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Stability and Asymtotic Behavior of the Causal Operator Dynamical Systems Using Nonlinear Variation of Parameters.

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ABSTRACT: The operator T from a domain D into the space of measurable functions is called a nonanticipating operator if the past informations is independent from the future outputs. We will use the solution to the operator differential equation $y\prime(t) = A(t)y(t) + f(t, y(t), T(y)(t))$ to analyze the solution of this operator differential equation which is generated by a perturbation $\zeta(t) = g(t, y_t, T_2(y_t))$. When this perturbation is from a measurable space then the existence and uniqueness of the solution to the operator differential equation will be studied. Finally, we use the nonlinear variation of parameters for nonanticipating operator differential equations to study the stability and asymptotic behavior of the equilibrium solution.

Keywords: Nonlinear Operator Differential Equations (NODE), Variation of parameters, Nonanticipating (Causal), Alekseev Theorem, k-Norm Stability, and Uniform Stabilty,.

Sec. (1) introduction and Basic Definitions: A mapping T from the space of functions Y into itself is said to be a nonanticipating mapping if for every fixed s in the real line R,

$(Tx)(t) = (Ty)(t)$ for all $t < s$, whenever $\phi(t) = y(t)$ for all $t < s$.

In other words, if the inputs x and y agree up to some time N, then the outputs T(x) and T(y) agree up to time N. In particular, T(x) and T(y) agree up to time N no matter what the inputs x and y are in the future beyond N. **The events in the past and present are independent from the future.**

Preliminary Conditions and Notations. The following are our elementary definitions and conditions for operator differential systems.

C1: Let S be the interval of all real numbers $t < t_0$. For real number a, let I be the compact interval $[t_0, a]$, and define $J = S \cup I$. Assume (Y, |. |), (Z, |.|), $and (U, |.|) are Banach spaces$.

C2: Let M(I,Y) be the **space of all essentially bounded Bochner measurable functions** with respect to classical Lebesgue measure from the interval I into the Banach space Y.

C3: Denote by L(J,Y) the space of all Lipschitzian functions **y** $\in M(I, Y)$ **strongly differentiable almost everywhere from J into Y.**

C4: Let ϕ **be a fixed initial function** from the space $L(S, Y)$. Denote by $D(\phi, Y)$ the subset of the lip-space $L(J, Y)$ consisting of all functions y such that

$y(t) = \phi(t)$ for all t in S.

According to these two definitions, $D(\phi, Y) \subset L(J, Y)$.

C5: For any Banach space Y and Z, let $Lip(I; Y, Z)$ denote the space of all functions $f(t, y)$ from the product $I \times Y$ into Z, Lipschitzian in y, and for every fixed y the function $f(., y)$ belongs to the space $M(I, Z)$. This space is called Lip-space.

C6: We apply the definition of nonanticipating operators in section 1.1 to the initial domain. An operator T from **the initial domain D**$(\phi,$ **Y**$)$ into $M(I, Z)$ will be called a **nonanticipating** operator if for every two functions y and z in $D(\phi, Y)$ and every point $s \in I$, the fact that

$y(t) = z(t)$ for almost all $t < s$ implies that $T(y)(t) = T(z)(t)$ for almost all $t < s$.

C7: An operator P from a subset D of Y into Z is said to be Lipschitzian if there exists a constant b such that

$$|P(y_1) - P(y_2)| \leq b|y_1 - y_2| \tag{1.1}$$

for every $y_1, y_2 \in Y$.

For $f \in Lip\,(I, Y; Z)$ the operator

$$F : M(I, Y) \to M(I, Z) \;\Leftrightarrow\; F(y)(t) = f(t, y(t)) \tag{1.2}$$

is called the operator induced by f and the operator F is called **Induced Operator generated by the function f**.

Lipschitzian Space (or simply the **Lip-Space**), denoted by $Lip\,(K, Y; Z)$, is the set of all functions $f : K \times Y \to Z$ such that $f\,(t, y)$ is uniformly bounded by ω, Lipschitzian in y and is measurable in t. That is,

$Lip(K, Y; Z) = \{f : K \times Y \to Z \mid f$ is Lipschitzian in y and $f\,(., y) \in M\,(K, Z)\}$.

The infimum of all Lipschitzian constants L will be denoted by $\|f\|$.

On the space M(I,Y), we shall introduce a family of norms, called **k-norm** by the formula

$$\| y \|_k = ess.\sup\{e^{-kt}|y(t)| : t \in I\} \tag{1.3}$$

for any fixed real number k. Observe that from this definition follows the inequality

$$|y(t)| \leq \| y \|_k \, e^{kt} \tag{1.4}$$

for almost all t in I. Notice that for every k, the k norms $\| \;\|_k$and $\| \;\|_0$ are equivalent.

A Lipschitzian operator T from a subset D of the measurable space M(J,Y) into the space M(I,U) is called **an operator of exponential type** if for some constants b and k_0,

$$\| T(y) - T(z) \|_k \leq b \,\| y - z) \|_k \tag{1.5}$$

for all y and z in the domain D and all $k \geq k_0$.

Nonanticipating Deterministic Dynamical System: Assume that the operator T is nonanticipating and Lipscitzian. The behavior of a dynamical system

$$y\prime(t) = f(t, y(t), T(y)(t)) \tag{1.6}$$

is known as an after effect differential equation with the initial domain $D(\phi, Y)$.

Given that $f \in Lip(I, Y \times Z; Y)$, **there exists a unique solution y** to the system (1.6).

Equations of this type arises in many mathematical modeling problems. The following is a single species growth model with time delay**.**

Example 1: Assume that T is a constant delay operator $T(y)(t) = y(t - r)$ for a constant real number r in the following differential equation. One can verify the existence and uniqueness of the solution of the system

$$\begin{cases} y\prime(t) = (y)(t - r), & t > t_0 \\ y(t) = \phi(t), & t \leq t_0 \end{cases} \tag{1.7}$$

with the initial data function $y(t) = \phi(t)$ for $-r \leq t < t_0$.

Our goal is to investigate the conditions which guarantee the solution of the system (1.6) when there is a random perturbation in the system.

Solution to the Nonanticipating Operator differential Equations:

The following operator differential equation when G is a nonanticipating operator from the initial domain D(ϕ, Y) to the Banach space Z is called nonanticipating differential equation,

$$\begin{cases} y\prime(t) = G(y)(t), & t > t_0 \\ y(t) = \phi(t), & t \leq t_0 \end{cases} \tag{1.8}$$

for almost all t in the interval I. We define **that a function y from the space M(I,Y) is a solution to the nonanticipating operator differential equation if it is strongly differentiable and satisfies the system (1.8)** *(Ahangar 1989-2017, [1],...[6])*. We accept the following theorem without proof.

Theorem 1.1: Given a nonanticipating and Lipschitzian operator G from the initial domain $D(\phi, Y)$ into the space of Bochner measurable functions $M(I, Y)$. Then there exists a unique solution $y \in D(\phi, Y)$ such that $y\prime(t) = G(y)(t)$, for almost all t in I.

Sec. (2): Strong Solution to the Perturbed Nonanticipating Operator Differential Equations:

Definition 2.1: By Nant-Lip we mean nonanticipating and Lipschitzian operators. The operator G in the system (1.8) has this property that is nonanticipating and Lipschitzian. We need to clarify the meaning of the solution to the nonlinear system of operator differential equations (1.8). The important part is when we accept some other principles indirectly hidden in the proof of the Theorem (1.1). In fact we use the equivalent relationship between (1.8) and the integral

$$y(t) = \phi(t) + \int_{t_0}^{t} G(y)(s)ds. \tag{2.1}$$

Notice that this equivalent relation requires the absolute continuity of function y and summability of G which implies the differentiability of y. The nonlinear operator system similar to (1.8) could be presented by the following operator differential equation

$$x\prime(t) = f(t, x(t), T(x)(t)), \qquad \text{for almost all } t > t_0 \tag{2.2}$$

for almost all t in I, which contain the initial function ϕ for the past time interval $S = \{t \in R : t \leq t_0\}$. The system (2.2) having a solution $x(t, t_0, \phi)$ which is called the strong solution to the system needs to be redefined.

