Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION

Volume 12, Issue 4
Published online: October 14, 2017|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Multiple solutions for a kind of periodic boundary value problems via variation approach [1][2]

Li Zhang[a,b], Nao Li[a], Yujie Xu[b], Qian Xu[b], Chong Li[b]
[a] Collaborative Innovation Center of eTourism, Beijing Union University, Beijing 100101, China
[b] Department of Foundation Courses, Beijing Union University, Beijing 100101, China

Abstract

In this paper, the existence of at least two solutions for periodic boundary value problems is obtained by the critical point theory. The interest is that the nonlinear term includes the first-order derivative and may not satisfy the classical Ambrosetti-Rabinowitz condition.

Keywords: p −laplacian; boundary value problem; variational; periodic solution.

Mathematics Subject Classification 2000: 34B24.

1. Introduction

This paper considers the existence of at least two periodic solutions for the following kind of boundary value problems (for short BVPs)

$$\begin{cases}(\phi_p(x'(t)))' = (a(t)\phi_p(x(t)) + f(t,x(t)))g(x'(t)), \ t \in [0,T]\backslash\{t_1,\cdots,t_k\} \\ \Delta G(x'(t_i)) = I_i(x(t_i)), i = 1,\cdots,k(1.2) \\ x(0) - x(T) = 0, \quad x'(0) - x'(T) = 0.\end{cases} \tag{1.1}$$

Here, $p > 1$, $T > 0$, $\phi_p(x) = |x|^{p-2}x$, $G(x) = \int_0^x \frac{(p-1)|s|^{p-2}}{g(s)} ds$.

The periodic BVPs have received a lot of attention. Many works have been carried out to discuss the existence of at least one solution, multiple solutions. The methods therein mainly depend on lower and upper solutions with monotone iterative[1] and fixed point theorems[2], etc. Reciently, variational methods have been used to study the existence of periodic solutions, such as [3,4]. Moreover, Ambrosetti and Rabinowitz[10] established the existence of nontrival solutions for Dirichlet problems under the well known Ambrosetti-Rabinowitz condition: there exist some $\mu > 2$ and $R > 0$ such that

$$0 < \mu \int_0^x f(t,s)ds \le f(t,x)x$$

for all $t \in [0,T]$ and $|x| \ge R$. Since then, the AR-condition has been used extensively. By the usual AR-condition, it is easy to show that the Euler-Lagrange functional associated with the system has the mountain pass geometry and the Palais-Smale sequence is bounded.

Since it is not easy to verify the corresponding Euler functional satisfying (PS)-condition, few papers consider the boundary value problems with the nonlinear term including the first-order derivative. G. A. Afrouzi and S. Heidarkhani[8] proved the existence of at least three weak solutions for the Dirichlet problem

[1] supported by Beijing Natural Science Foundation (1152002), National Natural Science Foundation of China(11501030, 11501031, 11471221, 41101111), the Scientific Research Project of Beijing Municipal Education Commission (No. KM201711417002)
[2]Corresponding author:Li Zhang, lytzhangli@buu.edu.cn

$$\begin{cases} y'' + \lambda h(y')f(t,y) = 0, \ a < t < b \\ y(a) = y(b) = 0, \end{cases}$$

where $a, b \in \mathbb{R}$, $\lambda > 0$. Based on the three critical points theorem in [5], the authors did not need to verify (PS)-condition of the corresponding Euler functional.

To the best of our knowledge, no people consider the existence of at least two solutions for periodic BVP with the nonlinearity including x' by using variational methods. As a result, the goal of this paper is to fill the gap in this area.

The outline of the paper is as follows. In the forthcoming section, we present some general results. In section 3, we exhibit the existence of at least two solutions. Thoughout, assume $a(t) \in C([0,T];(0,+\infty))$, $F(t,x) = \int_0^x f(t,u)du$, $G(x) = \int_0^x \frac{(p-1)|s|^{p-2}}{g(s)} ds$ and

$(A_1) f(t,x): [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

$(A_2) g(x): \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $M > m > 0$ such that $M \geq g(x) \geq m$, $x \in \mathbb{R}$.

