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On a Generalized Tribonacci Sequence

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Abstract

The well-known Tribonacci sequence is a third order recurrence sequence. In this paper, we define other generalized Tribonacci sequence and establish some properties of this sequence using matrix methods.

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1. Introduction and Preliminaries

In this paper, we define a generalized Tribonacci sequence and we obtain some properties of this sequence using matrix methods. First, we give some background about Tribonacci and Tribonacci-Lucas numbers. Tribonacci sequence $\{T_n\}_{n\geq 0}$ (sequence A000073 in [5]) and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ (sequence A001644 in [5]) are defined by the third-order recurrence relations

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad T_0 = 0, T_1 = 1, T_2 = 1, \quad \#(1.1)$$

and

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, \quad K_0 = 3, K_1 = 1, K_2 = 3, \quad \#(1.2)$$

respectively. Tribonacci concept was introduced by M. Feinberg [2] in 1963. Basic properties of it is given in [1], [3], [4], [6] and [7].

The sequences $\{T_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}$$

and

$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}$$

for $n = 1,2,3,\ldots$ respectively. Therefore, recurrences(1.1) and (1.2) hold for all integer n.
It is well-known that for all integers n, usual Tribonacci and Tribonacci-Lucas numbers can be expressed using Binet's formulas

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \quad \#(1.3)$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n \quad \#(1.4)$$

respectively, where α, β and $\gamma$are the distinct roots of the cubic equation $x^3 - x^2 - x - 1 = 0$. Furthermore,

$$\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3},$$

$$\beta = \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3},$$

$$\gamma = \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3},$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = exp(2\pi i/3),$$

is a primitive cube root of unity.

2. Main Results

We consider the generalized Tribonacci sequence defined by

$$E_n = E_{n-1} + E_{n-2} + E_{n-3}, \quad E_0 = 3, E_1 = 1, E_2 = 0. \quad \#(2.1)$$

Obviously, $x^3 - x^2 - x - 1 = 0$ is also the characteristic equation of the (2.1) and it produces three roots as α, β and γ which are given above.

We define the square matrix E of order 3 as:

$$E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $detE = 1$. $E$is called the generating matrix for the sequence (2.1).

Theorem 2.1

(a) For $n \geq 1$, we have

$$\begin{pmatrix} E_{n+2} \\ E_{n+1} \\ E_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_{n+1} \\ E_n \\ E_{n-1} \end{pmatrix}. \quad \#(2.2)$$

(b) For $n \geq 0$, we have

$$\begin{pmatrix} E_{n+2} \\ E_{n+1} \\ E_n \end{pmatrix} = E^n \begin{pmatrix} E_2 \\ E_1 \\ E_0 \end{pmatrix}. \quad \#(2.3)$$

Proof. (a) and (b) can be proved by using induction on n.
Next, we present Binet formula for the generalized Tribonacci sequence $\{E_n\}$.

Theorem 2.2 (Binet formula for the Generalized Tribonacci Sequence)

$$E_n = \left(\frac{\alpha^{n-2}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n-2}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n-2}}{(\gamma - \alpha)(\gamma - \beta)}\right)$$

$$+4\left(\frac{\alpha^{n-1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n-1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n-1}}{(\gamma - \alpha)(\gamma - \beta)}\right)$$

$$= T_{n-3} + 4T_{n-2}.$$

Proof. The general form of the generalized Tribonacci sequence can be expressed in the following form

$$E_n = A\alpha^n + B\beta^n + C\gamma^n$$

where A, B and C are constants that can be determined by the initial conditions. Thus putting the values $n = 0, n = 1$ and $n = 2$ in (2.1), we obtain

$$A + B + C = 3$$

$$A\alpha + B\beta + C\gamma = 1$$

$$A\alpha^2 + B\beta^2 + C\gamma^2 = 0.$$

Solving the above system of equations for A, B and C, we get

$$A = -\frac{\beta + \gamma - 3\beta\gamma}{\alpha^2 - \alpha\beta - \alpha\gamma + \beta\gamma} = -\frac{\beta + \gamma - 3\beta\gamma}{(\alpha - \beta)(\alpha - \gamma)},$$

$$B = -\frac{\alpha + \gamma - 3\alpha\gamma}{\beta^2 - \alpha\beta + \alpha\gamma - \beta\gamma} = -\frac{\alpha + \gamma - 3\alpha\gamma}{(\beta - \alpha)(\beta - \gamma)},$$

$$C = -\frac{\alpha + \beta - 3\alpha\beta}{\gamma^2 + \alpha\beta - \alpha\gamma - \beta\gamma} = -\frac{\alpha + \beta - 3\alpha\beta}{(\gamma - \alpha)(\gamma - \beta)}.$$