Definition 2.2: A function x(t) is said to be a strong solution to the system (2.2) if it satisfies the following conditions:

i) x(t) is strongly differentiable,

ii) x(t) satisfies the system (2.2) almost everywhere in the interval I,

iii) there exists a function $\phi \in D(J, Y)$ such that

$$x(t) = \phi(t), \qquad \text{for a. a. } t \leq t_0.$$

The following proposition will show the existence and uniqueness of the solution to the perturbed operator differential equation (2.2).

Proposition 2.1: Assume that the operator T is Nant-Lip and functions f and g belong to the Lip-space which is

$f \in Lip(I, Y \times Z; Y)$ and $g \in Lip(I, Y; Y)$.

i) If g is the perturbation to the equation (2.2) then **there is a unique strong solution** y(t) in the initial domain $D(\phi, Y)$ satisfying the differential equation

$$y\prime(t) = f(t, y(t), T(y)(t)) + g(t, y(t)) \tag{2.3}$$

ii) Given a solution $x(t, t_0, \phi)$ of (2.2) then the solution to the perturbed equation will satisfy the integral equation

$$y(t) = x(t, t_0, \phi) + \int_{t_0}^{t} g(s, y(s))ds. \tag{2.4}$$

Proof:(i) Let us assume that operators $P_1(y)(t) = f(t, y(t), T(y)(t))$ and $P_2(y)(t) = g(t, y(t))$. Define the direct sum operator $G = P_1 \oplus P_2$. It can be verified that the direct sum of two Nant-Lip operator is also a Nant-Lip. As a result the operator G which is equivalent to the perturbed system will be Nant-Lip and the differential equation (2.3) will be equivalent to the equation (2.1) that is

$$y'(t) = G(y)(t) \tag{2.5}$$

for almost all t in I. According to Bogdan's theorem (*see Bogdan 1981 and 1982, [7],[8]*), there exists a unique solution y(t) in $D(\phi, Y)$ to the equation (2.5).

The equivalent integral equation of (2.5) will be $y(t) = \phi(t) + \int_{t_0}^{t} G(y)(s)ds$ Applying the direct sum operators $P_1$and P_2 we get the conclusion which is (2.4).

Proof of ii) Substitute this unperturbed solution

$$x(t, t_0, \phi) = \phi(t) + \int_{t_0}^{t} f(s, y(s), T(y)(s))ds$$

in (2.4) as a solution of (2.2) we will get the following

$$y\prime(t) = \phi(t) + \int_{t_0}^{t} f(s, y(s), T(y)(s))ds + \int_{t_0}^{t} g(s, y(s))ds. \tag{2.6}$$

This completes the proof of part (ii).Q.E.D

Sec. (3)-Variation of Parameters for Generalized Perturbed Operator Differential Equations:

Suppose X is a Banach space, $A : D(A) \subset X \to X$ is the generator of a C^0- semigroup on X, $U \subset R \times X$ open and $f : U \to X$ is a continuous function such that $x \to f(t, x)$ is differentiable and $(t, \phi_0) \to D_x f(t, \phi_0)$ is continuous in U.

For $(t_0, x_0) \in U$, we denote by $x(t, t_0, \phi_0)$ the mild solution to the Cauchy problem

$x' = Ax + f(t, x(t), T(x)(t)), \quad for \quad t > t_0,$
and $x(t) = \phi(t), \quad for \quad t \leq t_0.$

These solutions are then related by the evolutionary property

$$x(t; t_0, \phi_0) = x(t; s, x(s; t_0, \phi_0))$$

for all $t_0 \leq s \leq t$. The initial function ϕ depends on t, t_0, and x_0. It is denoted by $\phi(t, t_0, x_0)$. The solution to the system says that **the future is determined completely by the present, with the past being involved only in that**

it determines the present. This is a deterministic version of the Markov property.

We make use of the following theorem in developing the variation formula for nonlinear differential equations. The Alekseeve's formula for $C^\circ - semigroups$ was generalized by *Hale 1977,[9] and Hale 1992, [10].* We will use the same idea to develop the operator differential equations.
Let X be a Banach space, operator $A : D(A) \subset X \rightarrow X$ is generator of a $C^\circ - semigroup$ on X, $f \in Lip(I, Y \times Z; Y)$ which is continuously differentiable with respect x.

Assume that $x(t, t_0, \phi)$ is a mild solution of the system

$$\begin{cases} x\prime = Ax + f(t, x(t), T(x)(t)), & t > t_0 \\ x(t) = \phi(t), & t \leq t_0, \end{cases} \tag{3.1}$$

In a finite dimensional operator A the following variational formula holds

$$\partial_1 x(t, t_0, \phi_0) = -\partial_2 x(t, t_0, \phi_0) \cdot x'(t). \tag{3.2}$$

When A is unbounded, one cannot expect to derive the same result for any $x_0 \in X$ since $x(t,t_0, \phi_0)$ in general is not differentiable with respect to t_0. We also need the differentiability of the solution $x(t, t_0, x_0)$ with respect to t.

The relation (3.2) has been generalized in *Hale 1992, [9]* for infinite dimensional variational operators when $f \in C^1$

$$\partial_1 x(t, t_0, \phi_0) = -\partial_2 x(t, t_0, \phi_0)[Ax_0 + f(t_0, x_0, T(x_0))]. \tag{3.3}$$

Assume that $(\partial/\partial\phi)y(t, t_0, \phi) \equiv U((t, t_0, \phi)$ exists then

$$U' = \frac{dU}{dt} = \frac{d}{dt}(\frac{\partial y}{\partial \phi}) = \frac{\partial}{\partial \phi}\frac{dy}{dt} = \frac{\partial}{\partial \phi}(f(t, y, T(y)) = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} = f_y.U.$$

This argument can lead us to the fact that if the operator $f_y \in Lip(I, Y \times Z; Y)$ then the solution to the system

$$U\prime = f_y[t, y(t, t_0, \phi)]U, \qquad t \geq t_0 \\ U(t_0) = I, \qquad for \qquad t \leq t_0. \tag{3.4}$$

has a unique solution. The system (3.4) is called **the variational equation.**

Notice that for all $t \leq t_0$, $\quad y(t, t_0, \phi) = \phi(t)$ then

$$U((t, t_0, \phi) = (\partial/\partial\phi)y(t, t_0, \phi) = (\partial/\partial\phi)\phi(t) = I.$$

Using chain rule for abstract functions, we get

$$\frac{d}{ds}y[t, t_0, \phi_0 + s(\psi_0 - \phi_0)] = U(t, t_0, \phi_0 + s(\psi_0 - \phi_0)](\psi_0 - \phi_0).$$

Thus, integrating the above system produces

$$y(t, t_0, \psi) - y(t, t_0, \phi) =$$

$$\int_0^1 U(t, t_0, \phi_0 + s(\psi_0 - \phi_0)](\psi_0 - \phi_0)ds. \tag{3.5}$$

The proof of the following proposition, theorems, and Lemmas are presented in *Ahangar 2017, [6].*

Proposition 3.1 (**Alexeeve's Theorem for Operator Differential Equations**): Suppose $f : U \subset R \times X \rightarrow X$ and $g : U \subset R \times X \rightarrow X$ are of class C^1. If $x(t, t_0, \phi_0)$ is the solution of equation
(3.1) through the initial state (t_0, ϕ_0) and $y(t, t_0, \psi_0)$ is the solution of

$$y\prime(t) = Ay + f(t, y(t), T(y(t)) + g(t, y(t)), \qquad t > t_0, \qquad (3.6)$$

$$y(t) = \phi(t), \qquad for \qquad t \leq t_0,$$

through (t_0, ϕ_0), then, for any $\phi_0 \in D(A) \cap D(\phi, Y)$ we have

$y(t, t_0, \phi_0) =$

$$x(t, t_0, \phi_0) + \int_{t_0}^{t} \frac{\partial}{\partial \phi_0} x(t, s, y(s, t_0, \phi_0)).g(s, y(s, t_0, \phi_0))ds. \qquad (3.7)$$

The next theorem will provide the **variation of parameters formula for operator differential equations.**

Theorem 3.1 (Variation of Parameters for NODE): The solution of the system (3.1) and (3.2) satisfy the following

$y(t, t_0, \phi(t)) = x(t, t_0, \phi(t)) +$

$$\int_{t_0}^{t} U(t, s, y(s, t_0, \phi(t)).g(s, y(s, t_0, \phi(t))ds, \qquad (3.8)$$

where $U(t, s, y(s, t_0, \phi(t)) = \frac{\partial x(t, t_0, \phi(t))}{\partial \phi}$, and assume that the inverse matrix $U^{-1}(t, t_0, v(t))$ exists.