$(A_3) I_i(x): \mathbb{R} \to \mathbb{R}$ is continuous, $i = 1, \cdots, k$.

Remark 1.1 From the expression of $G(x)$, one has $\frac{d}{dt}G(x) = \frac{(p-1)|x|^{p-2}}{g(x)}\frac{dx}{dt} = \frac{\frac{d}{dt}(\phi_p(x(t)))}{g(x)}$, $\frac{1}{M}|x|^p \leq G(x)x \leq \frac{1}{m}|x|^p$, $\frac{1}{Mp}|x|^p \leq \int_0^x G(s)ds \leq \frac{1}{mp}|x|^p$.

2. Preliminary

The Sobolev space $W^{1,p}[0,T]$ is defined by

$W^{1,p}[0,T] = \{x: [0,T] \to \mathbb{R} | x \text{ is absolutely continuous}, x' \in L^p(0,T;\mathbb{R}), x(0) = x(T)\}$ (2.1)

and is endowed with the norm

$$\| x \| = \left(\int_0^T |x(t)|^p dt + \int_0^T |x'(t)|^p dt\right)^{\frac{1}{p}}. \tag{2.2}$$

Then, from [3], $W^{1,p}[0,T]$ is a sparable and reflexive Banach space. Next, we show some basic knowledge.

Theorem 2.1[6] For the functional $F: M \subset X \to [-\infty, +\infty]$ with $M \neq \emptyset$, $\min_{u \in M} F(u) = \alpha$ has a solution in case the following holds:

(i) X is a real reflexive Banach space;

(ii) M is bounded and weak sequentially closed, i.e., for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we always have $u \in M$;

(iii) F is weakly sequentially lower semi-continuous on M.

Theorem 2.2[3] Let E be a Banach space and $\varphi \in C^1(E, \mathbb{R})$ satisfy (PS)-condition. Assume there exist $x_0, x_1 \in E$, and a bounded open neighborhood Ω of x_0 such that $x_1 \setminus \bar{\Omega}$ and

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x \in \partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h | h: [0,1] \to E \text{ is continuous and } h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} \varphi(h(s)).$$

Then, c is a critical value of φ, that is, there exists $x^* \in E$ such that $\varphi'(x^*) =$ and $\varphi(x^*) = c$, where $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

For $x \in C[0,T]$, suppose $\| x \|_\infty = \max_{t \in [0,T]} |x(t)|$, $\| x \|_m = \min_{t \in [0,T]} |x(t)|$.

Lemma 2.1 If $x \in W^{1,p}[0,T]$, then, $\| x \|_\infty \leq C \| x \|$ where $C = T^{-\frac{1}{p}} + T^{\frac{1}{q}}$.

Lemma 2.2[7] There exists a positive constant c_p such that

$$(|x|^{p-2}x-|y|^{p-2}y,x-y)\geq\begin{cases}c_p|x-y|^p, & p\geq 2,(2.3)\\ c_p\dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & 1<p<2(2.4)\end{cases}\qquad(2.3)$$

for any $x,y\in\mathbb{R}^N$, $|x|+|y|\neq 0$. Here, $(x,y)=x\cdot y^T$.

In the following, we state the (C) condition[8]. Let φ be a Frechet differential functional and H be a Banach space,

(C) Every sequence $(x_n)_{n\in N}\subset H$ such that the following conditions hold:

$(i)(\varphi(x_n))_{n\in N}$ is bounded,

$(ii)(1+\parallel x_n\parallel_H)\parallel\varphi'(x_n)\parallel_{H^*}\to 0$ as $n\to\infty$

has a subsequence which converges strongly in H.

