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Cubic BOI-ideal of BOI-algebra

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Abstract:

In this paper, the notion of cubic BOI-ideal of BOI-algebra is introduced. Some characterization of cubic BOI-ideal of BOI-algebra is given. Several theorems are presented in this regard. The homomorphic image and inverse image of cubic BOI-ideal are studied.

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1. Introduction:

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [7]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [6], Q. P. Hu and X. Li introduced a wide class of abstract: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [16], J. Neggers, S. S. Ahn and H. S. Kim introduced Q-algebras which is a generalization of BCK / BCI-algebras and obtained several results. In 2002, Neggers and Kim [15] introduced a new notion, called a B-algebra, and obtained several results. In 2007, Walendziak [18] introduced a new notion, called a BF-algebra, which is a generalization of B-algebra. In [14], C. B. Kim and H.S. Kim introduced BG-algebra as a generalization of B-algebra. In 2012, R. K. Bandaru [1] introduces a new notion, called BRK-algebra which is a generalization of BCK / BCI / BCH / Q / QS / BM-algebras. Jana et al. ([8-10]) have done a lot of works on *BCK*/*BCI* –algebra. Fuzzy sets, which were introduced by Zadeh [19], deal with possibilistic uncertainty, connected with the imprecision of states, perceptions, and preferences. In [4] we introduce the notion of fuzzy BRK-ideal of BRK-algebra. Several basic properties that are related to fuzzy BRK-ideals are investigated, also see ([2], [3]). In [5] we presented a new notion, called a BOI-algebra which is a generalization of BCI/BCH –algebras. This algebra is related to several classes of algebra, for example, Q/BG/BH/BF-algebras. We define BOI-ideal of BOI-algebra, and then we consider the fuzzification of BOI-ideal of BOI-algebra. Several basic properties that are related to fuzzy BOI-ideals are investigated. Based on the interval-valued fuzzy sets, Jun et al. [12] introduced the notion of cubic subalgebra/ideals in BCK/BCI-algebras, and they investigated several properties. They discussed the relationship between a cubic subalgebra and a cubic ideal. Also, they provided characterizations of a cubic subalgebra/ideal and considered a method to make a new cubic subalgebra from the old one also see [13]. In 2015, expanding the concept of cubic set, C. Jana et al. [11] introduced G-subalgebras of G-algebras. Senapati et al. [17] applied cubic intuitionistic theory to q- ideals of BCI-algebras. The aim of this paper is to apply the concept of cubic set to BOI-ideal of BOI-algebra. The notions of cubic BOI-subalgebra and cubic BOI-Ideal are defined, and a lot of properties are investigated. We discussed the relationship between the cubic level set, cubic BOI-ideal, and BOI-ideal. The homomorphic image and the inverse image of cubic BOI-ideal are studied. Several theorems and basic properties that are related to cubic BOI-ideal are investigated. In section 5, we conclude and present some topics for future research.

2. Preliminaries

Definition 2.1 [5]. A BOI-algebra $(X;*,0)$ (i.e., a nonempty set X with a binary operation "$*$" and a constant 0) satisfying the following axioms:

(M_1) $x*x=0$,

(M_2) $x*(x*y)=y$,

(M_3) $(x*y)*z=(x*z)*y$, for all $x,y,z\in X$.

In X, we can define a binary relation "$\leq$" by $x\leq y$ if and only if $x*y=0$.

Proposition 2.2 [5]. If $(X;*,0)$ is a BOI-algebra, then following conditions hold:

(1) $x*0=x$,

(2) $0*(x*y)=y*x$,

(3) $(z*x)*(z*y)=y*x$

(4) $(x*z)*(y*z)=x*y$,

(5) $(x*y)*(0*y)=x$,

(6) $x*(x*(x*y))=x*y$,

(7) $(x*y)*x=0*y$,

(8) $x*(y*z)=(x*y)*(0*z)$.

Example 2.3. Let $X = \{0,1,2\}$. Define "$*$" on X as the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X;*,0)$ is a BOI-algebra.

Definition 2.4 [5]. A non-empty subset I of a BOI-algebra X is said to be BOI-subalgebra of X, if $x, y \in I$, implies $x * y \in I$.

