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Lyapunov-type inequalities for higher order difference equations with anti-periodic boundary conditions

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Abstract. In this paper, some new Lyapunov-type inequalities for higher order difference equations with anti-periodic boundary conditions are established. The obtained results are used to obtain the lower bounds for the eigenvalues of corresponding equations.

Keywords: Difference equation; Anti-periodic boundary conditions; Lyapunov-type inequalities.

1. Introduction

The standard Lyapunov-type inequality [4] is useful in many applications, including eigenvalue problems, oscillation theory, disconjugacy. Although there are many literatures on the Lyapunov-type inequality for various classes of differential equations [1,3,6,8,9], there is not much done for the difference equations.

In 1983, Cheng [2] first established the following Lyapunov-type inequality which is a discrete analogue of the classical Lyapunov-type inequality:

If the second order difference equation

$$\begin{cases} \Delta^2 x(n) + q(n)x(n+1) = 0, \\ x(a) = x(b) = 0, x(n) \not\equiv 0, \ n \in \mathbb{Z}[a,b], \end{cases} \tag{1.1}$$

has a nonzero solution, then

$$F(b-a)\sum_{n=a}^{b-2} q(n) \geq 4,$$

where

$$F(m) = \begin{cases} \dfrac{m^2-1}{m}, & \text{if } m-1 \text{ is even}, \\ m, & \text{if } m-1 \text{ is odd}, \end{cases}$$

$a, b \in \mathbb{Z}$and $\mathbb{Z}[a,b] = \{a, a+1, \cdots, b-1, b\}$. Some other related results about discrete Lyapunov-type inequalities can be found in [5,7,10].

In this paper, we will consider the following difference equation

$$\Delta\left(|\Delta^m x(n)|^{p-2}\Delta^m x(n)\right) + r(n)|x(n)|^{p-2}x(n) = 0, \tag{1.2}$$

with the anti-periodic boundary conditions

$$\Delta^i x(a) + \Delta^i x(b) = 0, \ \ i = 0,1,\cdots,m, \tag{1.3}$$

where$x(n) \not\equiv 0, n \in \mathbb{Z}[a,b], m \in \mathbb{N}, p > 1, r(n)$is a real-valued function defined on$\mathbb{Z}$.

Furthermore, we will establish a new Lyapunov-type inequality for the following equation

$$|\Delta^m x(n)|^{p-2}\Delta^m x(n)+r(n)|x(n)|^{p-2}x(n)=0, \quad (1.4)$$

with the anti-periodic boundary conditions

$$\Delta^i x(a)+\Delta^i x(b)+\Delta^i x(c)=0,\ \ i=0,1,\cdots,m-1, \quad (1.5)$$

where $x(n)\not\equiv 0, n\in\mathbb{Z}[a,c], a<b<c, m\in\mathbb{N}, p>1, r(n)$ is a real-valued function defined on $\mathbb{Z}$.

2. Main results

In this section, we give two main results and some corollaries.

Theorem2.1. If (1.2) has a nonzero solution $x(n)$ satisfying the anti-periodic boundary conditions (1.3), then

$$\sum_{n=a}^{b-1}|r(n)|\geq 2\left(\frac{2}{b-a}\right)^{m(p-1)}.$$

Proof. For $x(a)+x(b)=0$, then

$$x(n)=x(n)-\frac{1}{2}[x(a)+x(b)]$$

$$=\frac{1}{2}[x(n)-x(a)]-\frac{1}{2}[x(b)-x(n)]$$

$$=\frac{1}{2}\sum_{k=a}^{n-1}\Delta x(k)-\frac{1}{2}\sum_{k=n}^{b-1}\Delta x(k).$$

Moreover, we have

$$|x(n)|\leq\frac{1}{2}\sum_{k=a}^{n-1}|\Delta x(k)|+\frac{1}{2}\sum_{k=n}^{b-1}|\Delta x(k)|=\frac{1}{2}\sum_{k=a}^{b-1}|\Delta x(k)|,$$

then

$$|x(n)|\leq\frac{1}{2}(b-a)^{\frac{1}{q}}\left(\sum_{k=a}^{b-1}|\Delta x(k)|^p\right)^{\frac{1}{p}} \quad (2.1)$$

by Hölder's inequality, where $\frac{1}{p}+\frac{1}{q}=1$.