Note that we have the following identities:

$$\alpha + \beta + \gamma = 1,$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -1,$$

$$\alpha\beta\gamma = 1.$$

It now follows that

$$-(\beta + \gamma - 3\beta\gamma) = -\frac{1}{\alpha}(\alpha\beta + \alpha\gamma - 3\alpha\beta\gamma)$$

$$= -\frac{1}{\alpha}(-1 - \beta\gamma - 3)$$

$$= -\frac{1}{\alpha}(-\beta\gamma - 4)$$

$$= \frac{1}{\alpha}(\beta\gamma + 4)$$

$$= \frac{1}{\alpha^2}(\alpha\beta\gamma + 4\alpha)$$

$$= \frac{1}{\alpha^2}(1 + 4\alpha)$$

$$= \left(\frac{1}{\alpha^2} + \frac{4}{\alpha}\right),$$

$$-(\alpha+\gamma-3\alpha\gamma)=\left(\frac{1}{\beta^2}+\frac{4}{\beta}\right),$$

$$-(\alpha+\beta-3\alpha\beta)=\left(\frac{1}{\gamma^2}+\frac{4}{\gamma}\right).$$

Hence, we get

$$E_n=-\frac{(\beta+\gamma-3\beta\gamma)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}-\frac{(\alpha+\gamma-3\alpha\gamma)\beta^n}{(\beta-\alpha)(\beta-\gamma)}-\frac{(\alpha+\beta-3\alpha\beta)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

$$=\left(\frac{\alpha^{n-2}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n-2}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n-2}}{(\gamma-\alpha)(\gamma-\beta)}\right)$$

$$+4\left(\frac{\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}\right)$$

$$=T_{n-3}+4T_{n-2}.$$

Identities which is given in the following Lemma can be established easily.

Lemma 2.3

(a) $53T_n-E_n+3E_{n+1}-13E_{n+2}=0,$
(b) $E_n-2T_n-7T_{n+1}+4T_{n+2}=0,$
(c) $44E_n-38K_n+54K_{n+1}-24K_{n+2}=0,$
(d) $53K_n-41E_n-36E_{n+1}-3E_{n+2}=0.$

Next, we present the $n.th$ power of the generating matrix.

Theorem 2.4 For $n\geq 1$, we have

$$E^n=\frac{1}{53}\begin{pmatrix}23E_n+10E_{n-1}+24E_{n+1} & 24E_n+47E_{n-1}+33E_{n-2}+10E_{n-3} & 24E_n+23E_{n-1}+10E_{n-2}\\ 14E_n+13E_{n-1}+10E_{n+1} & 10E_n+24E_{n-1}+27E_{n-2}+13E_{n-3} & 10E_n+14E_{n-1}+13E_{n-2}\\ E_{n-1}-3E_n+13E_{n+1} & 13E_n+10E_{n-1}-2E_{n-2}+E_{n-3} & 13E_n-3E_{n-1}+E_{n-2}\end{pmatrix}\#(2.4)$$

Proof. It is well-known that

$$E^n=\begin{pmatrix}1&1&1\\1&0&0\\0&1&0\end{pmatrix}^n=\begin{pmatrix}T_{n+1}&T_n+T_{n-1}&T_n\\T_n&T_{n-1}+T_{n-2}&T_{n-1}\\T_{n-1}&T_{n-2}+T_{n-3}&T_{n-2}\end{pmatrix}.$$

Using Lemma (2.3), we obtain (2.4).

Note that we can also prove (2.4) by using induction on n.

Now, we state cubic identity.