Definition 2.5 [5]. A non-empty subset M of a BOI-algebra $(X;*,0)$ is called a BOI-ideal if for any $x, y, z \in X$:

B1) $0 \in M$,

B2) $x * y \in M$ and $(x * z) * y \in M$ $imply$ $z \in M$.

Obviously, $\{0\}$ and X are ideals of a BOI-algebra X. we call $\{0\}$ and X the zero ideal and the trivial ideal of X, respectively. An ideal M is said to be proper if $M \neq X$.

Definition 2.6 [5]. Let $(X;*,0)$ be a BOI-algebra. A fuzzy set μ in X is called a fuzzy BOI-ideal of X if it satisfies:

F1) $\mu(0) \geq \mu(x)$,

F2) $\mu(z) \geq \min\{\mu(x * y), \mu((x * z) * y)\}, for\ all\ x, y, z \in X$.

Example 2.7. Let $X = \{0,1,2,3\}$ Define "$*$" on X as the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X;*,0)$ is a BOI-algebra, $\{0,1\}$ is a subalgebra and BOI-ideal of X.

Definition 2.8 [Homomorphism of BOI-algebra]. Let $(X;*,0)$ and $(Y;*',0')$ be BOI-algebras. A mapping $f: X \to Y$ is said to be a homomorphism if $f(x * y) = f(x) *' f(y)$, for all $x, y \in X$.

Proposition 2.9[5]. Let $(X;*,0)$ and $(Y;*',0')$ be BOI-algebras, and the mapping $f: X \to Y$ be a homomorphism of BOI-algebras, then the $Kerf$ is a BOI-ideal.

An interval-valued fuzzy set (briefly i-v fuzzy set)

An i-v fuzzy set A on the set X ($\neq \phi$) is given by $A = \{(x, [\mu_A^L(x), \mu_A^U(x)]), x \in X\}$. (briefly, it is denoted by $A = [\mu_A^L, \mu_A^U]$) where μ_A^L and μ_A^U are any two fuzzy sets in X such that $\mu_A^L \leq \mu_A^U$.

Let $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, and let $D[0,1]$ be the family of all closed sub-interval of $[0,1]$. It is clear if $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$, then $\tilde{\mu}_A(x) = [c, c]$ is in $D[0,1]$. Thus $\tilde{\mu}_A(x) \in [0,1]$, for all $x \in X$. Then the i-v fuzzy set A is given by $A = \{(x, \tilde{\mu}_A(x)), x \in X\}$, where $\tilde{\mu}_A : X \to D[0,1]$. Now we define the refined minimum (briefly r min) and order "$\leq$" on the subintervals $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as follows: $r\min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$, $D_1 \leq D_2 \Leftrightarrow a_1 \leq a_2$ and $b_1 \leq b_2$. Similarly, we can define $\geq$ and $=$.

3. Cubic BOI-ideal

Definition 3.1[12]. Let X be a nonempty set. A cubic set Ψ in a set X is a structure $\Psi = \{\langle x, A(x), \lambda(x)\rangle : x \in X\}$ which is briefly denoted by $\Psi = \langle A, \lambda\rangle$ where $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy set in X and λ is a fuzzy set in X.

Definition 3.2 [12]. A cubic set $\Psi = \langle A, \lambda\rangle$ in a BCK\BCI-algebra X is called a cubic ideal of X if it satisfies:

(a) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$,

(b) $\lambda(0) \leq \lambda(x)$,

(c) $\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$,

(d) $\lambda(x) \leq \max\{\lambda(x * y), \lambda(y)\}$, for all $x, y \in X$.

Definition 3.3. A cubic set $\Psi = \langle A, \lambda\rangle$ in X is called a cubic subalgebra of a BOI-algebra X if it satisfies:

(a) $\tilde{\mu}_A(x * y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$,

(b) $\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\}$, for all $x, y \in X$.

Definition 3.4. Let $(X; *, 0)$ be a BOI-algebra. A cubic set $\Psi = \langle A, \lambda\rangle$ in X is called cubic BOI-ideal of X if it satisfies the following conditions:

(O$_1$) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$,

(O$_2$) $\lambda(0) \leq \lambda(x)$,

(O$_3$) $\tilde{\mu}_A(z) \geq r\min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A((x * z) * y)\}$,

(O$_4$) $\lambda(z) \leq \max\{\lambda(x * y), \lambda((x * z) * y)\}$, for all $x, y, z \in X$.