Using the boundary conditions (1.3), we obtain

$$\left|\Delta^i x(n)\right|\leq\frac{1}{2}(b-a)^{\frac{1}{q}}\left(\sum_{k=a}^{b-1}\left|\Delta^{i+1}x(k)\right|^p\right)^{\frac{1}{p}}, \quad (2.2)$$

$$|\Delta^i x(n)|^p\leq(\frac{1}{2})^p(b-a)^{\frac{p}{q}}\sum_{k=a}^{b-1}\left|\Delta^{i+1}x(k)\right|^p,$$

$$\sum_{n=a}^{b-1}|\Delta^i x(n)|^p\leq(\frac{1}{2})^p(b-a)^p\sum_{k=a}^{b-1}\left|\Delta^{i+1}x(k)\right|^p,$$

$$(\sum_{n=a}^{b-1}\left|\Delta^i x(n)\right|^p)^{\frac{1}{p}}\leq\frac{1}{2}(b-a)(\sum_{k=a}^{b-1}\left|\Delta^{i+1}x(k)\right|^p)^{\frac{1}{p}}. \quad (2.3)$$

Combining inequalities (2.1) and (2.3), we have

$$|x(n)| \leq \frac{1}{2}(b-a)^{\frac{1}{q}}\left(\sum_{k=a}^{b-1}|\Delta x(k)|^p\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{2}(b-a)^{\frac{1}{q}}(\frac{b-a}{2})^{m-1}\left(\sum_{k=a}^{b-1}|\Delta^m x(k)|^p\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{2}\right)^m (b-a)^{m-\frac{1}{p}}\left(\sum_{k=a}^{b-1}|\Delta^m x(k)|^p\right)^{\frac{1}{p}},$$

therefore

$$|x(n)|^{p-1} \leq \left(\frac{1}{2}\right)^{m(p-1)} (b-a)^{\left(m-\frac{1}{p}\right)(p-1)}\left(\sum_{k=a}^{b-1}|\Delta^m x(k)|^p\right)^{\frac{p-1}{p}}. \qquad (2.4)$$

Moreover, by the boundary conditions (1.3), we have the following equality

$$\sum_{k=a}^{b-1}|\Delta^m x(k)|^p = \sum_{k=a}^{b-1}|\Delta^m x(k+1)|^p.$$

Combining inequalities (2.2) and (2.4),

$$|x(n)|^{p-1}|\Delta^{m-1}x(n+1)| \leq \left(\frac{1}{2}\right)^{m(p-1)} (b-a)^{\left(m-\frac{1}{p}\right)(p-1)}\left(\sum_{k=a}^{b-1}|\Delta^m x(k+1)|^p\right)^{\frac{p-1}{p}}$$

$$\cdot\frac{1}{2}(b-a)^{\frac{1}{q}}\left(\sum_{k=a}^{b-1}|\Delta^m x(k+1)|^p\right)^{\frac{1}{p}}$$

$$= \mathrm{C}\sum_{k=a}^{b-1}|\Delta^m x(k+1)|^p,$$

where$\mathrm{C} = \frac{1}{2}\cdot(\frac{b-a}{2})^{m(p-1)}$.

Multiplying (1.2) by $\Delta^{m-1}x(n+1)$ and summing the obtained equation from $a$to$b-1$, then

$$\sum_{n=a}^{b-1}|\Delta^m x(n+1)|^p = \sum_{n=a}^{b-1} r(n)|x(n)|^{p-2}x(n)\Delta^{m-1}x(n+1)$$

$$\leq \sum_{n=a}^{b-1}|r(n)||x(n)|^{p-1}|\Delta^{m-1}x(n+1)|$$

$$\leq \mathrm{C}\left(\sum_{n=a}^{b-1}|\Delta^m x(n+1)|^p\right)\sum_{n=a}^{b-1}|r(n)|,$$

Noting that $\Delta^m x(a) + \Delta^m x(b) = 0$and $x(n) \not\equiv 0, n \in \mathbb{Z}[a,b]$, we have

$$\sum_{n=a}^{b-1} |\Delta^m x(n+1)|^p > 0,$$

therefore,

$$\sum_{n=a}^{b-1} |r(n)| \geq 2\left(\frac{2}{b-a}\right)^{m(p-1)}.$$

The proof is complete.□

Theorem 2.2. If (1.4) has a nonzero solution $x(n)$ satisfying the anti-periodic boundary conditions (1.5), then

$$\sum_{n=a}^{c-1} |r(n)|^q \geq \frac{3^{mp}}{2^{mp}(c-a)^{mp-1}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $x(a) + x(b) + x(c) = 0$, we have

