$ and $d = 0$ the left-hand side of (1) is

$$\sum_{i=0}^{b-1} 2T_{i+1}T_{b-i}\binom{2b-1}{2i+1}.$$

Replacing i with j via $i = b - j - 1$ gives,

$$\begin{aligned}\sum_{i=0}^{b-1} 2T_{i+1}T_{b-i}\binom{2b-1}{2i+1} &= \sum_{j=0}^{b-1} 2T_{b-j}T_{j+1}\binom{2b-1}{2(b-j-1)+1}\\ &= \sum_{j=0}^{b-1} 2T_{b-j}T_{j+1}\binom{2b-1}{2b-1-(2(b-j-1)+1)}\\ &= \sum_{j=0}^{b-1} 2T_{b-j}T_{j+1}\binom{2b-1}{2b-1-2b+2j+1}\\ &= \sum_{j=0}^{b-1} 2T_{b-j}T_{j+1}\binom{2b-1}{2j},\end{aligned}$$

which is equivalent to the left-hand side of (1) in the case $c = 0$ and $d = 1$. So, without loss of generality, we may assume that $c = 1$. Thus it suffices to show that, for all $b \in \mathbb{N}$ and $d \in \{0, 1\}$,

$$\sum_{i=0}^{b-1} 2T_{i+1}T_{b-i}\binom{2b-1+d}{2i+1} = (d+1)T_{b+1}. \tag{2}$$

If $d = 1$, equation (2) can be derived from the identity $\tan^2 x = \sec^2 x - 1$ using the corresponding Maclaurin series. If $d = 0$, it can be derived from the $d = 1$ case using Pascal's Rule, which is

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k},$$

for all $n, k \in \mathbb{N}$. □

Theorem 3.3 *For all $r, s, l \in \mathbb{N}$ and $m, n \in \mathbb{Z}$ such that $m > n$, the following identities hold in $U(O(\mathfrak{sl}_2))$.*

$$(i)_r \qquad x_n x_m^r = x_m^r x_n + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_{2m-n} x_m^{r-2j} - x_m^{r-2j} x_n \right) + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1} \binom{r}{2k+1} x_m^{r-2k-1} w_{m-n}.$$

$$(ii)_r \qquad x_n^r x_m = x_m x_n^r + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_m x_n^{r-2j} - x_n^{r-2j} x_{2n-m} \right) + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1} \binom{r}{2k+1} x_n^{r-2k-1} w_{m-n}.$$

$$(iii)_r \qquad w_l x_m^r = x_m^r w_l + 2 \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1} \binom{r}{2k+1} \left(x_{m+l} x_m^{r-2k-1} - x_m^{r-2k-1} x_{m-l} \right).$$

$$(iv)_s \qquad w_l^s x_m = \sum_{k=0}^{s} 2^{s-k} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^j x_{m+(s-k-2j)l} w_l^k.$$

Proof. We will proceed by induction first on r and then on s. The base cases $(i)_1$, $(ii)_1$, $(iii)_1$, and $(iv)_1$ can be shown by straightforward calculations.

Let $r \in \mathbb{N}$ and assume for all $u \in \mathbb{N}$ with $1 \leq u \leq r$, that $(i)_u$, $(ii)_u$ and $(iii)_u$ hold. Then, in order to prove $(iii)_{r+1}$, we have:

$$\begin{aligned}
w_l x_m^{r+1} &= w_l x_m^r x_m \\
&= x_m^r w_l x_m + 2 \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1} \binom{r}{2k+1} \left(x_{m+l} x_m^{r-2k-1} - x_m^{r-2k-1} x_{m-l} \right) x_m \qquad \text{by } (iii)_r \\
&= x_m^r \left(x_m w_l + 2x_{m+l} - 2x_{m-l} \right) \\
&\quad + 2 \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1} \binom{r}{2k+1} \left(x_{m+l} x_m^{r-2k} - x_m^{r-2k-1} x_{m-l} x_m \right) \\
&= x_m^{r+1} w_l + 2x_m^r x_{m+l} - 2x_m^r x_{m-l} \\
&\quad + 2 \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1} \binom{r}{2k+1} \left(x_{m+l} x_m^{r-2k} - x_m^{r-2k-1} \left(x_m x_{m-l} + w_l \right) \right)
\end{aligned}$$

