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# On McShane integrals of interval-valued functions and fuzzy-number-valued functions on Time Scales

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**Abstract.** In 2016, Hamid et al. [1] introduced the thought of the *AP*-Henstock integrals of interval-valued functions and fuzzy-number-valued functions and obtained a number of their properties. The aim of this paper is to introduce the thought of the McShane delta integrals of interval-valued functions and fuzzy-number-valued functions and discuss some of their properties.

**Keywords:** Fuzzy numbers; McShane delta integral of interval-valued functions; McShane delta integral of fuzzy-number-valued functions.

## 1 Introduction

The calculus on time scales was introduced for the firrst time in 1988 by Hilger [2] to unify the theory of difference equations and the theory of differential equations. The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [3] in 2006. In 2016, Hamid and Elmuiz [4] introduced the concept of the Henstock-Stieltjes (*HS*) integrals of interval-valued functions and fuzzy-number-valued functions and discussed a number of their properties.

In this paper, we introduce the notion of the McShane delta integrals of interval-valued functions and fuzzynumber-valued functions and investigate some of their properties.

The paper is organized as follows, in Section 2 we have a tendency to provide the preliminary terminology used in this paper. Section 3 is dedicated to discussing the McShane delta integral of interval-valued functions. In Section 4, we present the McShane delta integral of fuzzy-number-valued functions. The last section provides Conclusions.

## 2 Preliminaries

A time scale $\mathbf{T}$ is a nonempty closed subset of real number $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t \in \mathbf{T}$ we define the forward jump operator $\sigma(t) = \inf\{s \in \mathbf{T} : s > t\}$ where $\inf \phi = \sup\{\mathbf{T}\}$, while the backward jump operator $\rho(t) = \sup\{s \in \mathbf{T} : s < t\}$ where $\sup \phi = \inf\{\mathbf{T}\}$. If $\sigma(t) > t$, we say that $t$ is right-scattered, while if $\rho(t) < t$, we say that $t$ is left-scattered. If $\sigma(t) = t$, we say that $t$ is right-dense, while if $\rho(t) = t$, we say that $t$ is left-dense. The forward graininess function $\mu(t)$ of $t \in \mathbf{T}$ is defined by $\mu(t) = \sigma(t) - t$, whlie the backward graininess function $\nu(t)$ of $t \in \mathbf{T}$ is defined by $\nu(t) = t - \rho(t)$. For $a, b \in \mathbf{T}$ we denote the closed interval $[a, b]_{\mathbf{T}} = \{t \in \mathbf{T} : a \leq t \leq b\}$.

Throughout this paper, all considered intervals will be intervals in $\mathbf{T}$. A division $P$ of $[a, b]_{\mathbf{T}}$ is a finite collection of interval-point pairs $\{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^{n}$, where $\{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ and $\xi_i \in [a, b]_{\mathbf{T}}$ for $i = 1, 2, \cdots, n$. By $\Delta t_i = t_i - t_{i-1}$ we denote the length of $i$th subinterval in the division $P$. $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$ is a $\Delta$- gauge for $[a, b]_{\mathbf{T}}$ provided $\delta_L(\xi) > 0$ on $(a, b]_{\mathbf{T}}$, $\delta_R(\xi) > 0$ on $[a, b)_{\mathbf{T}}$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(b) \geq \mu(\xi)$

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for all $\xi \in [a,b)_{\mathbf{T}}$. We say that $P = \{([t_{i-1},t_i]_{\mathbf{T}};\xi_i)\}_{i=1}^n$ is a $\delta$-fine McShane division of $[a,b]_{\mathbf{T}}$ if $[t_{i-1},t_i]_{\mathbf{T}} \subset \big(\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)\big)_{\mathbf{T}}$ and $\xi_i \in [a,b]_{\mathbf{T}}$ for all $i = 1,2,\cdots,n$.

**Definition 2.1** [5] A real-valued function $f : [a,b] \to \mathbb{R}$ is said to be McShane $(M)$ integrable to $B$ on $[a,b]$ if for every $\varepsilon > 0$, there is a function $\delta(t) > 0$ such that for any $\delta$-fine McShane division $P = \{[u_i,v_i];\xi_i\}_{i=1}^n$ of $[a,b]$, we have

$$\Big|\sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - B\Big| < \varepsilon, \tag{2.1}$$

we write $(M)\int_a^b f(t)\mathrm{d}t = B$ , and $f \in M[a,b]$.