Theorem 2.5 (Cubic Identity) For$n\geq 1$, we have

$$E_{n-1}^3+E_{n-3}E_n^2-2E_nE_{n-1}E_{n-2}-E_{n+1}E_{n-3}E_{n-1}+E_{n+1}E_{n-2}^2=53.$$

Proof. Note that $det(E^n)=1$ for all $n\geq 1$. We will use this property and (2.4). Then,

$$det(E^n)=\frac{1}{53^2}\begin{vmatrix}23E_n+10E_{n-1}+24E_{n+1} & 24E_n+47E_{n-1}+33E_{n-2}+10E_{n-3} & 24E_n+23E_{n-1}+10E_{n-2}\\ 14E_n+13E_{n-1}+10E_{n+1} & 10E_n+24E_{n-1}+27E_{n-2}+13E_{n-3} & 10E_n+14E_{n-1}+13E_{n-2}\\ E_{n-1}-3E_n+13E_{n+1} & 13E_n+10E_{n-1}-2E_{n-2}+E_{n-3} & 13E_n-3E_{n-1}+E_{n-2}\end{vmatrix}$$

and so

$$1 = \frac{1}{53^3}(2809E_{n-1}^3 + 2809E_{n-3}E_n^2 - 5618E_nE_{n-1}E_{n-2} - 2809E_{n+1}E_{n-3}E_{n-1} + 2809E_{n+1}E_{n-2}^2)$$

$$= \frac{1}{53}(E_{n-1}^3 + E_{n-3}E_n^2 - 2E_nE_{n-1}E_{n-2} - E_{n+1}E_{n-3}E_{n-1} + E_{n+1}E_{n-2}^2).$$

Hence the result follows.

We now present Binet formula for the generalized Tribonacci sequence $\{E_n\}$ using matrix method.

Theorem 2.6 (Binet Formula) For $n \geq 0$, we have

$$E_n = \frac{1}{\lambda}\big((3\beta^2\gamma - \beta^2 - 3\beta\gamma^2 + \gamma^2)\alpha^n + (-3\alpha^2\gamma + \alpha^2 + 3\alpha\gamma^2 - \gamma^2)\beta^n + (3\alpha^2\beta - \alpha^2 - 3\alpha\beta^2 + \beta^2)\gamma^n\big)$$

where$\lambda = (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$.

Proof. The characteristic equation of the generating matrix E is

$$0 = |E - xI_3| = \left|\begin{pmatrix} 1-x & 1 & 1 \\ 1 & -x & 0 \\ 0 & 1 & -x \end{pmatrix}\right| = x^3 - x^2 - x - 1$$

where x is the eigenvalue of E and I_3 is the 3×3 unite matrix. Note that α, β and γ are the roots of the characteristic (cubic) equation $x^3 - x^2 - x - 1 = 0$and also α, β and γ are the three eigenvalues of the square matrix E. Next we find the eigenvalues corresponding to the eigenvalues $\alpha, \beta$and γ. We can find the eigenvector by solving the following system of linear equations:

$$(E - xI_3)u_x = 0$$

whereu_x is the column vector of order 3×1.First we find the eigenvector corresponding to the eigenvalue α. Then from

$$(E - \alpha I_3)u_\alpha = \begin{pmatrix} 1-\alpha & 1 & 1 \\ 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

we have the system

$$u_2 + u_3 - u_1(\alpha - 1) = 0$$

$$u_1 - \alpha u_2 = 0$$

$$u_2 - \alpha u_3 = 0.$$

If we take $u_3 = c$ in above system, we obtain $u_2 = \alpha c$, $u_1 = \alpha^2 c$. Thus the eigenvectors corresponding to α are of the form $\begin{pmatrix} \alpha^2 c \\ \alpha c \\ c \end{pmatrix}$ and in particular if we take $c = 1$, then the eigenvectors corresponding to α is $\begin{pmatrix} \alpha^2 \\ \alpha \\ 1 \end{pmatrix}$. Similarly, using the same technique, we see that the eigenvectors corresponding to β and γ are $\begin{pmatrix} \beta^2 \\ \beta \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \gamma^2 \\ \gamma \\ 1 \end{pmatrix}$ respectively. Let