$$
\begin{aligned}
&= x_m^{r+1} w_l + 2x_{m+l}x_m^r + 2\sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{m+l}x_m^{r-2j} - x_m^{r-2j}x_{m-l}\right) \\
&\qquad + 2\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1} x_m^{r-2k-1} w_l \\
&\qquad - 2x_m^r x_{m-l} + 2\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}\left(x_{m+l}x_m^{r-2k} - x_m^{r-2k}x_{m-l}\right) \\
&\qquad - 2\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1} x_m^{r-2k-1} w_l \qquad \text{by } (ii)_r \\
&= x_m^{r+1} w_l + 2\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}\left(x_{m+l}x_m^{r-2k} - x_m^{r-2k}x_{m-l}\right) \\
&\qquad + 2\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}\left(x_{m+l}x_m^{r-2k} - x_m^{r-2k}x_{m-l}\right) \\
&= x_m^{r+1} w_l + 2\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\left(\binom{r}{2k} + \binom{r}{2k+1}\right)\left(x_{m+l}x_m^{r-2k} - x_m^{r-2k}x_{m-l}\right) \\
&\qquad + 2((r+1) \mod 2) T_{\lfloor \frac{r}{2} \rfloor}\left(x_{m+l} - x_{m-l}\right) \\
&= x_m^{r+1} w_l + 2\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r+1}{2k+1}\left(x_{m+l}x_m^{r-2k} - x_m^{r-2k}x_{m-l}\right) \\
&\qquad + 2((r+1) \mod 2) T_{\lfloor \frac{r}{2} \rfloor}\left(x_{m+l} - x_{m-l}\right) \qquad \text{by Pascal's Rule} \\
&= x_m^{r+1} w_l + 2\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r+1}{2k+1}\left(x_{m+l}x_m^{r-2k} - x_m^{r-2k}x_{m-l}\right).
\end{aligned}
$$

This proves $(iii)_{r+1}$.

In order to prove $(i)_{r+1}$, we have

$$
\begin{aligned}
x_n x_m^{r+1} &= x_n x_m^r x_m \\
&= x_m^r x_n x_m + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r-2j} - x_m^{r-2j}x_n\right) x_m \\
&\qquad + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1} x_m^{r-2k-1} w_{m-n} x_m \text{ by } (i)_r
\end{aligned}
$$

$$= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r-2j}x_n x_m\right)$$

$$+ \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k-1}\left(x_m w_{m-n} + 2x_{2m-n} - 2x_n\right)$$

$$= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n - x_m^{r-2j}w_{m-n}\right)$$

$$+ \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}\left(x_m^{r-2k}w_{m-n} + 2x_m^{r-2k-1}x_{2m-n} - 2x_m^{r-2k-1}x_n\right)$$

$$= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n - x_m^{r-2j}w_{m-n}\right)$$

$$+ \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}\Bigg(x_m^{r-2k}w_{m-n} + 2x_{2m-n}x_m^{r-2k-1}$$

$$+ \sum_{j=1}^{\lfloor \frac{r-2k-1}{2} \rfloor} 2T_{j+1}\binom{r-2k-1}{2j}\left(x_{2m-n}x_m^{r-2k-2j-1} - x_m^{r-2k-2j-1}x_n\right)$$

$$+ \sum_{l=0}^{\lfloor \frac{r-2k-2}{2} \rfloor} 2T_{l+1}\binom{r-2k-1}{2l+1}x_m^{r-2k-2l-2}w_{m-n} - 2x_m^{r-2k-1}x_n\Bigg) \qquad \text{by } (ii)_{r-2k-1}$$

$$= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right)$$

$$+ \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} 2T_{k+1}\binom{r}{2k+1}\left(x_{2m-n}x_m^{r-2k-1} - x_m^{r-2k-1}x_n\right)$$

$$+ \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}\sum_{j=1}^{\lfloor \frac{r-2k-1}{2} \rfloor} 2T_{j+1}\binom{r-2k-1}{2j}\left(x_{2m-n}x_m^{r-2k-2j-1} - x_m^{r-2k-2j-1}x_n\right)$$

$$- \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}x_m^{r-2j}w_{m-n} + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n}$$

$$+ \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}\sum_{l=0}^{\lfloor \frac{r-2k-2}{2} \rfloor} 2T_{l+1}\binom{r-2k-1}{2l+1}x_m^{r-2k-2l-2}w_{m-n}$$