**Definition 2.2** [6] A function $f : [a,b]_{\mathbf{T}} \to \mathbb{R}$ is McShane delta integrable (McShane $\Delta$-integrable) on $[a,b]_{\mathbf{T}}$ if there exists a number $A \in \mathbb{R}$ such that for each $\varepsilon > 0$ there is a $\Delta$-gauge, $\delta$, on $[a,b]_{\mathbf{T}}$ such that

$$\Big|\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - A\Big| < \varepsilon \tag{2.2}$$

for each $\delta$-fine McShane division $P = \{([t_{i-1},t_i]_{\mathbf{T}};\xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbf{T}}$. $A$ is called McShane $\Delta$-integral of $f$ on $[a,b]_{\mathbf{T}}$, and we write $A = (M)\int_a^b f(t)\Delta t$.

**Theorem 2.1** If $f(t)$ and $g(t)$ are McShane $\Delta$-integrable on $[a,b]_{\mathbf{T}}$ and $f(t) \le g(t)$ almost everywhere on $[a,b]_{\mathbf{T}}$, then

$$(M)\int_a^b f(t)\Delta t \le (M)\int_a^b g(t)\Delta t. \tag{2.3}$$

**Proof** The proof follows easily from the same argument in Theorem 3.6 [5]. □

# 3 McShane delta integral of interval-valued functions on time scales

In this section, we introduce the notion of the McShane delta integral of interval-valued functions on time scales and investigate some of their properties.

**Definition 3.1** [7] Let $I_{\mathbb{R}} = \{I = [I^-, I^+] : I$ is the closed bounded interval on the real line $\mathbb{R}\}$.

For $A, B \in I_{\mathbb{R}}$, we define $A \le B$ iff $A^- \le B^-$ and $A^+ \le B^+$, $A + B = C$ iff $C^- = A^- + B^-$ and $C^+ = A^+ + B^+$, and $A \cdot B = \{a \cdot b : a \in A, b \in B\}$, where

$$(A \cdot B)^- = \min\{A^- \cdot B^-, A^- \cdot B^+, A^+ \cdot B^-, A^+ \cdot B^+\} \tag{3.1}$$

and

$$(A \cdot B)^+ = \max\{A^- \cdot B^-, A^- \cdot B^+, A^+ \cdot B^-, A^+ \cdot B^+\}. \tag{3.2}$$

Define $d(A,B) = \max(|A^- - B^-|, |A^+ - B^+|)$ as the distance between intervals $A$ and $B$.

**Definition 3.2** [8] Let $F : [a,b] \to I_{\mathbb{R}}$ be an interval-valued function. $I_0 \in I_{\mathbb{R}}$, for every $\varepsilon > 0$ there is a $\delta(t) > 0$ such that for any $\delta$-fine McShane division $P = \{([u_i,v_i],\xi_i)\}_{i=1}^n$, we have

$$d\big(\sum_{i=1}^{n} F(\xi_i)(v_i - u_i), I_0\big) < \varepsilon, \tag{3.3}$$

then $F(t)$ is said to be McShane integrable over $[a,b]$ and write $(IM)\int_a^b F(t)\mathrm{d}t = I_0$. For brevity, we write $F(t) \in IM[a,b]$.

**Definition 3.3** A interval-valued function $F : [a,b]_{\mathbf{T}} \to I_{\mathbb{R}}$ is McShane delta integrable to $I_0 \in I_{\mathbb{R}}$ on $[a,b]_{\mathbf{T}}$ if for every $\varepsilon > 0$ there exists a $\Delta$-gauge, $\delta$, on $[a,b]_{\mathbf{T}}$ such that

$$d\Big(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\Big) < \varepsilon, \tag{3.4}$$

whenever $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a $\delta$-fine McShane division of $[a,b]_{\mathbf{T}}$. We write $(IM)\int_a^b F(t)\Delta t = I_0$ and $F \in IM[a,b]_{\mathbf{T}}$.