The operator T in the differential equation (3.1) and (3.6) could be any delay, integral, composition, or Cartesian product of nonanticipating and Lipschitzian operators which will affect the nonperturbed solution $x(t, t_{0,} \phi(t))$. The variation formula (3.9) will be affected by the operator T through these changes.

Assuming that the variation of parameters is given, we will investigate some of the properties of this formula through the following conclusions for particular cases.

Corollary 3.1: Suppose that the conditions of H1 through H7 satisfy to guarantee the existence and uniqueness of the solution of the system (3.2). Assume also $\phi(t_0) = \phi_0$ is the initial state of the system $x' = Ax$. Then the relation (3.7) will be

$y(t, t_0, \phi_0) = x(t, t_0, \phi_0) + \int_{t_0}^{t} f(s, x(s), T(x)(s)]ds +$

$$\int_{t_0}^{t} \frac{\partial}{\partial x_0} x(t, s, y(s, t_0, \phi_0)).g(s, y(s, t_0, \phi_0))ds \qquad (3.9)$$

Proof: Assuming that $x(t, t_0, \phi_0)$ is a solution to the homogeneous equation $x\prime - Ax = 0$, then by the direct integration of the system (3.1) and applying the variation of parameters formula (2.1.3) to the nonlinear system (3.2), we will get the formula (3.9).

Corollary 3.2: Suppose that the conditions of H1 through H7 guarantee the existence and uniqueness of the solution of (3.1) and (3.6). Assume also in a particular case when $f \equiv 0$ and $g(t, x(t)) = g(t)$, the Alekseev's formula (3.7) *(see Alekseev, 1961 [11])* deduces the **variation of parameters formula**

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)g(s)ds \qquad (3.10)$$

for linear differential equation

$$x\prime(t) = A(t)x(t) + g(t). \qquad (3.11)$$

Corollary 3.3: Suppose that in the differential equation (3.1) $A = 0$, then the general solution of (3.5) about the equilibruim solution $y' = 0$ will be

$$y(t,t_0,x_0) = x_0 + \int_{t_0}^{t} f[s,x(s)]ds + \int_{t_0}^{t} g(s,y(s,t_0,x_0))ds. \qquad (3.12)$$

We will study the variation of parameters for operator differential equations with disturbed force operator functions. These nonlinear operators can involve the following types: delay, integrals, composition, or cartesian products of all nonanticipating and Lipschitzian operators.

Variation of Parameters for Nonanticipating Operator Differential Equations:

We will study the variation of parameters in operator differential equations. These nonlinear operators can involve the following types: delay, integrals, composition, or cartesian products of all nonanticipating and Lipschitzian operators.

Let us define the following hypothesis;
(H1) The operator A_t is a Semigroup.
(H2) Assume that functions f and g belong to the following Lip spaces.

$$f \in Lip(J, Y \times Z, Y), \qquad g \in Lip(J; Y, Y) \qquad (3.13)$$

(H3) Assume that $x(t,t_0,\phi_0)$ is a solution to the following operator differential equation

$$x'(t) = Ax + f(t, x(t), T(x)(t)), \qquad t > t_0 \qquad (3.14)$$
$$x(t,t_0,x_0) = \phi(t), \qquad t \leq t_0,$$

(H4) also let $y(t,t_0,\phi)$ be a solution to the following perturbed nonlinear operator differential equation

$$y'(t) = Ax + f(t, y(t), T(y)(t)) + g(t, y(t)), t > t_0 \qquad (3.15)$$

$y(t,t_0,\phi_0) = x(t,t_0,\phi), \qquad t \leq t_0,$
(H5) The inverse matrix $U^{-1}(t,t_0,v(t))$ exists.

The next theorem will provide the **variation of parameters formula for operator differential equations.**

Theorem 3.1 (Variation of Parameters for NODE): Given (H1) through (H5). The solution of the system (3.14) and (3.15) satisfy the following

$$y(t,t_0,\phi(t)) = x(t,t_0,\phi(t)) +$$
$$\int_{t_0}^{t} U(t,s,y(s,t_0,\phi(t)).g(s,y(s,t_0,\phi(t),T_2(y)(s,t_0,\phi)ds, \qquad (3.16)$$

where $U(t,s,y(s,t_0,\phi(t)) = \frac{\partial x(t,t_0,\phi(t))}{\partial \phi}$.

Proof: In a variation of parameters, we will determine a function $v(t)$ such that

$$y(t,t_0,\phi(t)) = x(t,t_0,v(t)), \qquad v(t_0) = \phi_0 \qquad (3.17)$$

is a solution process of (3.15).
From the system (3.15) and sample differentiation of (3.17) we get

$$y\prime(t) = \frac{\partial x(t,t_0,v(t))}{\partial t} + \frac{\partial x(t,t_0,v(t))}{\partial v}.\frac{\partial v(t)}{\partial t} =$$

$$f(t,y(t),T_1(y)(t)) + g(t,y(t),T_2(y)(t)) \qquad t \geq t_0. \qquad (3.18)$$

Since $x(t,t_0,v(t))$ is a solution of (2.2.5), then

$$g(t,y(t),T_2(y)(t)) = \frac{\partial x(t,t_0,v(t))}{\partial \phi}.v'(t). \qquad (3.19)$$

Since the inverse matrix $U^{-1}(t,t_0,v(t))$ exists, then

$$v'(t) = U^{-1}(t,t_0,v(t))g(t,x(t,t_0,v(t)),T_2(x)(t,t_0,v(t)) \qquad (3.20)$$

which implies

$$v(t) = \phi(t_0) + \int_{t_0}^{t} U^{-1}(s,t_0,v(s))g(s,x(s,t_0,v(s),T_2(x)(s,t_0,v(s))ds. \qquad (3.21)$$

Differentiation with respect to the second independent variable s when $t_0 \leq s \leq t$ implies that

$$\frac{dx(t,t_0,v(s))}{ds} = \frac{\partial x(t,t_0,v(s))}{\partial v}.\frac{\partial v(s)}{\partial s} = \frac{\partial x(t,t_0,v(t))}{\partial \phi}.v'(t) = U(t,t_0,v(s)).v'(t).$$

Substituting (3.20) for $v\prime(t)$ we will get the following for the right hand side

$$= U(t,t_0,v(s)).U^{-1}(s,t_0,v(s)).g(s,x(s,t_0,v(s)T_2(x)(s,t_0,v(s))$$

which implies

$$x(t,t_{0,}v(s)) = x(t,t_{0,}\phi(t)) + \int_{t_0}^{t} U(t,t_0,v(s)).U^{-1}(s,t_0,v(s)).g(s,x(s,t_0,v(s)),T_2(x)(s,t_0,v(s))ds.$$

Using variation definition (3.17) in the above relation, then we get the variation of parameters for nonlinear operator differential equation (3.15)

$$y(t,t_0,x_0) = x(t,t_{0,}\phi(t)) + \int_{t_0}^{t} U(t,t_0,v(s)).U^{-1}(s,t_0,v(s)).g(s,y(s,t_0,\phi(t)),T_2(y)(s,t_0,\phi(t)))ds \qquad (3.22)$$

The operator T in the differential equation (3.17) could be any delay, integral, composition, or cartesian product of nonanticipating and Lipschitzian operators which will effect the nonperturbed solution $x(t,t_{0,}\phi(t))$. The variation formula (3.22) will be effected by the operator T through these changes.

