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On Solving Comfortable Fractional Differential Equations

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Abstract

This paper adopts the relationship between conformable fractional derivative and the classical derivative. By using this relation, the comfortable fractional differential equation can transform to a classical differential equation such that the solution of these differential equations is the same. Two examples have been considered to illustrate the validity of our main results.

Keywords: Differential Equation; Conformable Fractional Differential Equations; Conformable Fractional Derivative.

1. Introduction

The famous letter written by L'Hopital to Leibniz in 30/9/1695 asking him about the possibility that n in Leibniz's rule be a fraction. "What if $n = \frac{1}{2}$." was a birthday of fractional calculus. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born [1-7]. Fractional differential equations are a generalization of the ordinary differential equation to arbitrary non-integer order . Fractional differential equations arise in many complex systems in nature and society with many dynamics, such as rheology, porous media, viscoelasticity, electrochemistry, electromagnetism, signal processing, dynamics of earthquakes, optics, geology, viscoelastic materials, biosciences, bioengineering, medicine, economics, probability and statistics, astrophysics, chemical engineering, physics, splines, tomography, fluid mechanics, electromagnetic waves, nonlinear control, control of power electronic, converters, chaotic dynamics, polymer science, proteins, polymer physics, electrochemistry, statistical physics, thermodynamics, neural networks and many more, see e.g. Schneider and Wyss [8], Mainardi [9], Magin et. al. [10], Magin [11], Metzler and Klafter [12], Beyer and Kempfle [13], Lederman et. al. [14], Bagley and Torvik [15], Riewe [16], Kulish and Lage [17], Wyss [18], Song and Wang [19], and the works by Diethelm and Freed cf. Keil et. al. [20].

The Leibniz rule and chain rule are not valid for Riemann–Liouville derivative and Caputo derivative, which sometimes prevent us from using these derivative in the physical models. Recently, Khalil et al introduced the new fractional derivative called conformable fractional derivative and integral [21]. This derivative is well-behaved and satisfied the Leibniz rule and chain rule. In this article, we explain the relationship between conformable fractional derivative and the classical derivative. This relation enable us to transform the comfortable fractional differential equation in to classical differential equation such that the solution of both these differential equations is the same.

2. Comfortable Derivative

The fractional derivative has different definitions [ ], and exploiting any of them depends on the boundary conditions and the specifics of the considered physical systems and processes. The first definition of fractional derivative which has been proposed in the literature is the so-called Riemann-Liouville definition which reads as follows.

Definition 2.1. (Conformable fractional derivative) [ ] Let $f:(0,\infty)\to R$ be a function. Then for all $t>0$, the conformable fractional derivative of $f(t)$ of order α is

$$T_\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},\quad \alpha\in(0,1).$$

If $f(t)$ is $\alpha-$differentiable in some $(0,a)$, $a>0$ and $\lim_{t\to 0^+} f^{(a)}(t)$ exists, then it can be defined

$$f^{\alpha}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t).$$

Definition 2.2. (sequential conformable fractional derivative) [22] Let $f:(0,\infty)\to R$ be a a function. Then for all $t>0$, the sequential conformable fractional derivative of $f(t)$ of order (α,n) is

$$^{n}T_\alpha f(t)=\underbrace{T_\alpha T_\alpha \cdots T_\alpha}_{n\,\text{times}} f(t)$$

Definition 2.3. (higher order conformable fractional derivative) [22] Given a function $f:(0,\infty)\to R$. Let $n<\alpha\le n+1$ and $\beta=\alpha-n$ Then the conformabl fractional derivative of $f(t)$ of order α, where $f^{(n)}(t)$ exists, is

$$T_\alpha(f(t))=T_\beta(f^{(n)}(t))$$

Definition 2.4. (Conformable fractional integral) [21] Let $f:(0,\infty)\to R$ be a a function. Then for all $t>0$, $\alpha\in(0,1)$ the conformable fractional integral of $f(t)$ of order α is

$$(I_\alpha^a f)(t)=\int_a^t f(x)d_\alpha(x)=\int_a^t x^{\alpha-1}f(x)dx,\quad a<t$$

where the integral is the usual Riemann improper integral.