3. Existence of at least two solutions

Define

$$\varphi(x)=\int_0^T[F(t,x)+\frac{1}{p}a(t)|x(t)|^p+\int_0^{x'}G(u)du]dt+\sum_{i=1}^k\int_0^{x(t_i)}I_i(t)dt,\qquad(3.1)$$

Obviously, φ is continuously differentiable on $W^{1,p}[0,T]$ and by computation, one has

$$<\varphi'(x),y\geq\int_0^T f(t,x)ydt+\int_0^T a(t)\phi_p(x)ydt+\int_0^T G(x')y'dt$$
$$+\sum_{i=1}^k I_i(x(t_i))y(t_i),\quad x,y\in W^{1,p}[0,T].(3.2)$$

Lemma 3.1 If $x\in W_T^{1,p}$ is a critical point of φ, then, x is a solution to BVP(1.1).

Lemma 3.2 Assume (A_1), (A_2), and the following conditions

(B_1) there exists some constants $\mu>p$ and γ_1,γ_2 satisfying

$$\mu F(t,x)-xf(t,x)\geq\gamma_1x^p-\gamma_2,\quad x\in\mathbb{R};$$

$(B_2)\lim_{x\to+\infty}\frac{f(t,x)}{|x|^{p-1}}=-\infty$, $\lim_{x\to-\infty}\frac{f(t,x)}{|x|^{p-1}}=+\infty$, $t\in[0,T]$;

$$(B_3)\mu\min\{\parallel a\parallel_m,\frac{1}{M}\}>p\max\{\parallel a\parallel_\infty,\frac{1}{m}\}$$

$$(B_4)0\leq I_i(x)\cdot x, I_i(x)\leq b_i+c_ix^{\tau-1},\tau<p,b_i,c_i\in R^+,i=1,\cdots,k$$

hold, then the functional φ satisfies (C)-condition.

Proof. Let $(x_n)_{n\in N}$ be a sequence in $W^{1,p}[0,T]$ such that $\varphi(x_n)$ is bounded and $\parallel\varphi'(x_n)\parallel\times(1+\parallel x_n\parallel)\to 0$ as $n\to\infty$. Then, there exist a constant $C_1>0$ and a sequence $(\varepsilon_n)_{n\in N}$, $\varepsilon_n\to 0$ as $n\to\infty$ such that

$$|\varphi(x_n)|\leq C_1,\quad|<\varphi'(x_n),x_n>|\leq\varepsilon_n.\qquad(3.3)$$

Suppose $\parallel x_n\parallel\to\infty$, $n\to\infty$. Set $y_n=\frac{x_n}{\parallel x_n\parallel}$ for all $n\geq 1$. Obviously, $\parallel y_n\parallel=1$, that is, $(y_n)_{n\in N}$ is a bounded sequence in $W^{1,p}[0,T]$. Going to a subsequence if necessary, we may assume that

$$y_n\rightharpoonup y\ \text{ in } W^{1,p}[0,T],\ \ y_n(t)\to y(t)\ \text{ in } [0,T].$$

From (3.3),

$$\left|\frac{\int_0^T a(t)|x_n|^p dt+\int_0^T G(x_n')x_n'dt}{\parallel x_n\parallel^p}+\int_0^T\frac{f(t,x_n)y_n}{\parallel x_n\parallel^{p-1}}dt+\frac{\sum_{i=1}^k I_i(x_n(t_i))x_n(t_i)}{\parallel x_n\parallel^p}\right|\leq\frac{\varepsilon_n}{\parallel x_n\parallel^p}.\qquad(3.4)$$

Moreover,

$\min\{\frac{1}{M}, \| a \|_m\} \le |\frac{\int_0^T a(t)|x_n|^p dt + \int_0^T G(x_n{}')x_n{}' dt}{\|x_n\|^p}| \le \max\{\frac{1}{m}, \| a \|_\infty\}$.(3.5)

Let $\Omega_0^+ = \{t \in [0,T], y(t) > 0\}$, $\Omega_0^- = \{t \in [0,T], y(t) < 0\}$, then, $x_n(t) \to +\infty$ for $t \in \Omega_0^+$, $x_n(t) \to -\infty$ for $t \in \Omega_0^-$, $n \to \infty$. By the hypothesis,

$$\frac{f(t, x_n(t))}{(x_n(t))^{p-1}} \to -\infty, \quad t \in \Omega_0^+, \ n \to \infty.$$