Example 3.5. Consider a BOI-algebra X = {0, 1, 2, 3} in which the "$*$" operation is given by example 2.7. Define $A = [\mu_A^L, \mu_A^U]$ and λ by

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.5,0.9] & [0.1,0.3] & [0.2,0.4] & [0.2,0.4] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.2 & 0.2 & 0.5 & 0.5 \end{pmatrix}.$$

Then $\Psi = \langle A, \lambda \rangle$ is a cubic BOI-ideal of X.

Example 3.6. Consider a BOI-algebra X = {0, 1, 2, 3} in which the "$*$" operation is given by example 2.7. Define $A = [\mu_A^L, \mu_A^U]$ and λ by

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.4,0.8] & [0.4,0.8] & [0.1,0.3] & [0.1,0.3] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.2 & 0.2 & 0.6 & 0.6 \end{pmatrix}.$$

Then $\Psi = \langle A, \lambda \rangle$ is a cubic BOI-ideal of X.

Lemma 3.7. Let $\Psi = \langle A, \lambda \rangle$ be a cubic BOI-ideal of BOI-algebra X. If $y * x \leq x$ holds in X, then $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y * x)$, and $\lambda(x) \leq \lambda(y * x)$.

Proof. Assume that $y * x \leq x$ holds in X. Then $(y * x) * x = 0$ and by (O$_3$),

$$\tilde{\mu}_A \geq rmin\{\tilde{\mu}_A(x * y), \tilde{\mu}_A((x * z) * y)\}.$$

Thus

$$\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(y * x), \tilde{\mu}_A((y * x) * x)\} = r\min\{\tilde{\mu}_A(y * x), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(y * x). \quad \text{(A)}$$

Similarly, by (O$_4$), $\lambda(z) \leq \max\{\lambda(x * y), \lambda((x * z) * y)\}$. Thus

$$\lambda(x) \leq \max\{\lambda(y * x), \lambda((y * x) * x)\} = \max\{\lambda(y * x), \lambda(0)\} = \lambda(y * x) \quad \text{(B)}$$

From (A) and (B), we get $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y * x)$, and $\lambda(x) \leq \lambda(y * x)$.

This completes the proof.

Lemma 3.8. Let $\Psi = \langle A, \lambda \rangle$ be a cubic BOI-ideal of BOI-algebra *X*. If $x \leq y$ holds in *X*, then $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(0 * y)$ and $\lambda(x) \leq \lambda(0 * y)$.

Proof. If $x \leq y$, then $x * y = 0$. This together with $x * x = 0$, $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, and $\lambda(0) \leq \lambda(x)$. Thus

$$\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A((x * x) * y)\} = r\min\{\tilde{\mu}_A(0), \tilde{\mu}_A((x * x) * y)\}$$

$$= r\min\{\tilde{\mu}_A(0), \tilde{\mu}_A(0 * y)\} = \tilde{\mu}_A(0 * y).$$

Also

$$\lambda(x) \leq \max\{\lambda(x * y), \lambda((x * x) * y)\} = \max\{\lambda(0), \lambda((x * x) * y)\}$$

$$= \max\{\lambda(0), \lambda(0 * y)\} = \lambda(0 * y).$$

Note 3.9. Let $\Psi = \langle A, \lambda \rangle$ and $\Omega = \langle R, \alpha \rangle$ be two cubic sets in a BOI-algebra *X*. Then

$$\Psi \cap \Omega = \{\langle x, r\min\{\mu_A(x), \mu_R(x)\}, \max\{\lambda(x), \alpha(x)\}\rangle : x \in X\}$$
$$= \{\langle x, \mu_A(x) \cap \mu_R(x), \lambda(x) \cup \alpha(x)\}\rangle : x \in X\}.$$

Proposition 3.10. Let $\{\Psi_j\}_{j \in J}$ be a family of cubic BOI-ideals of a BOI-algebra *X*. Then $\bigcap_{j \in J} \Psi_j$ is a cubic BOI-ideal of *X*.

