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# Opial Type Inequalities for Conformable Fractional Derivative and Integral of Two Functions

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**Abstract**

In this paper, we establish the Opial type inequalities for conformable fractional derivative and integral of two functions and give some results in special cases of $\alpha$ .

**Keywords:** Opial type inequality; Conformable fractional derivative; Conformable fractional integral.

## 1. Introduction

Mathematical inequalities which involve derivatives and integrals of functions are of great interest. Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Many mathematicians gave the improvements and generalizations in last few decades to add the considerable contribution in the literature (see, for instance, [1]-[8]).

The paper is motivated by the work of Mehmet Zeki Sarikaya and Hüseyin Budak [9] and their study of Opial type inequalities involving conformable fractional derivative and integral. We will prove some new Opial type inequalities for conformable fractional derivative and integral of two functions. The paper is organized as follows. In the next section, we present some concepts related to conformable fractional derivative and integral. In section 3, we will give some new Opial type inequalities which involve conformable fractional derivative and integral.

## 2. Preliminaries

The following definitions and Lemmas with respect to conformable fractional derivative and integral were referred in (see, [9], [11]-[13]).

**Definition 2.1 (Conformable fractional derivative).** Given a function $f:[0,\infty)\to R$ . Then the "conformable fractional derivative" of $f$ of order $\alpha$ is defined by

$$D_\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for all $t>0$, $\alpha\in(0,1)$. If $f$ is $\alpha$-differetiable in some $(0,a)$, $a>0$, $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t).$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha f(t)$ to denote the conformable fractional derivative of $f$ of order $\alpha$ . In addition, if the the conformable fractional derivative of $f$ of order $\alpha$ exists, then we simply say $f$ is $\alpha$ -differentiable.

**Lemma 2.2.** Let $\alpha\in(0,1)$ and $f,g$ be $\alpha$ -differentiable at a point $t>0$. Then

(1) $D_\alpha(af+bg)=aD_\alpha(f)+bD_\alpha(g)$, for all $a,b\in R$,

(2) $D_\alpha(\lambda)=0,$ for all constant functions $f(t)=\lambda,$

(3) $D_\alpha(fg)=fD_\alpha(g)+gD_\alpha(f),$

(4) $D_\alpha(f^\gamma)=\gamma f^{\gamma-1}D_\alpha(f),$

(5) $D_\alpha(\frac{f}{g})=\frac{fD_\alpha(g)-gD_\alpha(f)}{g^2}.$

If $f$ is differentiable, then

$$D_\alpha f(t)=t^{1-\alpha}\frac{df}{dt}(t). \quad (2.1)$$

**Definition 2.3 (Conformable fractional integral).** Let $\alpha\in(0,1)$ and $0\le a<b$. A function $f:[a,b]\to R$ is $\alpha$-fractional integrable on $[a,b]$ if the integral

$$I_\alpha f=\int_a^b f(x)d_\alpha x:=\int_a^b f(x)x^{\alpha-1}dx \quad (2.2)$$

exists and finite. All $\alpha$-fractional integrable on $[a,b]$ is indicated by $L_\alpha^1([a,b])$.

**Remark.**

$$I_\alpha^a f(t)=I_1^a(t^{\alpha-1}f)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx, \quad (2.3)$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

**Lemma 2.4.** Let $f:[a,b]\to R$ be differentiable and $0<\alpha\le1$. Then, for all $t>a$ we have

$$I_\alpha^a D_\alpha^a f(t)=f(t)-f(a). \quad (2.4)$$

**Lemma 2.5 (Integration by parts).** Let $f,g:[a,b]\to R$ be two functions such that $fg$ is differentiable. Then

$$\int_a^b f(x)D_\alpha^a g(x)d_\alpha x=fg|_a^b-\int_a^b g(x)D_\alpha^a f(x)d_\alpha x. \quad (2.5)$$

**Lemma 2.6 (Hölder's inequality).** Let $f,g\in C[a,b],\ p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1,$ then

$$\int_a^b|f(x)g(x)|d_\alpha x\le\left(\int_a^b|f(x)|^p d_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b|g(x)|^q d_\alpha x\right)^{\frac{1}{q}}. \quad (2.6)$$

## 3. Main results

In this section, we give some new Opial-type inequalities for conformable fractional derivative and integral of two functions. We start with the following Lemma.