If $meas\Omega_0^+ > 0$, then,

$$(y_n(t))^p \frac{f(t, x_n(t))}{(x_n(t))^{p-1}} \to -\infty, \quad t \in \Omega_0^+, \ n \to \infty.$$

If $meas\Omega_0^- > 0$, then,

$$(y_n(t))^p \frac{f(t, x_n(t))}{(x_n(t))^{p-1}} = y_n(t)|y_n(t)|^{p-1} \frac{f(t, x_n(t))}{|x_n(t)|^{p-1}} \to -\infty, \quad t \in \Omega_0^-, \ n \to \infty.$$

Moreover,

$$\int_0^T \frac{f(t, x_n)y_n}{\| x_n \|^{p-1}} dt = \int_{\Omega_0^+} (y_n(t))^p \frac{f(t, x_n(t))}{(x_n(t))^{p-1}} dt + \int_{\Omega_0^-} (y_n(t))^p \frac{f(t, x_n(t))}{(x_n(t))^{p-1}} dt$$
$$+ \int_{[0,T]\setminus(\Omega_0^+ \cup \Omega_0^-)} \frac{f(t, x_n(t))x_n(t)}{\| x_n \|^p} dt.$$

Then, from (B_2), there exists some constant $k > 0$, $f(t,x)x < 0$ for $|x| > k$. For $|x| \le k$, $f(t,x)$ is bounded by continuity, hence, $\int_{[0,T]\setminus(\Omega_0^+ \cup \Omega_0^-)} \frac{f(t,x_n(t))x_n(t)}{\|x_n\|^p} dt$ is bounded or negative. By (B_3) and the discussion above, one has

$$\int_0^T \frac{f(t, x_n)y_n}{\| x_n \|^{p-1}} dt \to -\infty, \quad n \to \infty.$$

This reaches a contradiction, that is, $meas\Omega_0^+ = meas\Omega_0^- = 0$. Then, one can conclude that $y(t) = 0$ for a.e. $t \in [0,T]$. Hence, $y(t) \equiv 0$, $t \in [0,T]$.

Moreover,

$$\frac{1}{p}\min\{ \| a \|_m, \frac{1}{M}\} \| x_n \|^p \le \int_0^T [\frac{1}{p} a(t)|x_n(t)|^p + \int_0^{x_n{}'} G(u)du]dt$$
$$\le \frac{1}{p}\max\{ \| a \|_\infty, \frac{1}{m}\} \| x_n \|^p,$$

$$\min\{ \| a \|_m, \frac{1}{M}\} \| x_n \|^p \le \int_0^T [a(t)|x_n(t)|^p + G(x_n{}')x_n{}']dt \le \max\{ \| a \|_\infty, \frac{1}{m}\} \| x_n \|^p.$$

$$0 \le I_i(x_n(t_i)) \cdot x_n(t_i) \le b_i|x_n(t_i)| + c_i|x_n^\tau(t_i)| \le cb_i \| x_n \| + c^\tau c_i \| x_n \|^\tau$$

$$|\int_0^{x_n(t_i)} I_i(t)dt| = |I_i(\zeta) \cdot x_n(t_i)| \le b_i|x_n(t_i)| + c_i|x_n^\tau(t_i)| \le cb_i \| x_n \| + c^\tau c_i \| x_n \|^\tau$$