$$
\begin{aligned}
&= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j\binom{r}{2j-1}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor}\sum_{j=1}^{\lfloor \frac{r-2k-1}{2} \rfloor} 2T_{k+1}T_{j+1}\binom{r}{2k+1}\binom{r-2k-1}{2j}\left(x_{2m-n}x_m^{r-2k-2j-1} - x_m^{r-2k-2j-1}x_n\right) \\
&\quad - \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n} + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n} \\
&\quad + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor}\sum_{l=0}^{\lfloor \frac{r-2k-2}{2} \rfloor} 2T_{k+1}T_{l+1}\binom{r}{2k+1}\binom{r-2k-1}{2l+1}x_m^{r-2k-2l-2}w_{m-n} \\
&= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j\binom{r}{2j-1}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor}\sum_{l=1}^{\lfloor \frac{r-1}{2} \rfloor - k} 2T_{k+1}T_{l+1}\binom{r}{2k+1}\binom{r-2k-1}{2l}\left(x_{2m-n}x_m^{r-2k-2l-1} - x_m^{r-2k-2l-1}x_n\right) \\
&\quad - \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n} + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n} \\
&\quad + \sum_{j=0}^{\lfloor \frac{r-1}{2} \rfloor}\sum_{l=0}^{\lfloor \frac{r}{2} \rfloor - j - 1} 2T_{j+1}T_{l+1}\binom{r}{2j+1}\binom{r-2j-1}{2l+1}x_m^{r-2j-2l-2}w_{m-n}
\end{aligned}
$$

$$
\begin{aligned}
&= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j\binom{r}{2j-1}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor}\sum_{j=k+2}^{\lfloor \frac{r+1}{2} \rfloor} 2T_{k+1}T_{j-k}\binom{r}{2k+1}\binom{r-2k-1}{2(j-k-1)}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad - \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n} + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n} \\
&\quad + \sum_{j=0}^{\lfloor \frac{r-1}{2} \rfloor}\sum_{k=j+1}^{\lfloor \frac{r}{2} \rfloor} 2T_{j+1}T_{k-j}\binom{r}{2j+1}\binom{r-2j-1}{2(k-j)-1}x_m^{r-2k}w_{m-n} \\
&= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j\binom{r}{2j-1}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{j=2}^{\lfloor \frac{r+1}{2} \rfloor}\sum_{k=0}^{j-2} 2T_{k+1}T_{j-k}\binom{r}{2k+1}\binom{r-2k-1}{2(j-k-1)}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad - \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n} + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n} \\
&\quad + \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor}\sum_{j=0}^{k-1} 2T_{j+1}T_{k-j}\binom{r}{2j+1}\binom{r-2j-1}{2(k-j-1)+1}x_m^{r-2k}w_{m-n}
\end{aligned}
$$

$$
\begin{aligned}
&= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor}\sum_{k=0}^{j-1} 2T_{k+1}T_{j-k}\binom{r}{2k+1}\binom{r-2k-1}{2(j-k-1)}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad - \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n} + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n} \\
&\quad + \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor}\sum_{j=0}^{k-1} 2T_{j+1}T_{k-j}\binom{r}{2j+1}\binom{r-2j-1}{2(k-j-1)+1}x_m^{r-2k}w_{m-n} \\
&= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1}\binom{r}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} T_{j+1}\binom{r}{2j-1}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad - \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n} + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n} \\
&\quad + \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} 2T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n} \text{ by Lemma 3.2} \\
&= x_m^{r+1}x_n + x_m^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} T_{j+1}\left(\binom{r}{2j} + \binom{r}{2j-1}\right)\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n} + \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n} \\
&= x_m^{r+1}x_n + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} T_{j+1}\left(\binom{r}{2j} + \binom{r}{2j-1}\right)\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} T_{k+1}\binom{r}{2k+1}x_m^{r-2k}w_{m-n} + \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1}\binom{r}{2k}x_m^{r-2k}w_{m-n}
\end{aligned}
$$

$$= x_m^{r+1}x_n + \sum_{j=1}^{\lfloor\frac{r+1}{2}\rfloor} T_{j+1}\left(\binom{r}{2j} + \binom{r}{2j-1}\right)\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right)$$

$$+ \sum_{k=0}^{\lfloor\frac{r}{2}\rfloor} T_{k+1}\left(\binom{r}{2k+1} + \binom{r}{2k}\right) x_m^{r-2k}w_{m-n}$$

$$= x_m^{r+1}x_n + \sum_{j=1}^{\lfloor\frac{r+1}{2}\rfloor} T_{j+1}\binom{r+1}{2j}\left(x_{2m-n}x_m^{r+1-2j} - x_m^{r+1-2j}x_n\right)$$

$$+ \sum_{k=0}^{\lfloor\frac{r}{2}\rfloor} T_{k+1}\binom{r+1}{2k+1} x_m^{r-2k}w_{m-n} \text{ by Pascal's Rule.}$$

This completes the proof of $(i)_{r+1}$.