**Lemma 3.1[9].** Let $\alpha\in(0,1]$ and $f$ be an $\alpha$-fractional differentiable function on $(0,h)$ with $f(0)=f(h)=0$. Then, the following inequality holds

$$\int_0^h|f(t)D_\alpha f(t)|d_\alpha t\le\frac{h^\alpha}{4\alpha}\int_0^h|D_\alpha f(t)|^2 d_\alpha t. \quad (3.1)$$

**Lemma 3.2[9].** Let $\alpha\in(0,1]$ and $\omega$ be a nonnegative and continuous function on $[0,h]$. Let $f$ be an $\alpha$-fractional differentiable function on $(0,h)$ with $f(0)=f(h)=0$. Then, the following inequalities hold

$$\int_0^h\omega(t)|f(t)|^2 d_\alpha t\le\frac{h^\alpha}{4\alpha}\left(\int_0^h\omega(t)d_\alpha t\right)\left(\int_0^h|D_\alpha f(t)|^2 d_\alpha t\right) \quad (3.2)$$

and

$$\int_0^h \omega(t)|f(t)||D_\alpha f(t)|d_\alpha t \le \left(\frac{h^\alpha}{4\alpha}\int_0^h \omega^2(t)d_\alpha t\right)^{\frac{1}{2}}\left(\int_0^h |D_\alpha f(t)|^2 d_\alpha t\right). \tag{3.3}$$

**Lemma 3.3[9].** Let $\alpha \in (0,1]$ and $p \ge 0$, $q \ge 1$, $r \ge 0$, $m \ge 1$ be real numbers. If $f$ is an $\alpha$-fractional differentiable function on $(0,h)$ with $f(0) = f(h) = 0$. Then, the following inequality holds

$$\int_0^h |f(t)|^{m(p+q)}|D_\alpha f(t)|^{mr} d_\alpha t \le [(p+q)^m I(m)]^{p+q}\int_0^h |D_\alpha f(t)|^{m(p+q+r)} d_\alpha t, \tag{3.4}$$

where

$$I(m) = \frac{1}{\alpha^{m-1}}\int_0^h \left[t^{(1-m)\alpha} + (h^\alpha - t^\alpha)^{1-m}\right]^{-1} d_\alpha t. \tag{3.5}$$

**Theorem 3.4.** Let $\alpha \in (0,1]$ and $p \ge 0$, $q \ge 1$, $r \ge 0$, $m \ge 1$ be real numbers. If $f$ is an $\alpha$-fractional differentiable function on $(0,h)$ with $f(0) = f(h) = 0$, and $g$ is an $\alpha$-fractional differentiable function on $(0,h)$ with $g(0) = g(h) = 0$. Then the following inequality holds

$$\begin{aligned}&\int_0^h [|f(t)|^{m(p+q)}|D_\alpha g(t)|^{mr} + |g(t)|^{m(p+q)}|D_\alpha f(t)|^{mr}]d_\alpha t\\ &\le 2[(p+q)^m I(m)]^{p+q}\int_0^h [|D_\alpha f(t)|^{m(p+q+r)} + |D_\alpha g(t)|^{m(p+q+r)}]d_\alpha t,\end{aligned} \tag{3.6}$$

where $I(m)$ defined by (3.5).