Proof. Let $\{\Psi_j\}_{j \in J}$ be a family of cubic BOI-ideals of a BOI-algebra *X*. Then for any $x, y, z \in X$,

$$\left(\cap \tilde{\mu}_{A_j}\right)(0) = \inf\left(\tilde{\mu}_{A_j}(0)\right) \geq \inf\left(\tilde{\mu}_{A_j}(x)\right) = \left(\cap \tilde{\mu}_{A_j}\right)(x),$$

$$\left(\cap \tilde{\mu}_{A_j}\right)(z) = \inf\left(\tilde{\mu}_{A_j}(z)\right) \geq \inf\left(r\min\{\tilde{\mu}_{A_j}(x * y), \tilde{\mu}_{A_j}((x * z) * y)\}\right)$$

$$= r\min\left(\inf\{\tilde{\mu}_{A_j}(x * y), \tilde{\mu}_{A_j}((x * z) * y)\}\right),$$

$$= r\min\left\{\inf\left(\tilde{\mu}_{A_j}(x * y)\right), \inf\left(\tilde{\mu}_{A_j}((x * z) * y)\right)\right\},$$

$$= r\min\left\{\left(\cap \tilde{\mu}_{A_j}\right)(x * y), \left(\cap \tilde{\mu}_{A_j}\right)((x * z) * y)\right\}.$$

Also

$$\left(\cup \lambda_j\right)(0) = \sup\left(\lambda_j(0)\right) \leq \sup\left(\lambda(x)\right) = \left(\cup \lambda_j\right)(x),$$

$$(\cup \lambda_j)(z) = \sup (\lambda_j(z)) \leq \sup \left(\max\{ \lambda(x*y), \lambda((x*z)*y)\}\right)$$

$$= \max \left(\sup\{ \lambda(x*y), \lambda((x*z)*y)\}\right)$$

$$= \max \left\{\sup \left(\lambda(x*y)\right), \sup \left(\lambda((x*z)*y)\right)\right\}$$

$$= \max \left\{ \left(\cup \lambda_j\right)(0*(x*y)), \left(\cup \lambda_j\right)((x*z)*y)\right\}.$$

This completes the proof.

Definition 3.11. For any $i \in D[0,1]$, and $s \in [0,1]$, let $\Psi = \langle A, \lambda \rangle$ be a cubic set of BOI-algebra *X*. Then the set $\nu(\Psi; i, s) = \{x \in X : \mu_A(x) \geq i, \ \lambda(x) \leq s\}$ is called the cubic level set of $\Psi = \langle A, \lambda \rangle$.

Theorem 3.12. Let $\Psi = \langle A, \lambda \rangle$ be a cubic subset in *X*. If $\Psi = \langle A, \lambda \rangle$ is a cubic BOI-ideal of *X* then for all $i \in D[0,1]$, and $s \in [0,1]$, the set $\nu(\Psi; i, s)$ is either empty or a BOI-ideal of *X*.

Proof. Let $\Psi = \langle A, \lambda \rangle$ be a cubic BOI-ideal of *X*. And let $i \in D[0,1]$, $s \in [0,1]$ be such that $\nu(\Psi; i, s) \neq \phi$. And let $x \in X$ be such that $x \in \nu(\Psi; i, s)$. Then $\mu_A(x) \geq i$ and $\lambda(x) \leq s$.

By definition (2.1), we get

$$\tilde{\mu}_A(0) \geq r\min\{ \tilde{\mu}_A(x*x), \tilde{\mu}_A((x*0)*x)\} = r\min\{ \tilde{\mu}_A(0), \tilde{\mu}_A((x*x)*0)\}$$

$$= r\min\{ \tilde{\mu}_A(0), \tilde{\mu}_A(0)\} \geq i .$$

And

$$\lambda(0) \leq \max\{ \lambda(x*x), \lambda((x*0)*x)\} = \max\{ \lambda(0), \lambda((x*x)*0)\}$$

$$= \max\{ \lambda(0), \lambda(0)\} \leq s .$$

Then $0 \in \nu(\Psi; i, s)$.

Now letting $(x*y)$ and $((x*z)*y) \in \nu(\Psi; i, s)$, that means

$\tilde{\mu}_A(x*y) \geq i$ and $\tilde{\mu}_A((x*z)*y) \geq i$.

Then $\tilde{\mu}_A(z) \geq r\min\{ \tilde{\mu}_A(x*y), \tilde{\mu}_A((x*z)*y)\} = i$.

Also $\lambda(z) \leq \max\{ \lambda(x*y), \lambda((x*z)*y)\} = s$.

Thus $z \in \nu(\Psi; i, s)$. Hence $\nu(\Psi; i, s)$ is a BOI-ideal of *X*.