In order to prove $(ii)_{r+1}$, we have

$$x_n^{r+1}x_m = x_n x_n^r x_m$$

$$= x_n x_m x_n^r + \sum_{j=1}^{\lfloor\frac{r}{2}\rfloor} T_{j+1}\binom{r}{2j}\left(x_n x_m x_n^{r-2j} - x_n^{r+1-2j}x_{2n-m}\right)$$

$$+ \sum_{k=0}^{\lfloor\frac{r-1}{2}\rfloor} T_{k+1}\binom{r}{2k+1} x_n^{r-2k}w_{m-n} \text{ by } (ii)_r$$

$$= x_m x_n^{r+1} + w_{m-n}x_n^r + \sum_{j=1}^{\lfloor\frac{r}{2}\rfloor} T_{j+1}\binom{r}{2j}\left(x_m x_n^{r+1-2j} + w_{m-n}x_n^{r-2j} - x_n^{r+1-2j}x_{2n-m}\right)$$

$$+ \sum_{k=0}^{\lfloor\frac{r-1}{2}\rfloor} T_{k+1}\binom{r}{2k+1} x_n^{r-2k}w_{m-n}$$

$$= x_m x_n^{r+1} + x_n^r w_{m-n} + 2\sum_{k=0}^{\lfloor\frac{r-1}{2}\rfloor} T_{k+1}\binom{r}{2k+1}\left(x_m x_n^{r-2k-1} - x_n^{r-2k-1}x_{2n-m}\right)$$

$$+ \sum_{j=1}^{\lfloor\frac{r}{2}\rfloor} T_{j+1}\binom{r}{2j}\Bigg(x_m x_n^{r+1-2j} + x_n^{r-2j}w_{m-n}$$

$$+ 2\sum_{k=0}^{\lfloor\frac{r-2j-1}{2}\rfloor} T_{k+1}\binom{r-2j}{2k+1}\left(x_m x_n^{r-2j-2k-1} - x_n^{r-2j-2k-1}x_{2n-m}\right) - x_n^{r+1-2j}x_{2n-m}\Bigg)$$

$$+ \sum_{k=0}^{\lfloor\frac{r}{2}\rfloor} T_{k+1}\binom{r}{2k+1} x_n^{r-2k}w_{m-n} \text{ by } (iii)_r \text{ and } (iii)_{r-2j}$$

$$= x_m x_n^{r+1} + x_n^r w_{m-n} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j \binom{r}{2j-1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right)$$

$$+ \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} x_n^{r-2j} w_{m-n}$$

$$+ \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} 2 \sum_{k=0}^{\lfloor \frac{r-2j-1}{2} \rfloor} T_{k+1} \binom{r-2j}{2k+1} \left(x_m x_n^{r-2j-2k-1} - x_n^{r-2j-2k-1} x_{2n-m}\right)$$

$$+ \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r}{2k+1} x_n^{r-2k} w_{m-n}$$

$$= x_m x_n^{r+1} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j \binom{r}{2j-1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right)$$

$$+ \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right)$$

$$+ \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{r-2j-1}{2} \rfloor} 2T_{j+1} T_{k+1} \binom{r}{2j} \binom{r-2j}{2k+1} \left(x_m x_n^{r-2j-2k-1} - x_n^{r-2j-2k-1} x_{2n-m}\right)$$

$$+ \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r}{2k+1} x_n^{r-2k} w_{m-n} + \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r}{2k} x_n^{r-2k} w_{m-n}$$

$$= x_m x_n^{r+1} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j \binom{r}{2j-1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right)$$

$$+ \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right)$$

$$+ \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{r-2i-1}{2} \rfloor} 2T_{i+1} T_{k+1} \binom{r}{2i} \binom{r-2i}{2k+1} \left(x_m x_n^{r-2i-2k-1} - x_n^{r-2i-2k-1} x_{2n-m}\right)$$

$$+ \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \left(\binom{r}{2k+1} + \binom{r}{2k}\right) x_n^{r-2k} w_{m-n}$$