**Proof.** Define

$$K(x) = \int_0^x [|D_\alpha f(t)|^{m(p+q+r)} + |D_\alpha g(t)|^{m(p+q+r)}]^{\frac{1}{m(p+q+r)}} d_\alpha t. \tag{3.7}$$

Then

$$\begin{aligned}D_\alpha K(x) &= [|D_\alpha f(x)|^{m(p+q+r)} + |D_\alpha g(x)|^{m(p+q+r)}]^{\frac{1}{m(p+q+r)}}\\ &\ge \max\{|D_\alpha f(x)|, |D_\alpha g(x)|\},\end{aligned} \tag{3.8}$$

and

$$K(x) \ge \int_0^x \{|D_\alpha f(t)|^{m(p+q+r)}\}^{\frac{1}{m(p+q+r)}} d_\alpha t \ge \left|\int_0^x D_\alpha f(t)d_\alpha t\right| = |f(x)|. \tag{3.9}$$

Similarly,

$$K(x) \ge |g(x)|. \tag{3.10}$$

By Lemma 3.3

$$\begin{aligned}&\int_0^h [|f(t)|^{m(p+q)}|D_\alpha g(t)|^{mr} + |g(t)|^{m(p+q)}|D_\alpha f(t)|^{mr}]d_\alpha t\\ &\le \int_0^h [|K(t)|^{m(p+q)}|D_\alpha g(t)|^{mr} + |K(t)|^{m(p+q)}|D_\alpha f(t)|^{mr}]d_\alpha t\\ &= \int_0^h |K(t)|^{m(p+q)}[|D_\alpha g(t)|^{mr} + |D_\alpha f(t)|^{mr}]d_\alpha t\\ &\le 2\int_0^h |K(t)|^{m(p+q)}|D_\alpha K(t)|^{mr} d_\alpha t\\ &\le 2[(p+q)^m I(m)]^{p+q}\int_0^h |D_\alpha K(t)|^{m(p+q+r)} d_\alpha t\\ &\le 2[(p+q)^m I(m)]^{p+q}\int_0^h [|D_\alpha f(t)|^{m(p+q+r)} + |D_\alpha g(t)|^{m(p+q+r)}]d_\alpha t.\end{aligned} \tag{3.11}$$

The proof is complete.

**Theorem 3.5.** Let $\alpha \in (0,1]$, $q \ge p > 1$ and $\omega$ be a nonnegative and continuous function on $[0,h]$. If $f$ is an $\alpha$-fractional differentiable function on $(0,h)$ with $f(0) = f(h) = 0$. Then

$$\int_0^h \omega(t)|f(t)|^p |D_\alpha f(t)|^q d_\alpha t \le \frac{p}{q} K^{\frac{p-q}{q}} \left( \frac{h^\alpha}{4\alpha} \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \int_0^h [D_\alpha f(t)]^{2q} d_\alpha t$$

$$+ \frac{q-p}{q} K^{\frac{p}{q}} \left( \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \left( \int_0^h [D_\alpha f(t)]^{2q} d_\alpha t \right)^{\frac{1}{2}}. \qquad (3.12)$$

for any constant $K > 0$, where $\varphi(t) = \left( \frac{t^\alpha}{\alpha} \right)^{\frac{p(q-1)}{q}}$.

**Proof.** Using (2.6) with indices $q$ and $\frac{q}{q-1}$, we have

$$|f(t)| \le \int_0^t |D_\alpha f(\tau)| d_\alpha \tau \le \left( \frac{t^\alpha}{\alpha} \right)^{\frac{q-1}{q}} \left( \int_0^t |D_\alpha f(\tau)|^q d_\alpha \tau \right)^{\frac{1}{q}}.$$

Let $g(t) = \int_0^t |D_\alpha f(\tau)|^q d_\alpha \tau$, then

$$D_\alpha g(t) = |D_\alpha f(t)|^q, \qquad |f(t)|^p \le \left( \frac{t^\alpha}{\alpha} \right)^{\frac{p(q-1)}{q}} g^{\frac{p}{q}}(t) = \varphi(t) g^{\frac{p}{q}}(t). \qquad (3.13)$$