This completes the proof.

4. Image and Inverse image of Cubic BOI-ideals

Definition 4.1. Let f be a mapping from a set X to a set Y. If $\Psi = \langle A, \lambda \rangle$ is a cubic subset of X, then the cubic subset $\Omega = \langle B, \eta \rangle$ of Y is defined by

$$\tilde{\mu}_A f^{-1}(y) = \tilde{\mu}_B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \tilde{\mu}_A(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases},$$

$$\lambda f^{-1}(y) = \eta(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \lambda(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 1 & \text{otherwise} \end{cases},$$

for all $y \in Y$, is called the image of $\Psi = \langle A, \lambda \rangle$ under *f*.

Similarly, if $\Omega = \langle B, \eta \rangle$ is a cubic subset of *Y*, then the cubic subset $\Psi = \Omega \circ f$ in *X* defined by $\tilde{\mu}_B(f(x)) = \tilde{\mu}_A(x)$ and $\eta(f(x)) = \lambda(x)$ for all $x \in X$, is said to be the inverse image of Ω under *f*.

Theorem 4.2. An onto homomorphic inverse image of a cubic BOI-ideal is also a Cubic BOI-ideal.

Proof. Let $f : X \to X'$ be an onto homomorphism of BOI-algebras, $\Omega = \langle B, \eta \rangle$ be a cubic BOI-ideal of X' and $\Psi = \langle A, \lambda \rangle$ be the inverse image of $\Omega = \langle B, \eta \rangle$ under *f*. Then $\tilde{\mu}_B(f(x)) = \tilde{\mu}_A(x)$ and $\eta(f(x)) = \lambda(x)$ for all $x \in X$. Let $x \in X$.

Then

$$\tilde{\mu}_A(0) = \tilde{\mu}_B(f(0)) \geq \tilde{\mu}_B(f(x)) = \tilde{\mu}_A(x),$$

and

$$\lambda(0) = \eta(f(0)) \leq \eta(f(x)) = \lambda(x).$$

Let $x, y \in X$. Then

$$\begin{aligned} \tilde{\mu}_A(z) = \tilde{\mu}_B(f(z)) &\geq r\min\{\tilde{\mu}_B(f(x) *' f(y)), \tilde{\mu}_B((f(x) *' f(z)) *' f(y))\} \\ &= r\min\{\tilde{\mu}_B(f(x * y)), \tilde{\mu}_B(f((x * z) * y))\} \\ &= r\min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A((x * z) * y)\}, \end{aligned}$$

and

$$\lambda(z)=\eta(f(z))\le \max\{\eta((f(x)*' f(y)),\ \eta((f(x)*' f(z))*' f(y))\}$$

$$=\max\{\eta(f(x*y)),\ \eta(f((x*z)*y))\}$$

$$=\max\{\lambda(x*y),\ \lambda((x*z)*y)\}.$$

This completes the proof.

Definition 4.3 ("sup" and "inf" properties):

A cubic subset $\Psi=\langle A,\lambda\rangle$ of X has "sup" and "inf" properties if for any subset K of X, there exist $m,n\in K$ such that $\tilde{\mu}_A(m)=\sup_{m\in K}\tilde{\mu}_A(m)$, and $\lambda(n)=\inf_{n\in K}\lambda(n)$.

Theorem 4.4. An onto homomorphic image of a cubic BOI-ideal is also a cubic BOI-ideal.

Proof. Let $f:X\to X'$ be an onto homomorphism of BOI-algebras $(X;*,0)$ and $(X';*,0')$, $\Psi=\langle A,\lambda\rangle$ be a cubic BOI-ideal of X with "sup" and "inf" properties and $\Omega=\langle B,\eta\rangle$ is the image of $\Psi=\langle A,\lambda\rangle$ under f. By definition 4.1 we get

$$\tilde{\mu}_B(y')=\tilde{\mu}_A\ f^{-1}(y')=\sup_{x\in f^{-1}(y)}\tilde{\mu}_A(x).$$