The next challenge is when the perturbation is effected by a delay or integral operator. In this case the argument of random operator T in the transformation

$$\zeta(t) = g(t,y(t),T(y)(t)) \qquad (3.23)$$

will appear in the variation formula.

Corollary 3.1: If the perturbation appeared in the system (3.14) of the Theorem (3.1) then the variation of parameters (3.16) will be

$$y(t,t_0,\phi(t)) = x(t,t_{0,}\phi(t)) + \int_{t_0}^{t} U(t,t_0,v(s)).U^{-1}(s,t_0,v(s)).g(s,y(s,t_0,\phi(t),T(y(s,t_0,\phi(t)))ds. \quad (3.24)$$

Sec. (4): Stability of the Zero Solutions of Operator Differential Equations

Consider a general deterministic operator differential equation

$$\begin{cases} y\prime(t) = Ay + f(t,y(t),T(y)(t)), & for \quad t > t_0, \\ y(t) = \phi(t), & for \quad t \leq t_0. \end{cases} \qquad (4.1)$$

$$\xi(t) = \begin{cases} \overline{y}(t) - y(t) & for\ almost\ all \quad t > t_0 and \\ \overline{\phi}(t) - \phi(t) & for\ almost\ all \quad t \leq t_0. \end{cases}$$

For particular value $t = t_0$

$\xi(t_0) = \overline{y}(t_0) - y(t_0) = \overline{\phi}_0 - \phi_0$. Substituting in the system yield
$\xi'(t) = \overline{y}'(t) - y'(t) = f(t, \overline{y}(t), T(\overline{y})(t)) - f(t, y(t), T(y)(t))$
$= f(t, y(t) + \xi(t), T(y + \xi)(t)) - f(t, y(t), T(y)(t))$

Given the solution y(t), the right hand side of this system will be in the following form,

$$\xi'(t) = \overline{f}(t, \xi(t), \overline{T}(\xi)(t)), \qquad t > t_0$$
$$\xi(t) = \overline{\phi}(t) - \phi(t) \qquad t \leq t_0.$$

Notice that the *trivial solution* will satisfy

$\overline{f}(t, 0, \overline{T}(0)(t)) = f(t, \xi(t), T(\xi)(t)) - f(t, y(t), T(y)(t)) = 0.$

This implies that the zero solution solves the new system. Thus study of the stability, uniform stability, and asymptotic stability of the equilibruim solution of the new system will be equivalent to the same stability of the system (4.1). Let us denote the perturbed system by

$$\begin{cases} z\prime(t) = Ay + f(t, z(t), T_1(z)(t)) + g(t, z(t), T_2(y)(t)), & t > t_0, \\ z(t) = \psi(t), & t \leq t_0 \end{cases} \quad (4.2)$$

Assume that the function f and operators $T_i (i = 1, 2)$ satisfy the conditions for existence of the unique solution. We would like to use the logarithmic norm and variation of parameters to study the stability of the system (4.2).

Notice that the interval $I = [t_0, \infty)$ where the instant moment s represents the present moment. The position of s determines where we are, before or after the initial point t_0.

Lemma 4.1: Let $f \in Lip(I, D \times Z; Y)$ and let f_y exist and be continuous for $y \in D$. Then for every y_1 and y_2 in the domain D

$$z_1 - z_2 = \int_0^t z_y[t, sy_1 + (1 - s)y_2](y_1 - y_2)ds \quad (4.3)$$

where $z(t, y) \equiv f(t, y(t), T(y)(t))$.

Proof: Let us define the operator W

$$z[t, sy_1 + (1 - s)y_2] = W(s) \qquad \Leftrightarrow \qquad z \equiv f(t, y(t), T(y)(t))$$

for $0 \leq s \leq 1$.

For the convex initial domain D this is a well defined operator. Using chain rule for Frechet derivatives, then

$$W\prime(s) = z_y[t, sy_1 + (1 - s)y_2](y_1 - y_2).$$

Since $W(0) = z(y_2)$ and $W(1) = z(y_1)$ and by integrating the above relation

$$W(1) - W(0) = \int_0^1 z_y[t, sy_1 + (1 - s)y_2](y_1 - y_2)ds$$
$$= z(y_1) - z(y_2)$$

or

$$f(t, y_1(t), T(y_1)) - f(t, y_2(t), T(y_2)) =$$
$$\int_0^1 f_y[t, sy_1 + (1 - s)y_2](y_1 - y_2)ds. \quad (4.4)$$

This completes the proof of (4.3)▶.

The stability of the linear system

$$y\prime = Ay + g \quad (4.5)$$

This completes the proof of (4.3)►.
The stability of the linear system

$$y\prime = Ay + g \tag{4.5}$$

depends only on the eigenvalues of A. If these eigenvalues are complex we would expect the system to have oscillatory solutions; if they have negative real parts we expect decaying solutions. This assertion will not be true in general for nonlinear system (3.1.1). That means **the negativity of the real part of eigenvalues does not imply the stability of a nonlinear system.**

Logarithmic Norm and Comparison theorem:

Definition 4.1: Let A be an operator from domain $D \subset M(I, Y) \to (Y, |\,|)$. The logarithmic k-norm of the operator A is defined by

$$\mu[A] = \lim_{h \to 0^+} (\| I + hA \|_k - 1)/h \tag{4.6}$$

where I is the identity matrix and $h \in R$. For more properties of logarithmic norm and the proof of the following lemma (see *Ladas and Lakshmikantham [12],[13], 14])* . In the following, we will study some properties of the logarithmic norm.

Assume that $(\partial/\partial\phi)y(t, t_0, \phi) \equiv U(t, t_0, \phi)$ exists then

$$U' = \frac{dU}{dt} = \frac{d}{dt}\left(\frac{\partial y}{\partial \phi}\right) = \frac{\partial}{\partial \phi}\frac{dy}{dt} = \frac{\partial}{\partial \phi}(f(t, y, T(y)) = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} = f_y.U.$$

This argument can lead to the fact that if the operator $f_y \in Lip(I, Y \times Z; Y)$ then the following system

$$\begin{cases} U\prime = f_y[t, y(t, t_0, \phi)]U, & \text{for } \ t > t_0 \\ \quad U(t_0) = I, \quad for & t \leq t_0. \end{cases} \tag{4.7}$$

has a unique solution. The system (4.7) is called **the variational equation.**
Notice that for all $t \leq t_0, \quad y(t, t_0, \phi) = \phi(t)$ then

$$U((t, t_0, \phi) = (\partial/\partial\phi)y(t, t_0, \phi) = (\partial/\partial\phi)\phi(t) = I.$$

Using chain rule for abstract functions, we get

$$\frac{d}{ds}y[t, t_0, \phi_0 + s(\psi_0 - \phi_0)] = U[t, t_0, \phi_0 + s(\psi_0 - \phi_0)](\psi_0 - \phi_0).$$

Thus by integrating the system

$$y(t, t_0, \psi) - y(t, t_0, \phi) =$$

$$\int_0^1 U(t, t_0, \phi_0 + s(\psi_0 - \phi_0)](\psi_0 - \phi_0)ds. \tag{4.8}$$

The following Lemma needed for further investigations.

Lemma 4.2: Let $A(t) \in B(Y)$ for each $t \in R^+$ and suppose y(t) is the solution of

$$y\prime = A(t)y, \quad for \quad t \geq t_0, \quad and$$
$$y(t) = \phi(t), \quad for. \quad t < t_0.$$

Then

$$\| y \|_k \leq \| \phi_0 \|_k \, e^{\int_{t_0}^t \mu[A(s)]ds}, \qquad t \geq t_0. \tag{4.9}$$

Proof: Let us denote the norm of the solution by $m(t) = |y(t)|$. Then for small $h > 0$

$$m(t + h) - m(t) = |y(t + h)| - |y(t)| \leq |y(t) + y'(t)h| + O(h) - |y(t)| \leq [(I + hA) - 1]|y(t)| + O(h)$$

where $\lim_{h \to 0^+} O(h)/h = 0$.