Now we need the simple inequality to complete our result:

$$a^{\frac{p}{q}} \le \frac{p}{q} K^{\frac{p-q}{q}} a + \frac{q-p}{q} K^{\frac{p}{q}} \qquad (3.14)$$

for any constant $K > 0$, where $a \ge 0$, $q \ge p > 0$.
Therefore from (3.13), (3.14) and Lemma 3.2, we conclude that

$$\int_0^h \omega(t)|f(t)|^p |D_\alpha f(t)|^q d_\alpha t$$

$$\le \int_0^h \omega(t)\varphi(t) g^{\frac{p}{q}}(t) D_\alpha g(t) d_\alpha t$$

$$\le \int_0^h \omega(t)\varphi(t) \left[ \frac{p}{q} K^{\frac{p-q}{q}} g(t) + \frac{q-p}{q} K^{\frac{p}{q}} \right] D_\alpha g(t) d_\alpha t$$

$$= \frac{p}{q} K^{\frac{p-q}{q}} \int_0^h \omega(t)\varphi(t) g(t) D_\alpha g(t) d_\alpha t + \frac{q-p}{q} K^{\frac{p}{q}} \int_0^h \omega(t)\varphi(t) D_\alpha g(t) d_\alpha t$$

$$\le \frac{p}{q} K^{\frac{p-q}{q}} \left( \frac{h^\alpha}{4\alpha} \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \int_0^h [D_\alpha f(t)]^{2q} d_\alpha t$$

$$+ \frac{q-p}{q} K^{\frac{p}{q}} \left( \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \left( \int_0^h [D_\alpha f(t)]^{2q} d_\alpha t \right)^{\frac{1}{2}}. \qquad (3.15)$$

for any constant $K > 0$. The proof is complete.

**Theorem 3.6.** Let $\alpha \in (0,1]$, $q \ge p > 1$ and $\omega$ be a nonnegative and continuous function on $[0,h]$. If $f$ is an $\alpha$-fractional differentiable function on $(0,h)$ with $f(0) = f(h) = 0$, and $g$ is an $\alpha$-fractional differentiable function on $(0,h)$ with $g(0) = g(h) = 0$. Then

$$\int_0^h \omega(t)\{|f(t)|^p |D_\alpha g(t)|^q + |g(t)|^p |D_\alpha f(t)|^q\} d_\alpha t$$
$$\leq \frac{2p}{q} K^{\frac{p-q}{q}} \left( \frac{h^\alpha}{4\alpha} \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \int_0^h \{[D_\alpha f(t)]^{2q} + [D_\alpha g(t)]^{2q}\} d_\alpha t$$
$$+ \frac{2(q-p)}{q} K^{\frac{p}{q}} \left( \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \left( \int_0^h \{[D_\alpha f(t)]^{2q} + [D_\alpha g(t)]^{2q}\} d_\alpha t \right)^{\frac{1}{2}}.$$

for any constant $K > 0$, where $\varphi(t) = \left( \frac{t^\alpha}{\alpha} \right)^{\frac{p(q-1)}{q}}$.

**Proof.** Define

$$Z(x) = \int_0^x [|D_\alpha f(t)|^{2q} + |D_\alpha g(t)|^{2q}]^{\frac{1}{2q}} d_\alpha t. \tag{3.16}$$

Then

$$D_\alpha Z(x) = [|D_\alpha f(x)|^{2q} + |D_\alpha g(x)|^{2q}]^{\frac{1}{2q}} \geq \max\{|D_\alpha f(x)|, |D_\alpha g(x)|\}, \tag{3.17}$$

and

$$Z(x) \geq \int_0^x \{|D_\alpha f(t)|^{2q}\}^{\frac{1}{2q}} d_\alpha t \geq \left| \int_0^x D_\alpha f(t) d_\alpha t \right| = |f(x)|. \tag{3.18}$$