Here $\boldsymbol{c} = \boldsymbol{T}^{-\frac{1}{p}} + \boldsymbol{T}^{\frac{1}{q}}$. Hence,

$$-C_1 - \frac{1}{p}\max\{ \| a \|_\infty, \frac{1}{m}\} \| x_n \|^p - \sum_{i=1}^{k} (cb_i \| x_n \| + c^\tau c_i \| x_n \|^\tau)$$

$$\leq \int_0^T F(t, x_n)dt \leq C_1 - \frac{1}{p}\min\{\| a \|_m, \frac{1}{M}\} \| x_n \|^p,$$

$$-\varepsilon_n + \min\{\| a \|_m, \frac{1}{M}\} \| x_n \|^p$$

$$\leq -\int_0^T f(t, x_n)x_n dt$$

$$\leq \varepsilon_n + \max\{\| a \|_\infty, \frac{1}{m}\} \| x_n \|^p + \sum_{i=1}^k (cb_i \| x_n \| + c^\tau c_i \| x_n \|^\tau),$$

Then, one has

$$-\frac{\mu C_1}{\| x_n \|^p} - \frac{\mu}{p}\max\{\| a \|_\infty, \frac{1}{m}\}$$

$$\leq \int_0^T \frac{\mu F(t, x_n)}{\| x_n \|^p} dt \leq -\frac{\mu}{p}\min\{\| a \|_m, \frac{1}{M}\} + \frac{\mu C_1}{\| x_n \|^p} - \sum_{i=1}^k (cb_i \frac{1}{\| x_n \|^{p-1}} - c^\tau c_i \frac{1}{\| x_n \|^{p-\tau}}),$$

$$-\frac{\varepsilon_n}{\| x_n \|^p} + \min\{\| a \|_m, \frac{1}{M}\}$$

$$\leq -\int_0^T \frac{f(t, x_n)x_n}{\| x_n \|^p} dt \leq \max\{\| a \|_\infty, \frac{1}{m}\} + \frac{\varepsilon_n}{\| x_n \|^p} + \sum_{i=1}^k (cb_i \frac{1}{\| x_n \|^{p-1}} + c^\tau c_i \frac{1}{\| x_n \|^{p-\tau}}).$$

Moreover,

$$-\frac{\mu}{p}\max\{\| a \|_\infty, \frac{1}{m}\} + \min\{\| a \|_m, \frac{1}{M}\}$$

$$\leq \underline{\lim}_{n\to+\infty} \int_0^T \frac{\mu F(t, x_n) - x_n f(t, x_n)}{|x_n|^p} |y_n|^p dt$$

$$\leq -\frac{\mu}{p}\min\{\| a \|_m, \frac{1}{M}\} + \max\{\| a \|_\infty, \frac{1}{m}\} < 0. \quad (3.6)$$

By (B_1),

$$\underline{\lim}_{n\to+\infty} \frac{\mu F(t, x_n) - x_n f(t, x_n)}{|x_n|^p} |y_n|^p \geq \underline{\lim}_{n\to+\infty} \frac{\gamma_1 x_n^p - \gamma_2}{|x_n|^p} |y_n|^p \to 0,$$

which is contradictive with (3.6). Hence, $(x_n)_{n\in N}$ is bounded in $W^{1,p}[0,T]$. By the compactness of the embedding $W^{1,p}[0,T] \hookrightarrow C[0,T]$, the sequence $(x_n)_{n\in N}$ has a subsequence, again denoted by $(x_n)_{n\in N}$ for convenience, such that

$$x_n \rightharpoonup x \text{ weakly in } W^{1,p}[0,T],$$

$$x_n \to x \text{ strongly in } C[0,T].$$

From (3.2), for $m, n \in N$,

$$< \varphi'(x_n) - \varphi'(x_m), x_n - x_m >$$

$$= \int_0^1 (f(t, x_n) - f(t, x_m))(x_n - x_m)dt + \int_0^1 a(t)(\phi_p(x_n) - \phi_p(x_m))$$

$$(x_n - x_m)dt + \int_0^1 (G(x_n{}') - G(x'{}_m))(x'{}_n - x'{}_m)dt$$
$$+\sum_{i=1}^{k} (I_i(x_n(t_i)) - I_i(x_m(t_i)))(x_n(t_i) - x_m(t_i)).$$