Theorem 2.1. [21] Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then

(1) $T_\alpha(af(t)+bg(t))=aT_\alpha(f(t))+bT_\alpha(g(t))$ for all $a,b\in R$.

(2) $T_\alpha(t^p)=pt^{p-\alpha}$ for all $p\in R$.

(3) $T_\alpha(c)=0$ for all constant c.

(4) $T_\alpha(f(t)g(t))=T_\alpha(f(t))g(t)+f(t)T_\alpha(g(t))$.

(5) $T_\alpha(f(t)/g(t))=\dfrac{T_\alpha(f(t))g(t)-f(t)T_\alpha(g(t))}{g^2(t)}$.

(6) In additional, if $f(t)$ is differentiable, then $T_\alpha(f(t))=t^{1-\alpha}\dfrac{df(t)}{dt}$.

Theorem 2.2. [21] Let $f(t)$ be any continuous function in the domain of I_α^a and $\alpha\in(0,1]$. Then $T_\alpha I_\alpha^a f(t)=f(t)$ for all $t\ge a$

Definition 2.2. (sequential conformable fractional derivative) [22] Let $f:(0,\infty)\to R$ be a a function. Then for all $t>0$, the sequential conformable fractional derivative of $f(t)$ of order (α,n) is

$$^{n}T_{\alpha}f(t)=\underbrace{T_{\alpha}T_{\alpha}\cdots}_{n\,\text{times}}T_{\alpha}f(t)$$

Definition 2.3. (higher order conformable fractional derivative) [22] Given a function $f:(0,\infty)\to R$. Let $n<\alpha\le n+1$ and $\beta=\alpha-n$ Then the conformabl fractional derivative of $f(t)$ of order α, where $f^{(n)}(t)$ exists, is

$$T_{\alpha}(f(t))=T_{\beta}(f^{(n)}(t))$$

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$$(I_{\alpha}^{a}f)(t)=\int_{a}^{t}f(x)d_{\alpha}(x)=\int_{a}^{t}x^{\alpha-1}f(x)dx,\quad a<t$$

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Theorem 2.1.[21] Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then

(7) $T_{\alpha}(af(t)+bg(t))=aT_{\alpha}(f(t))+bT_{\alpha}(g(t))$ for all $a,b\in R$.

(8) $T_{\alpha}(t^{p})=pt^{p-\alpha}$ for all $p\in R$.

(9) $T_{\alpha}(c)=0$ for all constant c.

(10) $T_{\alpha}(f(t)g(t))=T_{\alpha}(f(t))g(t)+f(t)T_{\alpha}(g(t))$.

(11) $T_{\alpha}(f(t)/g(t))=\dfrac{T_{\alpha}(f(t))g(t)-f(t)T_{\alpha}(g(t))}{g^{2}(t)}$.

(12) In additional, if $f(t)$ is differentiable, then $T_{\alpha}(f(t))=t^{1-\alpha}\dfrac{df(t)}{dt}$.

Theorem 2.2.[21] Let $f(t)$ be any continuous function in the domain of I_{α}^{a} and $\alpha\in(0,1]$. Then $T_{\alpha}I_{\alpha}^{a}f(t)=f(t)$ for all $t\ge a$.

3. Main Results

In this section we will give some result about the relationship between the conformable fractional derivative and the classical derivative.

Lemma 3.1. For any smooth function $y(t):(0,\infty)\to R$, the conformable fractional derivative satisfied $^{n}T_{\alpha}[y(t)]=\dfrac{d^{n}Y(x)}{dx^{n}}$,for all $\alpha\in(0,1]$ and for any positive integer number n, where $y(t)\equiv Y(x)$, and $x=\dfrac{t^{\alpha}}{\alpha}$.

Proof: The mathematical induction will be used to prove ${}^{n}T_{\alpha}[y(t)]=\frac{d^{n}Y(x)}{dx^{n}}$, for any positive integer number n as follows.