**Remark 3.1** If $F(t) \in IM[a,b]_{\mathbf{T}}$, then the integral value is unique.

**Theorem 3.1** An interval-valued function $F : [a,b]_{\mathbf{T}} \to I_{\mathbb{R}}$ is McShane delta integrable on $[a,b]_{\mathbf{T}}$ if and only if $F^-, F^+ \in M[a,b]_{\mathbf{T}}$ and

$$(IM)\int_a^b F(t)\Delta t = \Big[(M)\int_a^b F^-(t)\Delta t, (M)\int_a^b F^+(t)\Delta t\Big]. \tag{3.5}$$

**Proof** Let $F \in IM[a,b]_{\mathbf{T}}$, then there exists an interval $I_0 = [I_0^-, I_0^+]$ with the property that for any $\varepsilon > 0$ there exists a $\Delta$-gauge, $\delta$, on $[a,b]_{\mathbf{T}}$ such that

$$d\Big(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\Big) < \varepsilon, \tag{3.6}$$

whenever $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a $\delta$-fine McShane division of $[a,b]_{\mathbf{T}}$.

Since $t_i - t_{i-1} \geq 0$ for $1 \leq i \leq n$, we have

$$\begin{aligned} & d\Big(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\Big) \\ &= \max\left(\left|\Big[\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1})\Big]^- - I_0^-\right|, \left|\Big[\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1})\Big]^+ - I_0^+\right|\right) < \varepsilon. \\ &= \max\left(\left|\sum_{i=1}^{n} F^-(\xi_i)(t_i - t_{i-1}) - I_0^-\right|, \left|\sum_{i=1}^{n} F^+(\xi_i)(t_i - t_{i-1}) - I_0^+\right|\right) < \varepsilon. \end{aligned} \tag{3.7}$$

Hence $\left|\sum_{i=1}^{n} F^-(\xi_i)(t_i - t_{i-1}) - I_0^-\right| < \varepsilon$, $\left|\sum_{i=1}^{n} F^+(\xi_i)(t_i - t_{i-1}) - I_0^+\right| < \varepsilon$ whenever $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a $\delta$-fine McShane division of $[a,b]_{\mathbf{T}}$. Thus $F^-, F^+ \in M[a,b]_{\mathbf{T}}$ and

$$(IM)\int_a^b F(t)\Delta t = \Big[(M)\int_a^b F^-(t)\Delta t, (M)\int_a^b F^+(t)\Delta t\Big]. \tag{3.8}$$

Conversely, let $F^-, F^+ \in M[a,b]_{\mathbf{T}}$. Then there exists $M_1, M_2 \in \mathbb{R}$ with the property that given $\varepsilon > 0$ there exists a $\Delta$-gauge, $\delta$, on $[a,b]_{\mathbf{T}}$ such that

$$\left|\sum_{i=1}^{n} F^-(\xi_i)(t_i - t_{i-1}) - M_1\right| < \varepsilon, \quad \left|\sum_{i=1}^{n} F^+(\xi_i)(t_i - t_{i-1}) - M_2\right| < \varepsilon$$

whenever $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a $\delta$-fine McShane division of $[a,b]_{\mathbf{T}}$. We define $I_0 = [M_1, M_2]$, then if $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a $\delta$-fine McShane division of $[a,b]_{\mathbf{T}}$, we have

$$d\Big(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\Big) < \varepsilon. \tag{3.9}$$

Hence $F : [a,b]_{\mathbf{T}} \to I_{\mathbb{R}}$ is McShane delta integrable on $[a,b]_{\mathbf{T}}$. □

**Theorem 3.2** If $F(t), G(t) \in IM[a,b]_{\mathbf{T}}$ and $\beta, \gamma \in \mathbb{R}$. Then $\big[\beta F(t) + \gamma G(t)\big] \in IM[a,b]_{\mathbf{T}}$ and

$$(IM)\int_a^b (\beta F(t) + \gamma G(t))\Delta(t) = \beta(IM)\int_a^b F(t)\Delta(t) + \gamma(IM)\int_a^b G(t)\Delta(t). \tag{3.10}$$

**Proof** If $F(t), G(t) \in IM[a,b]_{\mathbf{T}}$, then $F^-(t), F^+(t), G^-(t), G^+(t) \in M[a,b]_{\mathbf{T}}$ by Theorem 3.1. Hence $\beta F^-(t) + \gamma G^-(t), \beta F^-(t) + \gamma G^+(t), \beta F^+(t) + \gamma G^-(t), \beta F^+(t) + \gamma G^+(t) \in M[a,b]_{\mathbf{T}}$.