When we take the limit $\quad dm(t)/dt \leq \mu[A(t)].m(t)$. Then

$$|y(t)| \leq |\phi_0(t)|.e^{\int_{t_0}^{t}\mu[A(s)]ds}.$$

Since $|\phi_0(t)| \leq\| \phi_0 \|_k \, e^{kt}$ therefore

$$\Rightarrow \qquad |y(t)| \leq\| \phi_0 \|_k \, e^{kt}.e^{\int_{t_0}^{t}\mu[A(s)]ds}$$

Multiply each side by e^{-kt} and take the ess.sup for all $t \in I = [s,\infty)$, then

$$\| y \|_k \leq\| \phi_0 \|_k \, e^{\int_{t_0}^{t}\mu[A(s)]ds}.$$

Lemma 4.3: Let | | be a norm in the space M(I,Y). Suppose further that there exists a function $\nu \in M(I,R)$ such that

$$\mu[\frac{\partial f}{\partial y}(t,y,T(y))] \leq \nu(t) \qquad (4.10)$$

for almost all t in I. Then for two initial functions ϕ and ψ;

i) $\qquad |y(t,t_0,\psi)-y(t,t_0,\phi)| \leq |(\psi_0-\phi_0)|.e^{\int_{t_0}^{t}\nu(s)ds}$

ii) and for any two solutions y and z of (4.1) and (4.1)'

$$\| y(t_2)-z(t_2) \|_k \leq\| y(t_1)-z(t_1) \|_k \, e^{\int_{t_1}^{t_2}\nu(s)ds} \qquad (4.11)$$

for almost all $t_i \in I(i=1,2)$ satisfying $t_1 \leq t_2$.

Proof: (i) According to the relation (4.8) we have
$|y(t,t_0,\psi)-y(t,t_0,\phi)| \leq$

$$\int_0^1 |U(t,t_0,\phi_0+s(\psi_0-\phi_0))|.|(\psi_0-\phi_0)|ds \qquad (4.12)$$

According to the Lemma 4.2, since $A=f_y$ and $U\prime = AU$ then
$\| U(t) \| \leq\| U(t_0) \| .e^{\int_{t_0}^{t}\mu[f_y]ds}$.
Using the hypothesis (4.10)

$$\underset{t\in[0,s]}{Sup}\{\| U(t,t_0,\phi_0+s(\psi_0-\phi_0)] \|\} \leq e^{\int_{t_0}^{t}\nu(s)ds}. \qquad (4.13)$$

Thus the relation (4.12) will be in the following form
$|y(t,t_0,\psi)-y(t,t_0,\phi)| \leq |(\psi_0-\phi_0)|.\int_0^1 \sup\{|U[t,t_0,\phi_0+s(\psi_0-\phi_0)]|\}.ds$
$\leq |(\psi_0-\phi_0)|.e^{\int_{t_0}^{t}\mu[f_y(s)]ds} \leq |(\psi_0-\phi_0)|.e^{\int_{t_0}^{t}\nu(s)ds}$.
If we follow the steps in derivation of (4.9) in Lemma 4.2, then we will have the k-norm of this conclusion
$\| y(t,t_0,\psi)-y(t,t_0,\phi) \|_k \leq\| (\psi_0-\phi_0) \|_k .e^{\int_{t_0}^{t}\nu(s)ds}$

ii) Let us prove the second part of the Lemma 4.3 by subtracting $y(t,t_0,\phi)$ from each side of the variation of parameters for nonlinear system (4.1)' ;
$\| z(t,t_0,\psi)-y(t,t_0,\phi) \| =\| y(t,t_0,\psi)-y(t,t_0,\phi) \| +$
$\int_{t_0}^{t} \| U[t,s,z(s,t_0,\psi)] \| . \| g[s,z(s,t_0,\psi)] \| ds$
$\leq\| y(t,t_0,\psi)-y(t,t_0,\phi) \| +$
$\int_{t_0}^{t} \max_{t_0\leq s\leq t} \| U[t,s,z(s,t_0,\psi)] \| . \| g[s,z(s,t_0,\psi)] \| ds.$

$$\int_{t_0}^{t} \exp[\int_{s}^{t} \nu(\xi) d\xi] \parallel g(s, z(s, t_0, \psi)) \parallel_k ds. \qquad (4.14)$$

Sec. (5)-Stability of Operator Differential Equations:

The stability of delay functional differential equations has been presented in *Hale 1977 [9]*. The uniform and asymptotic stability of the nonlinear systems have been presented by *Lord and Mischell 1978 [15], [16])* using the variation of parameters. Many authors used variation of parameters to demonstrate the stability of the nonlinear systems *(see Brauer 1966 and 1967, [17] , [18])*. We will expand the idea for the operator differential equations in the following theorems.

Theorem: 5.1: Assume that the conditions for variation of parameters exist then

i) the estimate

$$\parallel U^{-1}(t, t_0, \phi_0) g(t, x(t, t_0, \phi_0)) \parallel \leq h(t, \parallel \phi_0 \parallel) W \qquad (5.1)$$

holds for $h \in Lip(I \times R^+, R^+), h(t, 0) \equiv 0,$*(W is defined in Lemma 4.1)*.

ii) the trivial solution of

$$u\prime = h(t, u), \qquad u(t_0) = u_0 \geq 0 \qquad (5.2)$$

is stable(uniformly stable).

iii) Further assume that the trivial solution of the unperturbed system (5.1) is stable(uniformly stable). Then the trivial solution of perturbed system (5.2) is stable (uniformly stable).

Proof: By the result of the variation of parameters

$$y(t, t_0, \phi_0) = x(t, t_0, v(t)) \qquad (5.3)$$

where v(t) is a solution of

$$v\prime(t) = U^{-1}(t, t_0, v(t)) g(t, x(t, t_0, v(t)), v(t_0) = \phi_0. \qquad (5.4)$$

The assumption (5.1) and the setting $m(t) = \parallel v(t) \parallel$ imply that the following inequality

$$D^+ m(t) \leq h(t, \parallel v(t) \parallel) = h(t, m(t)). \qquad (5.5)$$

Thus m(t) is a solution to the system (5.2) and $r(t, t_0, \parallel \phi_0 \parallel)$ is a maximal solution of (5.2), by comparison's theorem the relation (5.5) implies

$$\parallel v(t) \parallel = m(t) \leq r(t, t_0, \parallel \phi(t) \parallel), \qquad t \geq t_0. \qquad (5.6)$$

By assumption (ii), given $\varepsilon > 0, t_0 \in R^+$ there exists a $\delta_1(\varepsilon) > 0$ such that

$$\parallel x(t, t_0, \phi_0) \parallel < \varepsilon, \qquad t > t_0, \qquad (5.7)$$

if and ony if $\parallel \phi_0 \parallel < \delta_1(\varepsilon)$.

Also the stability of the trivial solution of (5.2) implies that there is a δ depending on δ_1 and t_0,

$$r(t, t_0, \phi_0) < \delta_1 \qquad whenever \qquad \parallel \phi_0 \parallel < \delta. \qquad (5.8)$$

Thus $\| v(t) \| \leq r(t, t_0, \| \phi_0 \|) < \delta_1$ whenever $\| \phi_0 \| < \delta$. The perturbed solution using (5.3) will satisfy

$$\| y(t, t_0, \phi_0) \| = \| x(t, t_0, v(t)) \| < \varepsilon, \tag{5.9}$$

for all $t > t_0$, whenever $\| \phi_0 \| < \delta$. This completes the proof of the stability(uniform stability).

Asymptotic Behavior:We use the notion of asymptotic equivalence to study the asymptotic behavior of the nonlinear operator differential equations. Two systems (5.1) and (5.2) are said to be asymptotically equivalent if given a solution $y(t, t_0, \psi_0)$ of (5.2) there exists a solution $x(t, t_0, \phi_0)$ of (5.1) satisfying

$$\lim_{t \to \infty} y(t, t_0, \psi_0) = x(t, t_0, \phi_0) \tag{5.10}$$

and conversely, given that $x(t, t_0, \phi_0)$ is a solution of (5.1) there exists $y(t, t_0, \psi_0)$ a solution of (5.2) satisfying (5.10).