Since ${}^{0}T_{\alpha}[y(t)]=y(t)$ and $\frac{d^{0}Y(x)}{dx^{0}}=Y(x)$, it is clear that the statement holds for $n=0$.

By using the chain rule, one can have $\frac{dY(x)}{dx}=\frac{dy(t)}{dt}\frac{dt}{dx}=t^{1-\alpha}\frac{dy(t)}{dt}$

On the other hand, from theorem (2.1), we have $T_{\alpha}[y(t)]=t^{1-\alpha}\frac{dy(t)}{dt}$. So that the statement holds for $n=1$.

Now, let ${}^{k}T_{\alpha}[y(t)]=\frac{d^{k}Y(x)}{dx^{k}}=h(t)$ holds for $k\in N$. We need prove ${}^{k+1}T_{\alpha}[y(t)]=\frac{d^{k+1}Y(x)}{dx^{k+1}}$, so one compute ${}^{k+1}T_{\alpha}[y(t)]$ and $\frac{d^{k+1}Y(x)}{dx^{k+1}}$ as follows

$${}^{k}T_{\alpha}[y(t)]=T_{\alpha}[{}^{k}T_{\alpha}[y(t)]]=T_{\alpha}[h(t)]=t^{1-\alpha}\frac{dh(t)}{dt}$$

$$\frac{d^{k+1}Y(x)}{dx^{k+1}}=\frac{d}{dx}[\frac{d^{k}Y(x)}{dx^{k}}]=\frac{d}{dx}[h(t)]=\frac{dh(t)}{dt}\frac{dt}{dx}=t^{1-\alpha}\frac{dh(t)}{dt}$$

So, ${}^{k+1}T_{\alpha}[y(t)]=\frac{d^{k+1}Y(x)}{dx^{k+1}}$. Therefore, the statement holds for all positive integer number n.

Remark 3.1. For any smooth function $y(t):(0,\infty)\to R$, and by simple computation, one can have the following sequential conformable fractional derivative of $y(t)$,where $\alpha\in(0,1]$

1- $T_{\alpha}[y(t)]=t^{1-\alpha}y'(t)$

2- ${}^{2}T_{\alpha}[y(t)]=t^{2-2\alpha}y''(t)+(1-\alpha)t^{1-2\alpha}y'(t)$

3- ${}^{3}T_{\alpha}[y(t)]=t^{3-3\alpha}y'''(t)+3(1-\alpha)t^{2-3\alpha}y''(t)+(1-\alpha)(1-2\alpha)t^{1-3\alpha}y'(t)$

4- $${}^{4}T_{\alpha}[y(t)]=t^{4-4\alpha}y^{(4)}(t)+6(1-\alpha)t^{3-4\alpha}y^{(3)}(t)+(1-\alpha)(1-7\alpha)t^{2-4\alpha}y^{(2)}(t)$$
$$+(1-\alpha)(1-2\alpha)(1-3\alpha)t^{1-4\alpha}y'(t)$$

Lemma 3.2. For any smooth function $y(t):(0,\infty)\to R$, and $n<\alpha\le n+1$, the conformable fractional derivative of $y(t)$ order α satisfied $T_{\alpha}[y(t)]=t^{n+1-\alpha}f^{(n+1)}(t)$,where $n<\alpha\le n+1$.

Proof:

By using definition (2.3), one can have $T_{\alpha}[y(t)]=T_{\alpha-n}[y^{(n)}(t)]$

Since $0<\alpha-n\le 1$, we have $T_{\alpha}[y(t)]=t^{1-(\alpha-n)}\frac{dy^{(n)}(t)}{dt}=t^{1+n-\alpha}y^{(n+1)}(t)$.

Remark 3.2. For any smooth function $y(t):(0,\infty)\to R$, and by simple computation, one can have the following conformable fractional derivative of $y(t)$

1- $T_\alpha[y(t)] = t^{1-\alpha}y'(t),\quad 0<\alpha\le 1$.

2- $T_\alpha[y(t)] = t^{2-\alpha}y''(t), 1<\alpha\le 2$.