$$x(n) = x(n) - \frac{1}{3}[x(a) + x(b) + x(c)].$$

Case $n \in \mathbb{Z}[a, b]$, we have

$$x(n) = \frac{1}{3}[x(n) - x(a)] - \frac{1}{3}[x(b) - x(n)] - \frac{1}{3}[x(c) - x(n)]$$

$$= \frac{1}{3}\sum_{k=a}^{n-1} \Delta x(k) - \frac{1}{3}\sum_{k=n}^{b-1} \Delta x(k) - \frac{1}{3}\sum_{k=n}^{c-1} \Delta x(k)$$

$$= \frac{1}{3}\sum_{k=a}^{n-1} \Delta x(k) - \frac{1}{3}\sum_{k=n}^{b-1} \Delta x(k) - \frac{1}{3}\sum_{k=n}^{b-1} \Delta x(k) - \frac{1}{3}\sum_{k=b}^{c-1} \Delta x(k).$$

Then,

$$|x(n)| \leq \frac{1}{3}\sum_{k=a}^{c-1} |\Delta x(k)| + \frac{1}{3}\sum_{k=n}^{b-1} |\Delta x(k)|$$

$$\leq \frac{2}{3}\sum_{k=a}^{c-1} |\Delta x(k)|.$$

Case $n \in \mathbb{Z}[b, c]$, we have

$$x(n) = \frac{1}{3}[x(n) - x(a)] + \frac{1}{3}[x(n) - x(b)] - \frac{1}{3}[x(c) - x(n)]$$

$$= \frac{1}{3}\sum_{k=a}^{n-1} \Delta x(k) + \frac{1}{3}\sum_{k=b}^{n-1} \Delta x(k) - \frac{1}{3}\sum_{k=n}^{c-1} \Delta x(k)$$

$$= \frac{1}{3}\sum_{k=a}^{b-1} \Delta x(k) + \frac{1}{3}\sum_{k=b}^{n-1} \Delta x(k) + \frac{1}{3}\sum_{k=b}^{n-1} \Delta x(k) - \frac{1}{3}\sum_{k=n}^{c-1} \Delta x(k).$$

Then,

$$|x(n)| \le \frac{1}{3}\sum_{k=a}^{c-1}|\Delta x(k)| + \frac{1}{3}\sum_{k=b}^{n-1}|\Delta x(k)| \le \frac{2}{3}\sum_{k=a}^{c-1}|\Delta x(k)|.$$

Thus, we have

$$|x(n)| \le \frac{2}{3}\sum_{k=a}^{c-1}|\Delta x(k)| \le \ \frac{2}{3}(c-a)^{\frac{1}{q}}\left(\sum_{k=a}^{c-1}|\Delta x(k)|^p\right)^{\frac{1}{p}}$$

for $n \in \mathbb{Z}[a,c]$. The rest of the proof is similar to the Theorem 2.1, we omit it.□

Corollary 2.3. Consider the following eigenvalue problem

$$\begin{cases}\Delta\left(|\Delta^m x(n)|^{p-2}\Delta^m x(n)\right) + \lambda r(n)|x(n)|^{p-2}x(n) = 0,\\ \Delta^i x(a) + \Delta^i x(b) = 0, \qquad\qquad i = 0,1,\cdots,m,\end{cases}$$

where$x(n) \not\equiv 0$, $n \in \mathbb{Z}[a,b]$, $m \in \mathbb{N}$, $p > 1$, *r(n)*is a real-valued function defined on$\mathbb{Z}$, then we have

$$|\lambda| \ge 2\left(\frac{2}{b-a}\right)^{m(p-1)}\left(\sum_{n=a}^{b-1}|r(n)|\right)^{-1}.$$

Corollary 2.4. Consider the following eigenvalue problem

$$\begin{cases}|\Delta^m x(n)|^{p-2}\Delta^m x(n) + \eta r(n)|x(n)|^{p-2}x(n) = 0,\\ \Delta^i x(a) + \Delta^i x(b) + \Delta^i x(c) = 0, \quad i = 0,1,\cdots,m-1,\end{cases}$$

where$x(n) \not\equiv 0$, $n \in \mathbb{Z}[a,c]$, $a < b < c$,$m \in \mathbb{N}$, *r(n)* is a real-valued function defined on$\mathbb{Z}$, $p > 1$,$\frac{1}{p} + \frac{1}{q} = 1$,then we have

$$|\eta| \ge \frac{3^{m(p-1)}}{2^{m(p-1)}(c-a)^{(m-\frac{1}{p})(p-1)}\left(\sum_{n=a}^{c-1}|r(n)|^q\right)^{\frac{1}{q}}}.$$

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