The following theorem shows the characteristic behavior of the equoivalence in two systems.

Theorem 5.1 Assume that the hypothesis of the variation of parameters hold and $U(t, t_0, \phi_0)$ is bounded for $t \geq t_0$ and all initial functions ϕ are in the initial domain D. Let

i) the function h in

$$\| U^{-1}(t, t_0, \phi_0) g(t, x(t, t_0, \phi_0)) \| \leq h(t, \| \phi_0 \|)$$

hold for $h \in Lip(R^+ \times R^+, R^+), h(t, 0) \equiv 0$,is monotone and nondecreasing in u for each fixed t in R^+

ii) all solutions of

$$u\prime = h(t, u), \qquad u(t_0) = u_0 \geq 0 \tag{5.11}$$

are bounded on $t \geq t_0$.Then given that $y(t, t_0, \psi_0)$ is a solution of (5.2) there exists a solution $x(t, t_0, \phi_0)$ of (5.1) which is asymptotically equivalent.

Proof: By the relation of variation of parameters

$$y(t, t_0, \psi_0) = x(t, t_0, v(t))$$

where v(t) is a solution of the variational equation with $v(t_0) = \psi_0$,we use the mean value theorem for the variation of parameters and the boundedness of $U(t, t_0, \phi_0)$ in the following

$$\begin{aligned} \| y(t, t_0, \psi_0) - x(t, t_0, \phi_0) \| &= \| x(t, t_0, v(t)) - x(t, t_0, \phi_0) \| \\ &\leq \int_0^1 \| U(t, t_0, \phi_0) \| . \| v(t) - \phi_0 \| ds \\ &\leq \sup_{t \geq t_0, \phi_0 \in D} \{ \| U(t, t_0, \phi_0) \| \}. \| v(t) - \phi_0 \| \end{aligned}$$

$$\| y(t, t_0, \psi_0) - x(t, t_0, \phi_0) \| \leq K \| v(t) - \phi_0 \| . \tag{5.12}$$

When $t \to \infty$ the right hand side approaches zero and $y(t, t_0, \psi_0) \to x(t, t_0, \phi_0)$. This proves the asymptotic equivalence between the two solutions.

Sec. (6)- K-norm Stability and Variation of Initial Function

Consider the following abstract operator differential equation

$$y'(t) = f(t, y(t), T(y)(t), \qquad for \qquad t > t_0 \tag{6.1}$$
$$y(t) = \phi(t), \qquad for \qquad t \leq t_0 \tag{6.2}$$

for any initial function ϕ in the initial domain D, where y(t) is the unpertubed solution. Assume that a perturbation is imposed on the system (6.1)-(6.2) with initial function $\psi(t)$,

$$z'(\tau) = f(\tau, z(\tau), T(z)(\tau) + g(\tau, z(\tau)), \qquad \tau > \tau_0 \qquad (6.3)$$
$$z(\tau) = \psi(\tau), \qquad for \qquad \tau \leq \tau_0 \qquad (6.4)$$

for $f \in Lip(I, Y \times Z, Y)$, $g \in Lip(I, Y; Y)$, $\psi \in D$, and the operator T is nonanticipating and Lipschitzian. When we assume $f(t, 0) \equiv 0$, then the system (6.1)-(6.2) has a trivial solution.

The operator T in this type of differential equations can be assumed as fixed and does not vary, thus the solution $y(t, t_0, \phi, T(\phi))$ will be independent from T and is equivalent to $y(t, t_0, \phi)$. That means the parameters in this variation are (t_0, ϕ).

Definition 6.1 (k-norm Stability): The equilibruim solution of (6.1)-(6.2) is said to be

- k-norm **stable** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\| \phi \|_k < \delta$ implying that $\| y(t, t_0, \phi) \|_k < \varepsilon$ for all $t \geq t_0$;

- It is **asymptotically stable** if it is stable and there exists a $\delta_0 > 0$ such that

$$\| \phi \|_k < \delta_0 \quad \text{implies that} \quad \lim_{t\to\infty} y(t, t_0, \phi) = 0.$$

Definition 6.2: Stability of the Initial Solutions; The initial solution ϕ is stable, that is for every $\varepsilon > 0$ and every $t_0 \geq 0$ there exists a $\delta > 0$ (depends on ε, and possibly t_0) such that whenever $|\phi(t_0) - y_0| < \delta$, the solution $y(t, t_0, y_0)$ exists for $t > t_0$ and satisfies $|\phi(t) - y(t, t_0, y_0)| < \varepsilon$ for $t > t_0$.

The following lemma will be useful for the proof of the stability of the solutions to the system.

Lemma 6.1: Assume that two initial functions ϕ and ψ belong to the same initial domain D. Let $y = y(t, t_0, \phi)$ and $\overline{y} = y(t, t_0, \psi)$ be their associated solutions to the system (5.1) respectively. Then there exists a real number L such that

$$\| y(., t_0, \phi) - y(., t_0, \psi) \|_k \leq L \| \phi - \psi \|_k \qquad (6.5)$$

where $L = k/(k- \| f \| (1 + c))$, for some constant numbers k and c.

Proof: The system (6.1) is equivalent to its integral, that is

$$y(t, t_0, \phi) = \phi(t) + \int_{t_0}^{t} f(s, y(s), T(y)(s))ds. \qquad (6.6)$$

Since f belongs to the space $Lip(I, Y \times Z; Z)$, then

$$| y(t, t_0, \phi) - y(t, t_0, \psi) | \leq |\phi(t) - \psi(t)| + \int_{t_0}^{t} \| f \| \{|y(s) - \overline{y}(s)| + |T(y)(s) - T(\overline{y})(s)|\}ds.$$
$$\leq |\phi(t) - \psi(t)| + \int_{t_0}^{t} \| f \| e^{ks}\{\| y - \overline{y} \|_k + \| T(y) - T(\overline{y}) \|\}ds.$$

Since the operator T is an exponential type operator, there exists a constant number c such that $\| T \|_k < c$, for $k \geq k_0$. Thus

$$| y(t, t_0, \phi) - y(t, t_0, \psi) | \leq |\phi(t) - \psi(t)| + \qquad (6.8)$$
$$\| f \| e^{kt}\{\| y - \overline{y} \|_k + \| T \|_k . \| y - \overline{y} \|_k\}/k$$
$$\leq |\phi(t) - \psi(t)| + \| y - \overline{y} \|_k . \| f \| .(1 + c)e^{kt}/k.$$

Multipling both sides by e^{-kt} and taking ess.sup over the interval J the equality (6.8) will be

$$\| y - \overline{y} \|_k \leq \| \phi - \psi \|_k +$$

$$| y(t,t_0,\phi) - y(t,t_0,\psi) | \leq |\phi(t) - \psi(t)| + \qquad (6.8)$$

$$\| f \| e^{kt}\{\| y - \overline{y} \|_k + \| T \|_k . \| y - \overline{y} \|_k\}/k$$
$$\leq |\phi(t) - \psi(t)| + \| y - \overline{y} \|_k . \| f \| .(1+c)e^{kt}/k.$$

Multipling both sides by e^{-kt} and taking ess.sup over the interval J the equality (6.8) will be

$$\| y - \overline{y} \|_k \leq \| \phi - \psi \|_k +$$

$$\| y - \overline{y} \|_k . \| f \| .(1+c)/k. \qquad (6.9)$$

From the inequality (6.9) the k-norm of the difference between two solutions y and z will be calculated and measured by the following inequality

$$\| y - \overline{y} \|_k \leq L. \| \phi - \psi \|_k \qquad (6.10)$$

where $L = k/[k - \| f \| .(1+c)]$ for constants k and c. Q.E.D.