3- $T_\alpha[y(t)] = t^{3-\alpha}y'''(t),\quad 2<\alpha\le 3$.

4- $T_\alpha[y(t)] = t^{4-\alpha}y^{(4)}(t), 3<\alpha\le 4$.

Remark 3.3. From remark 3.1 and remark 3.2 one can see the following

1- $^{n}T_1[y(t)] = \dfrac{d^n y(t)}{dt^n}$ for $n=1,2,\ldots$

2- $T_n[y(t)] = \dfrac{d^n y(t)}{dt^n}$ for $n=1,2,\ldots$

3- $^{n}T_{\alpha-n}[y(t)] \neq T_\alpha[y(t)],$ for $n<\alpha\le n+1$ and $n=1,2,\ldots.$

Theorem 3.1. The relationship between the solution $y(t)$ of the conformable fractional ordinary differential equation of the type

$$F(^{n}T_\alpha[y(t)],\ ^{n-1}T_\alpha[y(t)],\ldots,T_\alpha[y(t)],y(t),t)=0,\ 0<\alpha\le 1$$

and the solution $Y(x)$ of the ordinary differential equation of the type

$$F(\frac{d^nY(x)}{dx^n},\frac{d^{n-1}Y(x)}{dx^{n-1}},\ldots,\frac{dY(x)}{dx},Y(x),x)=0$$

is $y(t)=Y(x)$, where $x=\dfrac{t^\alpha}{\alpha}$.

Theorem 3.2. The conformable fractional ordinary differential equation of the type

$$F(T_{\alpha_1}[y(t)],T_{\alpha_2}[y(t)],\ldots,T_{\alpha_n}[y(t)],y(t),t)=0,\ 0<\alpha_n\le\alpha_{n-1}\le\cdots\le\alpha_2\le\alpha_1$$

and the ordinary differential equation of the type

$$F(t^{[\alpha_1]-\alpha_1}y^{([\alpha_1])}(t),t^{[\alpha_2]-\alpha_2}y^{([\alpha_2])}(t),\ldots,t^{[\alpha_n]-\alpha_n}y^{([\alpha_n])}(t),Y(t),t)=0$$

have same solution.

Corollary 3.1. The relationship between the solution $y(t)$ of the conformable fractional ordinary differential equation of the type

$$^{n}T_\alpha[y(t)]+a_1\,^{n-1}T_\alpha[y(t)]+\ldots+a_{n-1}T_\alpha[y(t)]+a_ny(t)=f(t),\ 0<\alpha\le 1$$

and the solution $Y(x)$ of the ordinary differential equation of the type

$$\frac{d^nY(x)}{dx^n}+a_1\frac{d^{n-1}Y(x)}{dx^{n-1}}+\ldots+a_{n-1}\frac{dY(x)}{dx}+a_nY(x)=f((\alpha x)^{\frac{1}{\alpha}})$$

is $y(t)=Y(x)$, where $x=\dfrac{t^\alpha}{\alpha}$, and $a_1,a_2,\ldots,a_n$ are constant.

Corollary 3.2. The conformable fractional ordinary differential equation of the type

$$T_{\alpha_1}[y(t)]+a_1T_{\alpha_2}[y(t)]+\ldots+a_{n-1}T_{\alpha_n}[y(t)]+a_n y(t)=f(t),\ 0<\alpha_n\le\alpha_{n-1}\le\cdots\le\alpha_2\le\alpha_1$$

and the ordinary differential equation of the type

$$t^{[\alpha_1]-\alpha_1}y^{([\alpha_1])}(t)+a_1t^{[\alpha_2]-\alpha_2}y^{([\alpha_2])}(t)+\ldots+a_{n-1}t^{[\alpha_n]-\alpha_n}y^{([\alpha_n])}(t)+a_n y(t)=f(t)$$

have same solution.