(1) If $\beta > 0$ and $\gamma > 0$, then

$$\begin{aligned}(M)\int_a^b (\beta F(t) + \gamma G(t))^- \Delta t &= (M)\int_a^b (\beta F^-(t) + \gamma G^-(t))\Delta t \\ &= \beta (M)\int_a^b F^-(t)\Delta t + \gamma (M)\int_a^b G^-(t)\Delta t \\ &= \beta\Big((IM)\int_a^b F(t)\Delta t\Big)^- + \gamma\Big((IM)\int_a^b G(t)\Delta t\Big)^- \\ &= \Big(\beta(IM)\int_a^b F(t)\Delta t + \gamma(IM)\int_a^b G(t)\Delta t\Big)^-.\end{aligned}$$

(2) If $\beta < 0$ and $\gamma < 0$, then

$$\begin{aligned}(M)\int_a^b (\beta F(t) + \gamma G(t))^- \Delta t &= (M)\int_a^b (\beta F^+(t) + \gamma G^+(t))\Delta t \\ &= \beta (M)\int_a^b F^+(t)\Delta t + \gamma (M)\int_a^b G^+(t)\Delta t \\ &= \beta\Big((IM)\int_a^b F(t)\Delta t\Big)^+ + \gamma\Big((IM)\int_a^b G(t)\Delta t\Big)^+ \\ &= \Big(\beta(IM)\int_a^b F(t)\Delta t + \gamma(IM)\int_a^b G(t)\Delta t\Big)^-.\end{aligned}$$

(3) If $\beta > 0$ and $\gamma < 0$, (or $\beta < 0$ and $\gamma > 0$), then

$$\begin{aligned}(M)\int_a^b (\beta F(t) + \gamma G(t))^- \Delta t &= (M)\int_a^b (\beta F^-(t) + \gamma G^+(t))\Delta t \\ &= \beta (M)\int_a^b F^-(t)\Delta t + \gamma (M)\int_a^b G^+(t)\Delta t \\ &= \beta\Big((IM)\int_a^b F(t)\Delta t\Big)^- + \gamma\Big((IM)\int_a^b G(t)\Delta t\Big)^+ \\ &= \Big(\beta(IM)\int_a^b F(t)\Delta t + \gamma(IM)\int_a^b G(t)\Delta t\Big)^-.\end{aligned}$$

Similarly, for four cases above we have

$$(M)\int_a^b (\beta F(t) + \gamma G(t))^+ \Delta t = \Big(\beta(IM)\int_a^b F(t)\Delta t + \gamma(IM)\int_a^b G(t)\Delta t\Big)^+. \tag{3.11}$$

Hence by Theorem 3.1 $\beta F(t) + \gamma G(t) \in IM[a,b]_{\mathbf{T}}$ and

$$(IM)\int_a^b (\beta F(t) + \gamma G(t))\Delta t = \beta(IM)\int_a^b F(t)\Delta t + \gamma(IM)\int_a^b G(t)\Delta t. \tag{3.12}$$

□

**Theorem 3.3** If $F(t) \in IM[a,c]_{\mathbf{T}}$ and $F(t) \in IM[c,b]_{\mathbf{T}}$, then $F(t) \in IM[a,b]_{\mathbf{T}}$ and

$$(IM)\int_a^b F(t)\Delta t = (IM)\int_a^c F(t)\Delta t + (IM)\int_c^b F(t)\Delta t. \tag{3.13}$$

**Proof** If $F(t) \in IM[a,c]_{\mathbf{T}}$ and $F(t) \in IM[c,b]_{\mathbf{T}}$, then by Theorem 3.1 $F^-(t), F^+(t) \in M[a,c]_{\mathbf{T}}$ and $F^-(t), F^+(t) \in M[c,b]_{\mathbf{T}}$. Hence $F^-(t), F^+(t) \in M[a,b]_{\mathbf{T}}$ and

$$\begin{aligned}(M)\int_a^b F^-(t)\Delta t &= (M)\int_a^c F^-(t)\Delta t + (M)\int_c^b F^-(t)\Delta t \\ &= \left((IM)\int_a^c F(t)\Delta t + (IM)\int_c^b F(t)\Delta t\right)^-.\end{aligned}$$