$$
\begin{aligned}
&= x_m x_n^{r+1} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j \binom{r}{2j-1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) \\
&\quad + \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} \sum_{j=i+1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_{i+1} T_{j-i} \binom{r}{2i} \binom{r-2i}{2(j-i-1)+1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r+1}{2k+1} x_n^{r-2k} w_{m-n} \text{ by Pascal's Rule} \\
&= x_m x_n^{r+1} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2T_j \binom{r}{2j-1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} \sum_{i=1}^{j-1} 2T_{i+1} T_{j-i} \binom{r}{2i} \binom{r-2i}{2(j-i-1)+1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r+1}{2k+1} x_n^{r-2k} w_{m-n} \\
&= x_m x_n^{r+1} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} \sum_{i=0}^{j-1} 2T_{i+1} T_{j-i} \binom{r}{2i} \binom{r-2i}{2(j-i-1)+1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m}\right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r+1}{2k+1} x_n^{r-2k} w_{m-n}
\end{aligned}
$$

$$
\begin{aligned}
&= x_m x_n^{r+1} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} T_{j+1} \binom{r}{2j} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m} \right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} T_{j+1} \binom{r}{2j-1} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m} \right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r+1}{2k+1} x_n^{r-2k} w_{m-n} \text{ by Lemma 3.2} \\
&= x_m x_n^{r+1} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} T_{j+1} \left(\binom{r}{2j} + \binom{r}{2j-1} \right) \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m} \right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r+1}{2k+1} x_n^{r-2k} w_{m-n} \\
&= x_m x_n^{r+1} + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} T_{j+1} \binom{r+1}{2j} \left(x_m x_n^{r+1-2j} - x_n^{r+1-2j} x_{2n-m} \right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} T_{k+1} \binom{r+1}{2k+1} x_n^{r-2k} w_{m-n} \text{ by Pascal's Rule.}
\end{aligned}
$$

This completes the proof of $(ii)_{r+1}$.

We now prove $(iv)_s$ by induction on s.

$$
\begin{aligned}
w_l^{s+1} x_m &= w_l w_l^s x_m \\
&= \sum_{k=0}^{s} 2^{s-k} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^j w_l x_{m+(s-k-2j)l} w_l^k \text{ by } (iv)_s \\
&= \sum_{k=0}^{s} 2^{s-k} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^j \big(x_{m+(s-k-2j)l} w_l - 2x_{m+(s-k-2j-1)l} \\
&\qquad + 2x_{m+(s-k-2j+1)l} \big) w_l^k
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{s} 2^{s-k} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^j x_{m+(s-k-2j)l} w_l^{k+1} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^{j+1} x_{m+(s-k-2j-1)l} w_l^{k} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^{j} x_{m+(s-k-2j+1)l} w_l^{k} \\
&= \sum_{k=0}^{s} 2^{s-k} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^j x_{m+(s-k-2j)l} w_l^{k+1} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} \sum_{j=1}^{s-k+1} \binom{s-k}{j-1} (-1)^{j} x_{m+(s-k-2j+1)l} w_l^{k} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^{j} x_{m+(s-k-2j+1)l} w_l^{k} \\
&= \sum_{k=0}^{s} 2^{s-k} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^j x_{m+(s-k-2j)l} w_l^{k+1} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} \sum_{j=1}^{s-k} \binom{s-k}{j-1} (-1)^{j} x_{m+(s-k-2j+1)l} w_l^{k} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} \sum_{j=1}^{s-k} \binom{s-k}{j} (-1)^{j} x_{m+(s-k-2j+1)l} w_l^{k} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} x_{m+(s-k+1)} w_l^{k} + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} (-1)^{s-k+1} x_{m+(k-s-1)l} w_l^{k} \\
&= \sum_{k=0}^{s} 2^{s-k} \binom{s}{s-k} \sum_{j=0}^{s-k} \binom{s-k}{j} (-1)^j x_{m+(s-k-2j)l} w_l^{k+1} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} \sum_{j=1}^{s-k} \left(\binom{s-k}{j} + \binom{s-k}{j-1} \right) (-1)^{j} x_{m+(s-k-2j+1)l} w_l^{k} \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} x_{m+(s-k+1)} w_l^{k} + \sum_{k=0}^{s} 2^{s-k+1} \binom{s}{s-k} (-1)^{s-k+1} x_{m+(k-s-1)l} w_l^{k}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{s+1} 2^{s-k+1}\binom{s}{s-k+1}\sum_{j=0}^{s-k+1}\binom{s-k+1}{j}(-1)^j x_{m+(s-k-2j+1)l}w_l^k \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1}\binom{s}{s-k}\sum_{j=1}^{s-k}\binom{s-k+1}{j}(-1)^j x_{m+(s-k-2j+1)l}w_l^k \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1}\binom{s}{s-k}x_{m+(s-k+1)}w_l^k + \sum_{k=0}^{s} 2^{s-k+1}\binom{s}{s-k}(-1)^{s-k+1}x_{m+(k-s-1)l}w_l^k \\
&\quad \text{by Pascal's Rule.} \\
&= \sum_{k=1}^{s} 2^{s-k+1}\binom{s}{s-k+1}\sum_{j=1}^{s-k}\binom{s-k+1}{j}(-1)^j x_{m+(s-k-2j+1)l}w_l^k \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1}\binom{s}{s-k}\sum_{j=1}^{s-k}\binom{s-k+1}{j}(-1)^j x_{m+(s-k-2j+1)l}w_l^k \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1}\binom{s}{s-k}x_{m+(s-k+1)}w_l^k + \sum_{k=0}^{s} 2^{s-k+1}\binom{s}{s-k}(-1)^{s-k+1}x_{m+(k-s-1)l}w_l^k \\
&\quad + x_m w_l^{s+1} + \sum_{k=1}^{s} 2^{s-k+1}\binom{s}{s-k+1}(-1)^{s-k+1}x_{m+(k-s-1)l}w_l^k \\
&\quad + \sum_{k=1}^{s} s^{s-k+1}\binom{s}{s-k+1}x_{m+(s-k+1)l}w_l^k \\
&= \sum_{k=1}^{s} 2^{s-k+1}\binom{s}{s-k+1}\sum_{j=1}^{s-k}\binom{s-k+1}{j}(-1)^j x_{m+(s-k-2j+1)l}w_l^k \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1}\binom{s}{s-k}\sum_{j=0}^{s-k+1}\binom{s-k+1}{j}(-1)^j x_{m+(s-k-2j+1)l}w_l^k \\
&\quad + \sum_{k=1}^{s} 2^{s-k+1}\binom{s}{s-k+1}\left((-1)^{s-k+1}x_{m+(k-s-1)l}w_l^k + x_{m+(s-k+1)l}w_l^k\right) + x_m w_l^{s+1} \\
&= \sum_{k=1}^{s} 2^{s-k+1}\binom{s}{s-k+1}\sum_{j=0}^{s-k+1}\binom{s-k+1}{j}(-1)^j x_{m+(s-k-2j+1)l}w_l^k \\
&\quad + \sum_{k=0}^{s} 2^{s-k+1}\binom{s}{s-k}\sum_{j=0}^{s-k+1}\binom{s-k+1}{j}(-1)^j x_{m+(s-k-2j+1)l}w_l^k + x_m w_l^{s+1}
\end{aligned}
$$