Since $(x_n)_{n\in N}$ is a Cauchy sequence in $C[0,T]$, $f(t,x)$ is continuous in x, $G(x)$ is continuous, and $| < \varphi'(x_n) - \varphi'(x_m), x_n - x_m > | \leq (\| \varphi'(x_n) \| + \| \varphi'(x_m) \|) \| x_n - x_m \|$, $\| x_n - x_m \|$ is bounded in $W^{1,p}[0,T]$, $\| \varphi'(x_n) \| + \| \varphi'(x_m) \| \to 0$, $n,m \to \infty$, one has $< \varphi'(x_n) - \varphi'(x_m), x_n - x_m > \to 0$. Hence

$$\int_0^T (G(x_n{}') - G(x'{}_m))(x'{}_n - x'{}_m)dt \to 0, \quad n,m \to \infty.$$

Moreover,

$$\int_0^T (G(x_n{}') - G(x_m{}'))(x'{}_n - x_m{}')dt = \int_0^T (\int_{x_m{}'}^{x_n{}'} \frac{(p-1)|s|^{p-2}}{g(s)} ds)(x'{}_n - x_m{}')dt$$
$$\geq \frac{1}{M}\int_0^1 (\phi_p(x_n{}') - \phi_p(x_m{}'))(x_n{}' - x_m{}')dt$$
$$\geq 0$$

then,

$$\int_0^T (\phi_p(x_n{}') - \phi_p(x_m{}'))(x_n{}' - x_m{}')dt \to 0, \text{ as } n,m \to \infty. \tag{3.7}$$

If $p \geq 2$, from Lemma 2.2, there exists a positive constant c_p such that

$$\int_0^T (\phi_p(x_n{}') - \phi_p(x_m{}'))(x_n{}' - x_m{}')dt \geq c_p \int_0^T |x_n{}' - x_m{}'|^p dt. \tag{3.8}$$

If $p < 2$, by Lemma 2.2, Hölder inequality and the boundedness of $(x_n)_{n\in N}$ in $W^{1,p}$, one has

$$\int_0^T |x_n{}' - x_m{}'|^p dt = \int_0^T \frac{|x_n{}' - x_m{}'|^p}{(|x_n{}'| + |x_m{}'|)^{\frac{p(2-p)}{2}}} (|x_n{}'| + |x_m{}'|)^{\frac{p(2-p)}{2}} dt$$
$$\leq (\int_0^T \frac{|x_n{}' - x_m{}'|^2}{(|x_n{}'| + |x_m{}'|)^{2-p}} dt)^{\frac{p}{2}} (\int_0^T (|x_n{}'| + |x_m{}'|)^p dt)^{\frac{2-p}{2}}$$
$$\leq c_p^{\frac{-p}{2}} \left(\int_0^T (\phi_p(x_n{}') - \phi_p(x_m{}')\right)(x_n{}' - x_m{}')dt)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}}$$
$$(\int_0^T (|x_n{}'|^p + |x_m{}'|^p)dt)^{\frac{2-p}{2}}$$
$$\leq c_p^{\frac{-p}{2}} (\int_0^T (\phi_p(x_n{}') - \phi_p(x_m{}'))(x_n{}' - x_m{}')dt)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}}$$
$$(\| x_n \|^p + \| x_m \|^p))^{\frac{2-p}{2}}. \tag{3.9}$$

From (3.7)-(3.9), we have $\int_0^1 |x_n{}' - x_m{}'|^p dt \to 0$ as $n,m \to \infty$. Then, $\| x_n - x_m \| \to 0$, that is, $(x_n)_{n\in N}$ is a Cauchy sequence in $W^{1,p}[0,T]$. By the completeness of $W^{1,p}[0,T]$, one has $(x_n)_{n\in N}$ has a convergent subsequence.⊓⊔

Theorem 3.1 Assume (A_1), (A_2), (B_1), (B_2), (B_3) and the following condition

(B_5) there exist positive constants $l_1, l_2, R, \sigma \leq p$, such that for $\| x \| = R$,

$$-F(t,x) \leq l_1|x|^\sigma + l_2,$$

and

$$T^{1-\frac{\sigma}{p}} l_1 R^\sigma + l_2 T < \frac{min\{\| a \|_m, \frac{1}{M}\}}{p} R^p,$$

hold, then, BVP(1.1) has at least two periodic solutions x_0, x_1 and $\| x_0 \| < R$.