Similarly, $(M)\int_a^b F^+(t)\Delta t = \left((IM)\int_a^c F(t)\Delta t + (IM)\int_c^b F(t)\Delta t\right)^+$. Hence by Theorem 3.1 $F(t) \in IM[a,b]_{\mathbf{T}}$ and

$$(IM)\int_a^b F(t)\Delta t = (IM)\int_a^c F(t)\Delta t + (IM)\int_c^b F(t)\Delta t. \tag{3.14}$$

□

**Theorem 3.4** If $F(t) \le G(t)$ nearly everywhere on $[a,b]_{\mathbf{T}}$ and $F(t), G(t) \in IM[a,b]_{\mathbf{T}}$, then

$$(IM)\int_a^b F(t)\Delta t \le (IM)\int_a^b G(t)\Delta t. \tag{3.15}$$

**Proof** Let $F(t) \le G(t)$ nearly everywhere on $[a,b]_{\mathbf{T}}$ and $F(t), G(t) \in IM[a,b]_{\mathbf{T}}$. Then $F^-(t), F^+(t), G^-(t), G^+(t) \in M[a,b]_{\mathbf{T}}$ and $F^-(t) \le G^-(t)$, $F^+(t) \le G^+(t)$ nearly everywhere on $[a,b]_{\mathbf{T}}$. By Theorem 2.1 $(M)\int_a^b F^-(t)\Delta t \le (M)\int_a^b G^-(t)\Delta t$ and $(M)\int_a^b F^+(t)\Delta t \le (M)\int_a^b G^+(t)\Delta t$. Hence

$$(IM)\int_a^b F(t)\Delta t \le (IM)\int_a^b G(t)\Delta t, \tag{3.16}$$

by Theorem 3.1. □

**Theorem 3.5** Let $F(t), G(t) \in IM[a,b]_{\mathbf{T}}$ and $d(F(t),G(t))$ is Lebesgue integrable on $[a,b]_{\mathbf{T}}$. Then

$$d\big((IM)\int_a^b F(t)\Delta t, (IM)\int_a^b G(t)\Delta t\big) \le (L)\int_a^b d\big(F(t),G(t)\big)\Delta t. \tag{3.17}$$

**Proof** By definition of distance,

$$\begin{aligned}& d\big((IM)\int_a^b F(t)\Delta t, (IM)\int_a^b G(t)\Delta t\big) \\ &= \max\left(\left|\left((IM)\int_a^b F(t)\Delta t\right)^- - \left((IM)\int_a^b G(t)\Delta t\right)^-\right|, \left|\left((IM)\int_a^b F(t)\Delta t\right)^+ - \left((IM)\int_a^b G(t)\Delta t\right)^+\right|\right) \\ &= \max\left(\left|(M)\int_a^b (F^-(t) - G^-(t))\Delta t\right|, \left|(M)\int_a^b (F^+(t) - G^+(t))\Delta t\right|\right) \\ &\le \max\left((L)\int_a^b \left|F^-(t) - G^-(t)\right|\Delta t, (L)\int_a^b \left|F^+(t) - G^+(t)\right|\Delta t\right)\end{aligned}$$

$$\leq (L)\int_a^b \max\left(\left|F^-(t)-G^-(t)\right|\Delta t, \left|F^+(t)-G^+(t)\right|\Delta t\right)$$

$$= (L)\int_a^b d\big(F(t),G(t)\big). \tag{3.18}$$

□

# 4 McShane delta integral of fuzzy-number-valued functions on time scales

This section introduces the notion of the McShane delta integral of fuzzy-number-valued functions and discusses some of their properties.

**Definition 4.1** [9, 10, 11] Let $\tilde{A} \in F(\mathbb{R})$ be a fuzzy subset on $\mathbb{R}$. If for any $\lambda \in [0,1]$, $A_\lambda = [A_\lambda^-, A_\lambda^+]$ and $A_1 \neq \phi$, where $A_\lambda = \{t : \tilde{A}(t) \geq \lambda\}$, then $\tilde{A}$ is called a fuzzy number. If $\tilde{A}$ is (1) convex, (2) normal, (3) upper semi-continuous, (4) has the compact support, we say that $\tilde{A}$ is a compact fuzzy number.