Notice that the relation (6.5) could be verified by the properties of Nant-Lip operators. To show this, let us define the operator G which transforms the initial function into a solution $y(t,t_0,\phi)$ in the following form

$$y = G(\phi) \qquad \Leftrightarrow \qquad y(t) = \phi(t) + \int_{t_0}^{t} f(s, y(s), T(y)(s))ds.$$

We can define $\overline{y} = G(\psi)$ to represent a solution initiated by the function ψ. Since the operators are Nant-Lip then it is an operator of exponential type. Thus, there exist constant numbers L and k_0 such that

$$\| G(\phi) - G(\psi) \|_k \leq L \| \phi - \psi \|_k$$

for some $k \geq k_0$. Substitute the operator G, then

$$\| y(.,t_0,\phi) - y(.,t_0,\psi) \|_k \leq L \| \phi - \psi \|_k .$$

This is actually the conclusion of the Lemma 6.1.

Theorem 6.1: Suppose that

i) the conditions H1-H5 presented in sec. (1) for existence and uniqueness of the solution of the systems (6.1) and (6.2) hold (according to the Theorem (1.1) and Proposition 2.1); and

ii) $f(t, 0, T(0)) \equiv 0$,

iii) the equilibruim solution of the systems (6.1)-(6.2) is stable with respect to the variation of initial functions in the initial domain of attractions.
Then the trivial solution of the system (6.3)-(6.4) is k-norm stable.

Proof: Let us take $\psi \equiv 0$ in the relation (6.10). Thus by the relation (6.6) the solution $z(t)$ will be identically zero. This implies that $\| y \|_k \leq L. \| \phi \|_k$. For every $\varepsilon > 0$ there exists a number $\delta = \varepsilon/L$ such that

$\| \phi \|_k < \delta$ implies $\| y \|_k < \varepsilon$
for almost all $t \in I$. This proves that the solution y is **k-norm stable**.

Lemma (6.2): Assume that hypotheses H1-H5 hold to gaurantee the solution y of the systems (6.1) and (6.2), and the solution z of the systems (6.3) and (6.4). If the operator $U = \frac{\partial y}{\partial \phi}$ is Nant-Lip then we have the following estimate

$\| y(t,t_0,\phi) - z(t,t_0,\psi) \|_k = L \| \phi - \psi \|_k + be^{-kt}$.

Proof: Using Alekseev's variation of parameters
$z(t,t_0,\psi) - y(t,t_0,\phi) = y(t,t_0,\psi) - y(t,t_0,\phi(t)) +$

$$\int_{t_0}^{t} U[t,s,z(s,t_0,\psi)].g[s,z(s,t_0,\psi)]ds. \qquad (6.11)$$

Using the estimated relation in (6.10) and the calculus of Banach space $(Y,|.|)$

$$| y(t,t_0,\phi) - z(t,t_0,\psi) | \leq$$
$$|y(t,t_0,\phi) - y(t,t_0,\psi) | + \int_{t_0}^{t} |U[t,s,z(s,t_0,\psi)]|.|g[s,z(s,t_0,\psi)]|ds$$

$$\leq \| y - \overline{y} \|_k e^{kt} + \int_{t_0}^{t} |U[t,s,z(s,t_0,\psi)]|.e^{ks}. \| g[s,z(s,t_0,\psi)] \|_k ds \qquad (6.12)$$

for almost all t in the interval I. The solution z of the perturbed system (6.3) and (6.4) is in the initial domain and it is Nant-Lip. The perturbed function g is in the Lip space. Thus the composition operator

$$u = G(z) \qquad \Leftrightarrow \qquad u(s) = g[s,z(s,t_0,\psi)]$$

is Nant-Lip and it will be an operator of exponential type. This property implies that there exists a constant number c and k_0 such that $\| G \|_k \leq c$ for all $k \geq k_0$. The right hand side of the relation (6.12) will be

$$\leq \| y - \overline{y} \|_k e^{kt} + c\int_{t_0}^{t} |U[t,s,z(s,t_0,\psi)]|.e^{ks}ds$$

We multiply both sides by e^{-kt} and take the ess.sup over the interval I and use the relation (6.10) to obtain,

$$\| y - z \|_k \leq L \| \psi - \phi \|_k + ce^{-kt}\int_{t_0}^{t} |U(t,s,z(s,t_0,\psi))|e^{ks}.ds$$

$$\leq L \| \psi - \phi \|_k + c\int_{t_0}^{t} ess.\sup\{|U[t,s,z(s,t_0,\psi)]e^{-kt}\}|.e^{-ks}ds \qquad (6.13)$$

Since U is an operator of exponential type, there exists a constant number c such that $\| U \|_k < d$, for all $k \geq \overline{k_0}$. Thus,

$$\| y(.,t_0,\phi) - z(.,t_0,\psi) \|_k \leq$$

$$L \| \phi - \psi \|_k + c\int_{t_0}^{t} \| U \|_k .e^{-ks}ds \qquad (6.14)$$

for all $k \geq n_0 = \max\{k_0, \overline{k_0}\}$.

In the relation (6.14) if the operator U is an operator of exponential type then there exists a constant number d such that $\| U \|_k \leq d$, for some $k \geq n_0$, thus

$$\| y(.,t_0,\phi) - z(.,t_0,\psi) \|_k$$
$$\leq L \| \phi - \psi \|_k + c.d| \int_{t_0}^{t} e^{-ks}|ds$$
$$\leq L \| \phi - \psi \|_k + \tfrac{cd}{k}(e^{-kt} - e^{-kt_0})$$
$$\leq L \| \phi - \psi \|_k + \tfrac{cd}{k}(e^{-kt}) \qquad (6.15)$$

From the inequality (6.15) the k-norm of the variation on two solutions, that is y(t) and z(t) will be calculated as follows

$$\| y(.,t_0,\phi) - z(.,t_0,\psi) \|_k \leq L \| \phi - \psi \|_k + be^{-kt} \qquad (6.16)$$

where $b = \frac{cd}{k}$ for constants $k \geq n_0$. *Q.E.D.*

Remark: Let $f(t,0,T(0)) \equiv g(t,0) \equiv 0$, then the solution of the systems (6.3) and (6.4) is

$z\prime(t) = 0$ for $t \geq t_0$ and $z(t) = \psi(t) \equiv 0$ for $t < t_0$

will be the trivial solution $z(t,t_0,0) \equiv 0$.

Theorem 6.2: Assume that

i) the hypothesis of the Proposition 2.1 hold

ii) $f(t,0,T(0)) \equiv g(t,0) \equiv 0, \qquad for\ almost$ all t in R^+,

iii) f_y is a Nant-Lip,

iv) the equilibruim solution of (6.1) - (6.2) is asymptotically stable for variation of initial function in the initial domain of attractions.

Then the trivial solution of the system (6.3) and (6.4) is asymptotically stable for all choices of (t_0, ϕ).

Proof: Assume two solution functions y and z in the initial domain $D \subset M(I,Y)$ satisfying the operator dynamical systems (6.3) and (6.4) associated to initial functions ϕ and ψ respectively. If we take $\psi \equiv 0$ in the relation (6.10) we obtain the following

$$\| y(t,t_0,\phi) - z(t,t_0,0) \|_k \leq L \| \phi \|_k + be^{-kt}.$$

The relation $\| \phi \|_k \leq \delta$ implies the following limit

$$lim_{t\to\infty} \| y(t,t_0,\phi) - z(t,t_0,0) \|_k = 0$$

This proves that the trivial solution is asymptotically stable (we call $z(t,t_0,0)$ a zero solution).

Sec. (7)-Lyapanov Stability of the Operator Differential Equations:

We will try to extend the deterministic Lyapanov Stability for the operator differential equations.

Suppose that y and z are solutions to the systems (6.1) through (6.4). Define the **Lyapanov functional** $v(t) = V(t, y_t)$ as follows:

i) $V : (\alpha,\infty) \times D(\phi,Y) \to [0,\infty)$ is continuous and its first partial derivatives exist.

ii) V has partial derivatives and the total derivative of $V(t,y(t))$ will be $\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial V}{\partial y_i}$

where v_i and y_i are components of V and y.

iii) V is nonnegative in the region and vanishes at the origin.