$$= \sum_{k=1}^{s} 2^{s-k+1} \left(\binom{s}{s-k+1} + \binom{s}{s-k} \right) \sum_{j=0}^{s-k+1} \binom{s-k+1}{j} (-1)^j x_{m+(s-k-2j+1)l} w_l^k$$

$$+ 2^{s+1} \sum_{j=0}^{s+1} \binom{s+1}{j} (-1)^j x_{m+(s-2j+1)l} + x_m w_l^{s+1}$$

$$= \sum_{k=1}^{s} 2^{s-k+1} \binom{s+1}{s-k+1} \sum_{j=0}^{s-k+1} \binom{s-k+1}{j} (-1)^j x_{m+(s-k-2j+1)l} w_l^k$$

$$+ 2^{s+1} \sum_{j=0}^{s+1} \binom{s+1}{j} (-1)^j x_{m+(s-2j+1)l} + x_m w_l^{s+1} \text{ by Pascal's Rule.}$$

$$= \sum_{k=0}^{s+1} 2^{s-k+1} \binom{s+1}{s-k+1} \sum_{j=0}^{s-k+1} \binom{s-k+1}{j} (-1)^j x_{m+(s-k-2j+1)l} w_l^k.$$

This completes the proof of $(iv)_{s+1}$. □

4 Further Directions

The natural next step is to formulate and prove straightening identities for products of the form: $x_n^r x_m^s$ and $w_l^r x_m^s$. Although we have some partial results in this direction, we have not yet reached this goal.

Another direction in which one could head is to formulate straightening identities in the generalized Onsager algebras, $\mathcal{O}(\mathfrak{g})$, where $\mathfrak{g}$ is any simple complex Lie algebra. See Example 3.9 in [8] for a definition of $\mathcal{O}(\mathfrak{g})$.

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