And

$$\eta(y')=\lambda\ f^{-1}(y')=\inf_{x\in f^{-1}(y)}\lambda(x)$$

for all $y'\in Y$, $\sup\phi=0$ and $\inf\phi=1$. Since $\Psi=\langle A,\lambda\rangle$ is a cubic BOI-ideal of X, we have $\tilde{\mu}_A(0)\ge\tilde{\mu}_A(x)$ and $\lambda(0)\le\lambda(x)$ for all $x\in X$. Note that $0\in f^{-1}(0')$. Thus, for all $x\in X$,

$$\tilde{\mu}_B(0')=\tilde{\mu}_A f^{-1}(0')=\sup_{t\in f^{-1}(0')}\tilde{\mu}_A(t)=\tilde{\mu}_A(0)\ge\tilde{\mu}_A(x).$$

This implies that $\tilde{\mu}_B(0')\ge\sup_{t\in f^{-1}(x')}\tilde{\mu}_A(t)=\tilde{\mu}_B(x')$ for any $x'\in X'$.

On the other hand, for all $x\in X$, $\eta(0')=\lambda f^{-1}(0')=\inf_{t\in f^{-1}(0')}\lambda(t)=\lambda(0)\le\lambda(x)$.

This implies that $\eta(0')\le\inf_{t\in f^{-1}(x')}\lambda(t)=\eta(x')$ for any $x'\in X'$.

For any $x',y',z'\in X'$, let $x_0\in f^{-1}(x')$, $y_0\in f^{-1}(y')$, and $z_0\in f^{-1}(0')$ be such that

$$\tilde{\mu}_A(z_0)=\sup_{t\in f^{-1}(z')}\tilde{\mu}_A(t),$$

$$\tilde{\mu}_A(x_0 * y_0) = \sup_{t \in f^{-1}(x' * y')} \tilde{\mu}_A(t),$$

and

$$\begin{aligned}\tilde{\mu}_A((x_0 * z_0) * y_0) &= \tilde{\mu}_B\{f((x_0 * z_0) * y_0)\} \\ &= \tilde{\mu}_B((x_0' * z_0') * y_0') \\ &= \sup_{((x_0 * z_0) * y_0) \in f^{-1}((x' * z') * y')} \tilde{\mu}_A((x_0 * z_0) * y_0) \\ &= \sup_{t \in f^{-1}((x' * z') * y')} \tilde{\mu}_A(t).\end{aligned}$$

Thus

$$\begin{aligned}\tilde{\mu}_B(z') &= \sup_{t \in f^{-1}(z')} \tilde{\mu}_A(t) \\ &= \tilde{\mu}_A(z_0) \geq r\min\{\tilde{\mu}_A(x_0 * y_0), \tilde{\mu}_A((x_0 * z_0) * y_0)\} \\ &= r\min\{\sup_{t \in (x' * y')} \tilde{\mu}_A(t), \sup_{t \in ((x' * z') * y')} \tilde{\mu}_A(t)\} \\ &= r\min\{\tilde{\mu}_B(x' * y'), \tilde{\mu}_B((x' * z') * y')\}.\end{aligned}$$

Also

$$\lambda(z_0) = \inf_{t \in f^{-1}(z')} \lambda(t),$$

$$\lambda(x_0 * y_0) = \inf_{t \in f^{-1}(x' * y')} \lambda(t)$$

and

$$\begin{aligned}\lambda((x_0 * z_0) * y_0) &= \eta\{f((x_0 * z_0) * y_0)\} \\ &= \eta((x' * z') * y') \\ &= \inf_{((x_0 * z_0) * y_0) \in f^{-1}((x' * z') * y')} \lambda((x_0 * z_0) * y_0) \\ &= \inf_{t \in f^{-1}((x' * z') * y')} \lambda(t).\end{aligned}$$

So

$$\begin{aligned}\eta(z') &= \inf_{t \in f^{-1}(z')} \lambda(t) \\ &= \lambda(z_0) \leq \max\{\lambda(x_0 * y_0), \lambda((x_0 * z_0) * y_0)\} \\ &= \max\{\inf_{t \in (x' * y')} \lambda(t), \inf_{t \in ((x' * z') * y')} \lambda(t)\} \\ &= \max\{\eta(x' * y'), \eta((x' * z') * y'\}.\end{aligned}$$

Hence $\Omega = \langle B, \eta \rangle$ is a cubic BOI-ideal of X'.

This completes the proof.