Proof. First, we show for the constant $R > 0$ given in (B_4), φ has a local minimum point x_0 in $B_R = \{x \in W^{1,p}[0,T]: \ \| x \| < R\}$. There are two steps.

Step 1, we show $x_0 \in \bar{B_R}$. Obviously, $\bar{B_R}$ is bounded and weakly sequentially closed. In the following, we show φ is weakly sequently lower semi-continuous on $\bar{B_R}$. Let

$$\varphi_1(x) = \int_0^T [\frac{1}{p} a(t)|x(t)|^p + \int_0^{x'} G(u)du]dt, \quad \varphi_2(x) = \int_0^T F(t,x)dt + \sum_{i=1}^{k} \int_0^{x(t_i)} I_i(t)dt.$$

Assume $x_n \to x$ in $W^{1,p}[0,T]$, then, $\| x_n{}' - x' \|_{L^p} \to 0$, $\| x_n - x \|_{L^p} \to 0$. Hence,

$$|\varphi_1(x_n) - \varphi_1(x)| \leq \frac{1}{p} \| a \|_\infty \int_0^T ||x_n|^p - |x|^p|dt + \int_0^T |\int_{x'}^{x_n{}'} G(u)du|dt$$

$$\leq \frac{1}{p} \| a \|_\infty \int_0^T ||x_n|^p - |x|^p|dt + \frac{1}{mp} \int_0^T ||x'_n|^p - |x'|^p|dt.$$

Define $h(x) = |x|^p$, $p > 1$. It is obvious that $h \in C^1[0,T]$. By applying the Mid-value Theorem, there exists $\zeta(t)$ satisfying $0 \leq \zeta(t) \leq 1$ such that

$$||x'_n|^p - |x'|^p| = p|x'(t) + \zeta(t)(x_n{}'(t) - x'(t))|^{p-1} \times |x'_n(t) - x'(t)|.$$

Moreover, there exists $\delta > 0$, such that

$$|x'(t) + \zeta(t)(x_n{}'(t) - x'(t))|^{p-1} \leq \delta[|x'(t)|^{p-1} + |x_n{}'(t) - x'(t)|^{p-1}] \in L^q([0,T]).$$

Hence,

$$\int_0^T |\int_{x'}^{x_n{}'} G(u)du|dt \leq [\frac{\delta}{mp} \int_0^T |x'(t)|^{p-1}|x_n{}'(t) - x'(t)|dt + \int_0^T |x_n{}'(t) - x'(t)|^p]dt$$

$$\leq \frac{\delta}{mp} [(\int_0^T |x'(t)|^p dt)^{\frac{1}{q}} (\int_0^T |x_n{}'(t) - x'(t)|^p dt)^{\frac{1}{p}} + \int_0^T |x_n{}'(t) - x'(t)|^p]dt$$

$\to 0, n \to \infty$

Here, $\frac{1}{q} + \frac{1}{p} = 1$. With the same discussion above, one has $\int_0^T ||x_n|^p - |x|^p|dt \to 0$ as $n \to \infty$. Hence, φ_1 is continuous. Since φ_1 is convex, then, φ_1 is weakly lower semi-continuous. Assume $x_n \rightharpoonup x$ in $W^{1,p}[0,T]$, then, $x_n \to x$ in $C[0,T]$. Hence, φ_2 is weakly semi-continuous. Therefore, φ is weakly lower semi-continuous. From Theorem 2.1, φ has a local minimum $x_0 \in \bar{B_R}$.