Let $\tilde{\mathbb{R}}$ denote the set of all fuzzy numbers.

**Definition 4.2** [9] Let $\tilde{A}, \tilde{B} \in \tilde{\mathbb{R}}$, we define (1) $\tilde{A} \leq \tilde{B}$ iff $A_\lambda \leq B_\lambda$ for all $\lambda \in (0,1]$, (2) $\tilde{A} + \tilde{B} = \tilde{C}$ iff $A_\lambda + B_\lambda = C_\lambda$ for any $\lambda \in (0,1]$, (3) $\tilde{A} \cdot \tilde{B} = \tilde{D}$ iff $A_\lambda \cdot B_\lambda = D_\lambda$ for any $\lambda \in (0,1]$.

For $\tilde{A}, \tilde{B} \in \tilde{\mathbb{R}}^C$, then

$$D(\tilde{A},\tilde{B}) = \sup_{\lambda\in[0,1]} d(A_\lambda, B_\lambda), \tag{4.1}$$

is called the distance between $\tilde{A}$ and $\tilde{B}$.

**Lemma 4.1** [12] If a mapping $H : [0,1] \to I_{\mathbb{R}}$, $\lambda \to H(\lambda) = [m_\lambda, n_\lambda]$, satisfies $[m_{\lambda_1}, n_{\lambda_1}] \supset [m_{\lambda_2}, n_{\lambda_2}]$ when $\lambda_1 < \lambda_2$, then

$$\tilde{A} := \bigcup_{\lambda\in(0,1]} \lambda H(\lambda) \in \tilde{\mathbb{R}} \tag{4.2}$$

and

$$A_\lambda = \bigcap_{n=1}^{\infty} H(\lambda_n), \tag{4.3}$$

where $\lambda_n = [1 - \frac{1}{(n+1)}]\lambda$.

**Definition 4.3** Let $\tilde{F} : [a,b]_{\mathbf{T}} \to \tilde{\mathbb{R}}$. If the interval-valued function $F_\lambda(t) = [F_\lambda^-(t), F_\lambda^+(t)]$ is McShane delta integrable on $[a,b]_{\mathbf{T}}$ for any $\lambda \in (0,1]$, then $\tilde{F}(t)$ is called McShane delta integrable on $[a,b]_{\mathbf{T}}$ and the integral is defined by McShane delta integral is defined by

$$\begin{aligned}(FM)\int_a^b \tilde{F}(t)\Delta t &:= \bigcup_{\lambda\in(0,1]} \lambda (IM)\int_a^b F_\lambda(t)\Delta t \\ &= \bigcup_{\lambda\in(0,1]} \lambda\left[(M)\int_a^b F_\lambda^-(t)\Delta t, (M)\int_a^b F_\lambda^+(t)\Delta t\right].\end{aligned}$$

We write $\tilde{F}(t) \in FM[a,b]_{\mathbf{T}}$.

**Theorem 4.1** $\tilde{F}(t) \in FM[a,b]_{\mathbf{T}}$, then $(FM)\int_a^b \tilde{F}(t)\Delta t \in \tilde{\mathbb{R}}$ and

$$\left[(FM)\int_a^b \tilde{F}(t)\Delta t\right]_\lambda = \bigcap_{n=1}^{\infty} (IM)\int_a^b F_{\lambda_n}(t)\Delta t, \tag{4.4}$$

where $\lambda_n = [1 - \frac{1}{(n+1)}]\lambda$.

**Proof** Let $H : (0,1] \to I_{\mathbb{R}}$, be defined by $H(\lambda) = \big[(M)\int_a F_\lambda^-(t)\Delta t, (M)\int_a F_\lambda^+(t)\Delta t\big]$.