Similarly,

$$Z(x) \geq |g(x)|. \tag{3.19}$$

By (3.17), (3.18), (3.19) and Theorem 3.5, we have

$$\int_0^h \omega(t)\{|f(t)|^p |D_\alpha g(t)|^q + |g(t)|^p |D_\alpha f(t)|^q\} d_\alpha t$$
$$\leq \int_0^h \omega(t)\{|Z(t)|^p |D_\alpha g(t)|^q + |Z(t)|^p |D_\alpha f(t)|^q\} d_\alpha t$$
$$= \int_0^h \omega(t)|Z(t)|^p \{|D_\alpha g(t)|^q + |D_\alpha f(t)|^q\} d_\alpha t$$
$$\leq 2\int_0^h \omega(t)|Z(t)|^p |D_\alpha Z(t)|^q d_\alpha t$$
$$\leq \frac{2p}{q} K^{\frac{p-q}{q}} \left( \frac{h^\alpha}{4\alpha} \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \int_0^h [D_\alpha Z(t)]^{2q} d_\alpha t$$
$$+ \frac{2(q-p)}{q} K^{\frac{p}{q}} \left( \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \left( \int_0^h [D_\alpha Z(t)]^{2q} d_\alpha t \right)^{\frac{1}{2}}$$
$$\leq \frac{2p}{q} K^{\frac{p-q}{q}} \left( \frac{h^\alpha}{4\alpha} \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \int_0^h \{[D_\alpha f(t)]^{2q} + [D_\alpha g(t)]^{2q}\} d_\alpha t$$
$$+ \frac{2(q-p)}{q} K^{\frac{p}{q}} \left( \int_0^h \omega^2(t)\varphi^2(t) d_\alpha t \right)^{\frac{1}{2}} \left( \int_0^h \{[D_\alpha f(t)]^{2q} + [D_\alpha g(t)]^{2q}\} d_\alpha t \right)^{\frac{1}{2}}$$

for any constant $K > 0$. The proof is complete.

Next, we will use the experience of Sajid Iqbal, Josip Pečarić and Muhammad Samraiz [10] to establish the Opial-type inequalities for conformable fractional integral of two functions. By $L_p[a,b]$, $1 \leq p < \infty$, we denote the space of all Lebesgue measurable functions $f$ for which $|f^p|$ is Lebesgue integrable on $[a,b]$.

**Theorem 3.7.** Let $\alpha \in (0,1]$, $r > 1$, $r > q > 0$ and $p \geq 0$. If $\varphi > 0$, $\omega > 0$ are measurable functions on $[a, x]$, and $f_1, f_2 \in L_r[a, x]$, then the following inequality holds

$$\int_a^x \omega(t)\{|I_\alpha f_1(t)|^p |f_2(t)|^q + |I_\alpha f_2(t)|^p |f_1(t)|^q\} d_\alpha t$$

$$\leq 2^{1-\frac{q}{r}} \left(\frac{q}{p+q}\right)^{\frac{q}{r}} \left(d_{\frac{p}{q}} - 2^{-\frac{q}{p}}\right)^{\frac{q}{r}} \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x \varphi(t)[|f_1(t)|^r + |f_2(t)|^r] dt\right)^{\frac{p+q}{r}}, \tag{3.20}$$

where

$$h(t) = \varphi(t)[P(t)]^{\frac{p(r-1)}{r}} [\varphi(t)]^{-\frac{q}{r}}, \quad P(t) = \int_a^t (\tau^{\alpha-1})^{\frac{r}{r-1}} \varphi^{\frac{1}{1-r}}(\tau) d\tau,$$

and

$$d_{\frac{p}{q}} = \begin{cases} 2^{1-\frac{p}{q}}, & 0 \leq p \leq q; \\ 1, & p \geq q. \end{cases}$$

**Proof.** Since $I_\alpha f_i(t) = \int_a^t f_i(\tau)\tau^{\alpha-1} d\tau, (i = 1,2)$ and $\varphi > 0$, using the Hölder's inequality for $\{\frac{r}{r-1}, r\}$, we get that

$$|I_\alpha f_1(t)| \leq \left(\int_a^t (\tau^{\alpha-1})^{\frac{r}{r-1}} \varphi^{\frac{1}{1-r}}(\tau) d\tau\right)^{\frac{r-1}{r}} \left(\int_a^t \varphi(\tau)|f_1(\tau)|^r d\tau\right)^{\frac{1}{r}} \leq [P(t)]^{\frac{r-1}{r}} [G(t)]^{\frac{1}{r}}, \tag{3.21}$$

where $G(t) = \int_a^t \varphi(\tau)|f_1(\tau)|^r d\tau$.