Step 2. If $x_0 \in \partial B_R$,

$$\varphi(x_0) \geq \int_0^T F(t,x_0)dt + \frac{\| a \|_m}{p} \int_0^T |x_0(t)|^p dt + \frac{1}{pM} \int_0^T |x'_0|^p dt$$

$$\geq \int_0^T F(t,x_0)dt + \frac{min\{\|a\|_m, \frac{1}{M}\}}{p} \| x_0 \|^p. \tag{3.10}$$

Moreover, $\varphi(x_0) < \varphi(0) = \int_0^T F(t,0)dt = 0$, that is,

$$\int_0^T [-F(t,x_0)]dt \geq \frac{min\{\|a\|_m, \frac{1}{M}\}}{p} R^p. \tag{3.11}$$

From (B_4),

$$\int_0^T [-F(t,x_0)]dt \le l_1 \int_0^T |x_0|^\sigma dt + l_2 T \le T^{1-\frac{\sigma}{p}} l_1 \parallel x_0 \parallel^\sigma + l_2 T = T^{1-\frac{\sigma}{p}} l_1 R^\sigma + l_2 T,$$

together with (3.11), we reach a contradiction. Hence, $x_0 \in B_R$, that is x_0 is a critical point of φ.

In the following, we show there exists x_1 satisfying $\parallel x_1 \parallel > R$ and $\varphi(x_1) < \min_{x\in\partial B_R} \varphi(x)$. From (B_2), for $\forall M > 0$, there exists $C_M > 0$ such that

$$F(t,x) \le -M|x|^p, \quad |x| > C_M.$$

For large $\lambda > 0$, $M > 0$, one has

$$\varphi(\lambda) = \int_0^T F(t,\lambda)dt + \lambda^p \int_0^T \frac{1}{p} a(t)dt + \sum_{i=1}^k \int_0^\lambda I_i(t)dt$$

$$\le \lambda^p \int_0^T \frac{1}{p} a(t)dt - M\lambda^p T + \sum_{i=1}^k (b_i\lambda + c_i\lambda^2) \to -\infty.$$

Hence, by the mountain pass theorem, one obtain another periodic solution $\boldsymbol{x_1}$.

References

[1] W.Ding, M.Han, J.Yan, Periodic boundary vslue problems for the second order functional differential equations, J.Math.Anal.Appl,298(2004)341-351.

[2] H.Wang,Positive periodic solutions of functional differential equations,J.Differential Eqns. 202(2004)354-366.

[3] J.Mawhin, M.Willen. Critical Point Theorem and Hamiltonian Systems. Appl. Math. Sci., Vol. 74, Springer, New York; 1989.

[4] P.Rabinowitz, Periodic solutions of hamiltonian systems, Comm. Pure Appl. Math. 31(1978):157-184.

[5] Ricceri B.: On a three critical points theorem, Arch. Math. 75,220-226(2000).

[6] E.Zeidler, Nonlinear Functional Analysis and its Application, Vol.3, Springer,1985.

[7] Giovanna Cerami.: An existence criterion for the critical points on unbounded manifolds. Istit. Lombardo Accad. Sci. Lett. Rend. A, 112(2)(1979):332-336,1978.

[8] AfrouziG.A.,Heidarkhani S.: Three solutions for a quasilinear boundary value problem, Nonlinear Anal.69,3330-3336(2008).

[9] Y.Tian, W.Ge. Multiple positive solutions for periodic boundary value problem via vatiational methods. Tamkang J. Math., 39(2008):111-119.

[10] A.Ambrosetti, P.H.Rabinowitz. Dual variational methods in critical point theory and applications. J.Funct.Anal. 14(1973):349-381.

[11] Siegfried Carl, Dumitru Motreanu. Constant-sign and sign-changing solutions for nonlinear eigenvalue problems. Nonlinear Anal.,68(2008):2668-2676.

[12] Michael E.Filippakis, Nikolaos S.Papageorgiou. Multiple constant sign and nodal solutions for nonlinear elliptic with the p-Laplacian. J.Differential Equations. 245(2008):1883-1922.