5. Conclusion and Future Research

To investigate the structure of an algebraic system, it is clear that BOI-ideal with special properties plays an important role. In the present paper, we have applied the notion of the cubic set theory to BOI-ideal of BOI-algebra and investigated some of their useful properties. In the future, these definitions and fundamental results can be applied to some different algebraic structures. There are more topics that could take advantage of cubic BOI-ideal. Like for example cubic intuitionistic BOI-ideal of BOI-algebra, cubic fuzzy BOI-ideal of BOI-algebra, and cubic soft BOI-ideal in BOI-algebra. There are many other aspects which should be explored and studied in the area of BOI-algebra such as anti-fuzzy BOI-ideal of BOI-algebra, interval-valued fuzzy BOI-ideal of BOI-algebra, intuitionistic fuzzy BOI-ideal of BOI-algebra, fuzzy soft BOI-ideal of BOI-algebra, bipolar fuzzy BOI- ideal of BOI-algebra, doubt intuitionistic fuzzy BOI-ideal of BOI-algebra, fuzzy derivations BOI-ideal of BOI-algebra, and interval-valued intuitionistic fuzzy BOI-ideal of BOI-algebra. It is our hope that this work would other foundations for further study of the theory of BOI-algebra.

References

[1] R. K. Bandaru. On BRK-algebras, Int. J. Math. Mathematical Sci. (2012), ID: 952654 12pp.

[2] O. R. El-Gendy, Anti Fuzzy BRK-ideal of BRK-algebra, British J. math, and Comp. Science, 10 (6) (2015) 1-9.

[3] O. R. El-Gendy, Cubic BRK-ideal of BRK-algebra, Annals of Fuzzy Mathematics and Informatics, 12(2) (2016), 245-253.

[4] O. R. El-Gendy, Fuzzy BRK-ideal of BRK-algebra, JP Journal of Algebra, Number Theory, and Applications, 36 (3) (2015), 231-240.

[5] O. R. El-Gendy, On BOI-algebra, Int. J. Math & Comp. App. Res. (IJMCAR), 9(2) (2019), ID: IJMCARDEC20192,13-28.

[6] Q. P. Hu and X. Li, On BCH-algebras, Math. Sem. Notes Kobe Univ. 11(2) (1983), 313-320.

[7] Y. Imai and K. Iséki, on axiom systems of propositional calculi, XIV. proc. Japan Academy, 42 (1966),19-22.

[8] C. Jana and M. Pal, On $(\epsilon\alpha, \epsilon\alpha \vee q\beta)$ -fuzzy soft BCI-algebras, Missouri Journal of Mathematical Sciences, 29(2) (2017),197-215.

[9] C. Jana, M. Pal, and A.B. Saied, $(\epsilon, \epsilon \vee q)$-bipolar fuzzy BCK/BCI-algebras, Missouri Journal of Mathematical Sciences, 29(2) (2017), 139-160.

[10] C. Jana, T. Senapati, and M. Pal, $(\epsilon, \epsilon \vee q)$-intuitionistic fuzzy BCI-subalgebras of a BCI-algebra, Journal of Intelligent & Fuzzy Systems 31 (1) 2016, 613-621.

[11] C. Jana, T. Senapati, Cubic G-subalgebras of G-algebra, Annals of Pure and Applied Mathematics, 10 (1) 2015, 105-115.

[12] Y. B. Jun, C. S. Kim and M. S. Kang, Cubic Sub algebras and ideals of BCK/BCI-algebras, Far East. J. Math. Sci. (FJMS), 44 (2010), 239-250.

[13] Y. B. Jun, C. S. Kim, and K. O. Yang “Cubic sets”, Annals of fuzzy mathematics and informatics, 4(1) 2012, 83-98.

[14] C. B. Kim, H. S. Kim, On BG-algebras, math. J. 36 (1992), 15-20.

[15] J. Neggers and H. S. Kim, on B-algebras, 54 (2002), 21-29

[16] J. Neggers, S. S. Ahn and H. S. Kim, On Q-algebras, Int. J. Math. Sci. 27 (12) (2001), 749-757.

[17] T. Senapati, C. Jana, M. Pal, and Y.B. Jun, Cubic intuitionistic q-ideals of BCI- algebras, Symmetry, 10(12) (2018), 752.

[18] A. Walendziak, On BF- algebras, Math. Slovaca, 57 (2) (2007), 119-128.

[19] L. A. Zadeh, Fuzzy sets, Inform. And Control, 8 (1965), 338-353.