$$P = \begin{pmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \\ 1 & 1 & 1 \end{pmatrix},$$

i.e., P is a matrix of eigenvalues. Then, we get the inverse of P as

$$P^{-1} = \begin{pmatrix} \frac{1}{\alpha^2-\alpha\beta-\alpha\gamma+\beta\gamma} & -\frac{\beta+\gamma}{\alpha^2-\alpha\beta-\alpha\gamma+\beta\gamma} & \frac{\beta\gamma}{\alpha^2-\alpha\beta-\alpha\gamma+\beta\gamma} \\ \frac{1}{\beta^2-\alpha\beta+\alpha\gamma-\beta\gamma} & -\frac{\alpha+\gamma}{\beta^2-\alpha\beta+\alpha\gamma-\beta\gamma} & \frac{\alpha\gamma}{\beta^2-\alpha\beta+\alpha\gamma-\beta\gamma} \\ \frac{1}{\gamma^2+\alpha\beta-\alpha\gamma-\beta\gamma} & -\frac{\alpha+\beta}{\gamma^2+\alpha\beta-\alpha\gamma-\beta\gamma} & \frac{\alpha\beta}{\gamma^2+\alpha\beta-\alpha\gamma-\beta\gamma} \end{pmatrix}$$

$$= \frac{1}{\lambda}\begin{pmatrix} (\beta-\gamma) & -(\beta+\gamma)(\beta-\gamma) & \beta\gamma(\beta-\gamma) \\ -(\alpha-\gamma) & (\alpha+\gamma)(\alpha-\gamma) & -\alpha\gamma(\alpha-\gamma) \\ (\alpha-\beta) & -(\alpha+\beta)(\alpha-\beta) & \alpha\beta(\alpha-\beta) \end{pmatrix}$$

where $\lambda = (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)$. Now, let

$$D = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

i.e., D is the diagonal matrix in which the eigenvalues of E are on the main diagonal. Then using the diagonalization of the generating matrix E we obtain $E = PDP^{-1}$. So, we get

$$E^n = (PDP^{-1})^n = PD^nP^{-1}$$

$$= \frac{1}{\lambda}\begin{pmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha^n & 0 & 0 \\ 0 & \beta^n & 0 \\ 0 & 0 & \gamma^n \end{pmatrix}\begin{pmatrix} (\beta-\gamma) & -(\beta+\gamma)(\beta-\gamma) & \beta\gamma(\beta-\gamma) \\ -(\alpha-\gamma) & (\alpha+\gamma)(\alpha-\gamma) & -\alpha\gamma(\alpha-\gamma) \\ (\alpha-\beta) & -(\alpha+\beta)(\alpha-\beta) & \alpha\beta(\alpha-\beta) \end{pmatrix}$$

$$= \frac{1}{\lambda}\begin{pmatrix} \alpha^{n+2} & \beta^{n+2} & \gamma^{n+2} \\ \alpha^{n+1} & \beta^{n+1} & \gamma^{n+1} \\ \alpha^n & \beta^n & \gamma^n \end{pmatrix}\begin{pmatrix} (\beta-\gamma) & -(\beta+\gamma)(\beta-\gamma) & \beta\gamma(\beta-\gamma) \\ -(\alpha-\gamma) & (\alpha+\gamma)(\alpha-\gamma) & -\alpha\gamma(\alpha-\gamma) \\ (\alpha-\beta) & -(\alpha+\beta)(\alpha-\beta) & \alpha\beta(\alpha-\beta) \end{pmatrix}.$$

Using the above last equality and (2.3) and comparing the third row entries of the matrices, we obtain

$$E_n = \frac{1}{\lambda}(-(\beta+\gamma)(\beta-\gamma)\alpha^n + (\alpha+\gamma)(\alpha-\gamma)\beta^n - (\alpha+\beta)(\alpha-\beta)\gamma^n + 3\beta\gamma(\beta-\gamma)\alpha^n - 3\alpha\gamma(\alpha-\gamma)\beta^n + 3\alpha\beta(\alpha-\beta)\gamma^n).$$

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