5. Illustrated Examples:

Example 1 : Consider the following linear conformable fractional differential equation

$$T_\alpha y(x)+f(x)y(x)=g(x),\ \ 0<\alpha\le 1.$$

Assume that, one is looking for a differentiable $y(x)$, by applying the theorem (3.2), one have

$$x^{1-\alpha}\frac{dy(x)}{dx}+f(x)y(x)=g(x)$$

In fact, this is linear differential equation

$$\frac{dy(x)}{dx}+x^{\alpha-1}f(x)y(x)=x^{\alpha-1}g(x)$$

So, the general solution is

$$y(x)=e^{-\int x^{\alpha-1}f(x)dx}[\int x^{\alpha-1}g(x)e^{\int x^{\alpha-1}f(x)dx}dx+c]$$

Example 2 : we consider the homogeneous fractional differential equation

$$T_\alpha y(x)+y(x)=0,\quad 0<\alpha\le 1$$

$$x^{1-\alpha}\frac{dy(x)}{dx}+y(x)=0$$

$$\frac{dy(x)}{y(x)}=-x^{\alpha-1}dx$$

$$\ln(y(x)=-C\frac{x^\alpha}{\alpha}$$

$$y(x)=Ce^{\frac{-x^\alpha}{\alpha}}$$

References

[1] K. Oldham, J. spanier, The Fractional Calculus, Academic Press, New York, 1974.

[2] S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives, Gordon and Breach, New York, 1993.

[3] K. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[4] A. Kilbas, H. Strivatava, J. Trujillo, Theory and Application of Fractional Differential Equations,

Wiley, New York, 1993.

[5] I. Pdolubny, Fractional Differential Equations, Academic Press, New York, 1999.

[6] R. Hilfer, Application of fractional Calculus in Physics, World Scientific Publishing Company, Singapore, 2000.

[7] G. Zaslavsky, Hamiltonian Chaos and Fractional Dynamics, Oxford University Press, Oxford, 2005.

[8] W. Schneider and W. Wyss, Fractional diffusion and wave equations, J Math Phys 30 (1), 1989, 134–144.

[9] R. Gorenflo and F. Mainardi, Fractional calculus, 1997, Springer

[10] R.L. Magin, C. Ingo, L. Colon-Perez, W. Triplett and T.H. Mareci, Characterization of anomalous diffusion in porous biological tissues using fractional order derivatives and entropy, Microporous Mesoporous Mater 178, 2013, 39–43.

[11] R.L. Magin, Fractional calculus models of complex dynamics in biological tissues, Comput Math Appl 59 (5), 2010, 1586–1593.

[12] R. Metzler and J. Klafter, Boundary value problems for fractional diffusion equations, Physica A 278 (1), 2000, 107–125.

[13] H. Beyer, S. Kempfle, Definition of physically consistent damping laws with fractional derivatives, ZAMM-J Appl Math Mech/Zeitschrift fÄur Angewandte Mathematik und Mechanik 75 (8) (1995), 623–635.

[14] W. Lederman and Roquejoffre, Mathematical justification of a nonlinear integrodifferential equation for the propagation of spherical flames, Ann di Mate 183, 2004, 173–239.

[15] P.J. Torvik and R.L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, J Appl Mech 51 (2), 1984, 294–298.

[16] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, Phys Rev E 53 (2), 1996, 1890.

[17] V.V. Kulish and J.L. Lage, Application of fractional calculus to fluid mechanics, J Fluids Eng 124 (3), 2002, 803–806.

[18] W. Wyss, The fractional Black–Scholes equation, Fractional Calculus Appl Anal 3 (1), 2000, 51–62.

[19] L. Song and W. Wang, Solution of the fractional Black–Scholes option pricing model by finite difference method, Abstr. Appl. Anal 2013, 2013, Hindawi Publishing Corporation.

[20] F. Keil, W. Mackens, H. VoB and J. Werther, Scientific computing in chemical engineering II: computational fluid dynamics, reaction engineering, and molecular properties, 2012, Springer Science & Business Media.

[21] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014) 6570.

[22] E. Unal, A. Gokdogan I. Cumhur, The operator method for local fractional linear differential equations, Vol. 131, (2017), 986-993.