Let $F(t) = \int_a^t \varphi(\tau)|f_2(\tau)|^r d\tau$, then

$$|f_2(t)|^q = [\varphi(t)]^{-\frac{q}{r}} [F'(t)]^{\frac{q}{r}}. \tag{3.22}$$

Now (3.21) and (3.22) implies that for $\omega > 0$,

$$\omega(t)|I_\alpha f_1(t)|^p |f_2(t)|^q \leq \omega(t)[P(t)]^{\frac{p(r-1)}{r}} [G(t)]^{\frac{p}{r}} [\varphi(t)]^{-\frac{q}{r}} [F'(t)]^{\frac{q}{r}}$$

$$= h(t)[G(t)]^{\frac{p}{r}} [F'(t)]^{\frac{q}{r}}. \tag{3.23}$$

Now integrating over $[a, x]$ and using Hölder's inequality for $\{\frac{r}{r-q}, \frac{r}{q}\}$, we obtaint

$$\int_a^x \omega(t)|I_\alpha f_1(t)|^p |f_2(t)|^q \leq \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x [G(t)]^{\frac{p}{q}} F'(t) dt\right)^{\frac{q}{r}}. \tag{3.24}$$

Similarly we can write

$$\int_a^x \omega(t)|I_\alpha f_2(t)|^p |f_1(t)|^q \leq \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x [F(t)]^{\frac{p}{q}} G'(t) dt\right)^{\frac{q}{r}}. \tag{3.25}$$

Now we need the simple inequlities to complete our result:

$$c_\varepsilon (A+B)^\varepsilon \leq A^\varepsilon + B^\varepsilon \leq d_\varepsilon (A+B)^\varepsilon, (A, B \geq 0) \tag{3.26}$$

where

$$c_\varepsilon = \begin{cases} 1, & 0 \leq \varepsilon \leq 1; \\ 2^{1-\varepsilon}, & \varepsilon \geq 1, \end{cases} \quad \text{and} \quad d_\varepsilon = \begin{cases} 2^{1-\varepsilon}, & 0 \leq \varepsilon \leq 1; \\ 1, & \varepsilon \geq 1. \end{cases}$$

Therefore from (3.24), (3.25) and (3.26), with $r > q$, we conclude that

$$\int_a^x \omega(t)\{|I_\alpha f_1(t)|^p |f_2(t)|^q + |I_\alpha f_2(t)|^p |f_1(t)|^q\} d_\alpha t$$

$$\leq 2^{1-\frac{q}{r}} \left( \int_a^x [h(t)]^{\frac{r}{r-q}} dt \right)^{\frac{r-q}{r}} \left( \int_a^x \{[G(t)]^{\frac{p}{q}} F'(t) + [F(t)]^{\frac{p}{q}} G'(t)\} dt \right)^{\frac{q}{r}}. \quad (3.27)$$

Using $F(a) = G(a) = 0$ and (3.26), we conclude that

$$\int_a^x \{[G(t)]^{\frac{p}{q}} F'(t) + [F(t)]^{\frac{p}{q}} G'(t)\} dt$$

$$= \int_a^x \{[G(t)]^{\frac{p}{q}} + [F(t)]^{\frac{p}{q}}\}[F'(t) + G'(t)] dt - \int_a^x \{[G(t)]^{\frac{p}{q}} G'(t) + [F(t)]^{\frac{p}{q}} F'(t)\} dt$$

$$\leq d_{\frac{p}{q}} \int_a^x [G(t) + F(t)]^{\frac{p}{q}} [F(t) + G(t)]' dt - \frac{q}{p+q} \{[G(x)]^{\frac{p}{q}+1} + [F(x)]^{\frac{p}{q}+1}\}$$

$$= \frac{q}{p+q} d_{\frac{p}{q}} [G(x) + F(x)]^{\frac{p}{q}+1} - \frac{q}{p+q} \{[G(x)]^{\frac{p}{q}+1} + [F(x)]^{\frac{p}{q}+1}\}$$

$$= \frac{q}{p+q} (d_{\frac{p}{q}} - 2^{-\frac{q}{p}}) [G(x) + F(x)]^{\frac{p}{q}+1}. \quad (3.28)$$

Using (3.28) in (3.27), we can obtain (3.20).

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