For some $t_0 > \alpha$ let $y = y(.,t_0,\phi)$. We wish to show the stability of the trivial solution at t_0, so that if $v(t_0) = V(t, y(t_0))$ is sufficiently small then $v(t) = V(t, y_t)$ remains small for $t \geq t_0$.

Recall the notations for the interval $I = [0,a]$ for constant real number a, t_0, and $J = (-\infty, 0) \cup I$. We start with the following theorem.

Theorem 7.1: Let w and W be continuous nondecreasing functions from $I \to R^+$ such that $w(0) = W(0) = 0$. If there exists a nonnegative functional $V : (\alpha,\infty) \times D(\phi,Y) \to [0,\infty)$, such that

i) $V(t,\psi) \geq w(\| \psi(0) \|)$,

ii) $V(t,\psi) \leq W(\| \psi \|_k)$, and

iii) for every $(t_0,\phi) \in J \times D(\phi,Y)$ the Lyapanov function is nonincreasing in $t \in [t_0, a)$, then the trivial solution of (6.1) - (6.2) is uniformly stable.

Proof:

Let $\varepsilon > 0$ since w and W are nondecreasing thus

$$0 = w(0) < w(\varepsilon) \quad and \quad 0 = W(0) < W(\varepsilon).$$

Choose $\delta = \delta(\varepsilon) \in (0,\varepsilon)$ such that

$$W(\delta) < w(\varepsilon). \tag{7.1}$$

Let $(t_0, \phi) \in J \times D(\phi, Y)$ with $\| \phi \|_k < \delta$. By the existence and uniqueness theorem when T is a nonantipating and Lipschitzian operator and f belons to the space $Lip(J, Y \times Z, Y)$ and there exists a unique solution $y = y(., t_0, \phi)$ through (t_0, ϕ) to the system (6.1) in the initial domain $D(\phi, Y)$. Thus using hypothesis (i)-(iv) and condition (7.1) for $t_0 < t < a$,

$w(\| y(t) \|_k) \leq V(t, y_t) \leq V(t_0, \phi) \leq W(\| \phi \|_k) \leq W(\delta) < w(\varepsilon)$.

Since w is nondecreasing, for $t_0 < t < a$

$\| y_t \|_k < \varepsilon \Rightarrow w(\| y_t \|_k) < w(\varepsilon)$.

This shows that if $v(t_0) = V(t_0, \phi)$ is sufficiently small then $v(t) = V(t, y(t, t_0, \phi))$ remains small for $t \geq t_0$.

This result can be generalized for continuing the solution for $a \to \infty$. It will show that the trivial solution of (6.1) is uniformly stable.

Definition 7.1: (uniformly asymptotically stable) The trivial solution of (6.1) is said to be uniformly asymptotically stable if it is uniformly stable and furthermore, there exists δ_1(independent of t_0) such that whenever $t_0 > \alpha$ and

$$\| \phi \|_k < \delta_1 \quad \Rightarrow \quad \| y(t, t_0, \phi) \|_k \to 0, \quad \text{as} \quad t \to \infty.$$

The third condition of the Lyapanov functional can be substituted by a stronger condition

$\frac{d}{dt} V(t, y_t) \leq -h(\| y_t \|_k)$.

(*See Driver 1977, [19]*)

Theorem 7.2: If there exists a Lyapanov function $V(t, y(t))$ defined over the solution space and it satisfies the following conditions

i) $V \in Lip(I, Y; R)$

ii) $V(t, 0) \equiv 0$

iii) $V(t, y(t)) \geq h(y(t))$ for some function h.

iv) $V\prime(t, y) \leq 0$,

also assume that the equilibruim solution of the unperturbed system is stable, then the equilibruim solution of the perturbed equation

$$\begin{aligned} z'(t) &= f(t, z(t), T(z)(t)) + g(t, z(t), \qquad t > t_0 \\ z(t) &= \phi(t), \qquad t \leq t_0. \end{aligned} \tag{7.2}$$

is stable in the norm.

Proof: Let us define $\mathbf{V(t_0,}\phi(t)) = \Psi(t_0)$. We can write

$$V(t, z(t)) = \Psi(t_0) + \int_{t_0}^{t} V'(s, z(s)) ds. \tag{7.3}$$

Using the hypothesis inequality, there exists a constant m such that

$$V(t, z(t)) \geq m \| z \|_k$$

Thus $\Psi(t) + \int_{t_0}^{t} V'(s, z(s)) ds \geq m \| z \|_k$.

Since the integral is a negative number then $\Psi(t) \geq m \| z \|_k$. This will imply the following estimate

$$\| z \|_k \leq \frac{\Psi(t)}{m}. \tag{7.4}$$

The Lyapanov function $V \in Lip\,(\mathrm{R}^+, R)$, thus about the equilibruim solution $y \equiv 0$ the Lipschitzian inequality implies that there exists a constant real number L such that

$$\| V(t,z(t)) - V(t,y(t)) \|_k \leq L \| z - y \|_k \tag{7.5}$$

for some $k \geq k_0$. Since $V(t,0) = 0$ then

$$\| V(t,z(t)) \|_k \leq L \| z \|_k . \tag{7.6}$$

Thus

$$\| \Psi \|_k \leq L. \| z \|_k . \tag{7.7}$$

Since $z \in M(I,Z)$ assume that $0 < \| z \|_k < \gamma$. Now for every $\varepsilon > 0$ choose $\| z \|_k < \delta(\varepsilon)$ in such a way that

$\delta(\varepsilon) = \inf\{m\varepsilon/L, \gamma\}$.

It follows that for every initial function ϕ when $\| \Psi \|_k < \delta$ then using both inequalities (7.4) and (7.7) we have

$\| z \|_k \leq \frac{\|\Psi\|_k}{m} \leq \frac{L.\|z\|_k}{m} \leq \frac{L.\delta(\varepsilon)}{m} \leq \frac{Lm\varepsilon/L}{m} < \varepsilon.$

Hence this proves the stability of the equilibruim solution in k-norm.

Sec. (8)- Conclusion and Discussion:

The variation of parameters discovered by *Alekseeve 1961* is a great tool to study this kind of nonlinear system and use this conclusion for stability and asymptotic behavior of a nonlinear system. The solutions to a nonlinear operator differential equations of type (6.1 and 6.2) which includes all operators T satisfying nonanticipating and Lipschitzian conditions, also reviewed here, have a huge range of application.

We used a general method of variation of parameters of Alekseeve's type for a nonlinear operator differential equations (NODE) to study the stability and asymptotic behavior of the nonlinear system.

In this generalization, we assumed that $A(t)$ is Lebesgue summable from I into $M(I,Z)$ and $f \in Lip(I,Y;Z)$ is a perturbation in the system (7.1) then the solution process $y(t) = y(t,t_0,y_0)$ of the following nonlinear system

$$y\prime(t) = A(t)y(t) + f(t,y(t)), \qquad y(t_0) = y_0$$

for all $t \geq t_0$. These nonlinear operators can involve varieties of many types of operators like: delay, integrals, composition, or Cartesian products of all nonanticipating and Lipschitzian operators.

For operator in this paper we proved and demonstrated a general form of Alekseeve Theorem when a non linear perturbed system (7.1) includes a $C_0 - semigroup$ of opeartor A.

All important conditions in (H1) through (H5) are connecting the nonanticipating property of T, semigroup property of A_t, and Lipschitzian property of the functional f. The general form of Alekseev's theorem for variation of parameters helped us to find the solution to the purturbed system. This perturbed solution for nonanticipating dynamic systems can be used to study the stability and asymptotic behavior of the nonlinear operator system. Two major issues related to Stability and behavior using the Alekseev's type Variatiion of Parameters can be focused in the future.

First, is the numerical algorithm and computational program to produce the solution to such a general form of nonlinear variational of parameters method. Second, develop applications and numerical computation for the stability of the nonlinear system to operator differential equations.

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