Since $F_\lambda^-(t)$ and $F_\lambda^+(t)$ are increasing and decreasing on $\lambda$ respectively, therefore, when $0 < \lambda_1 \le \lambda_2 \le 1$, we have $F_{\lambda_1}^-(t) \le F_{\lambda_2}^-(t)$, $F_{\lambda_1}^+(t) \ge F_{\lambda_2}^+(t)$, on $[a,b]_{\mathbf{T}}$. From Theorem 3.4 we have

$$\left[(M)\int_a^b F_{\lambda_1}^-(t)\Delta t, (M)\int_a^b F_{\lambda_1}^+(t)\Delta t\right] \supset \left[(M)\int_a^b F_{\lambda_2}^-(t)\Delta t, (M)\int_a^b F_{\lambda_2}^+(t)\Delta t\right]. \tag{4.5}$$

Using Theorem 3.1 and Lemma 4.1 we obtain

$$(FM)\int_a^b \tilde{F}(t)\Delta t := \bigcup_{\lambda\in(0,1]} \lambda\left[(M)\int_a^b F_\lambda^-(t)\Delta t, (M)\int_a^b F_\lambda^+(t)\Delta t\right] \in \tilde{\mathbb{R}} \tag{4.6}$$

and for all $\lambda \in (0,1]$,

$$\big[(FM)\int_a^b \tilde{F}(t)\Delta t\big]_\lambda = \bigcap_{n=1}^{\infty}(IM)\int_a^b F_{\lambda_n}(t)\Delta t, \tag{4.7}$$

where $\lambda_n = [1 - \frac{1}{(n+1)}]\lambda$. □

**Theorem 4.2** If $\tilde{F}(t), \tilde{G}(t) \in FM[a,b]_{\mathbf{T}}$ and $\beta, \gamma \in \mathbb{R}$. Then $\beta\tilde{F}(t) + \gamma\tilde{G}(t) \in FM[a,b]_{\mathbf{T}}$ and

$$(FM)\int_a^b \big(\beta\tilde{F}(t) + \gamma\tilde{G}(t)\big)\Delta t = \beta(FM)\int_a^b \tilde{F}(t)\Delta t + \gamma(FM)\int_a^b \tilde{G}(t)\Delta t. \tag{4.8}$$

**Proof** If $\tilde{F}(t), \tilde{G}(t) \in FM[a,b]_{\mathbf{T}}$, then the interval-valued function $F_\lambda(t) = [F_\lambda^-(t), F_\lambda^+(t)]$ and $G_\lambda(t) = [G_\lambda^-(t), G_\lambda^+(t)]$ are McShane delta integrable on $[a,b]_{\mathbf{T}}$ for any $\lambda \in (0,1]$ and $(FM)\int_a^b \tilde{F}(t)\Delta t = \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b F_\lambda(t)\Delta$ and $(FM)\int_a^b \tilde{G}(t)\Delta t = \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b G_\lambda(t)\Delta t$. From Theorem 3.2 we have $\beta F_\lambda(t) + \gamma G_\lambda(t) \in IM[a,b]_{\mathbf{T}}$ and $(IM)\int_a^b \big(\beta F_\lambda(t) + \gamma G_\lambda(t)\big)\Delta t = \beta(IM)\int_a^b F_\lambda(t)\Delta t + \gamma(IM)\int_a^b G_\lambda(t)\Delta t$ for any $\lambda \in (0,1]$. Hence $\beta\tilde{F}(t) + \gamma\tilde{G}(t) \in FM[a,b]_{\mathbf{T}}$ and

$$\begin{aligned}
(FM)\int_a^b \big(\beta\tilde{F}(t) + \gamma\tilde{G}(t)\big)\Delta t &= \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b \big(\beta F_\lambda(t) + \gamma G_\lambda(t)\big)\Delta t \\
&= \bigcup_{\lambda\in(0,1]} \lambda\Big(\beta(IM)\int_a^b F_\lambda(t)\Delta t + \gamma(IM)\int_a^b G_\lambda(t)\Delta t\Big) \\
&= \beta\bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b F_\lambda(t)\Delta t + \gamma\bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b G_\lambda(t)\Delta t \\
&= \beta(FM)\int_a^b \tilde{F}(t)\Delta t + \gamma(FM)\int_a^b \tilde{G}(t)\Delta t.
\end{aligned}$$

□

**Theorem 4.3** If $\tilde{F}(t) \in FM[a,c]_{\mathbf{T}}$ and $\tilde{F}(t) \in FM[c,b]_{\mathbf{T}}$, then $\tilde{F}(t) \in FM[a,b]_{\mathbf{T}}$ and

$$(FM)\int_a^b \tilde{F}(t)\Delta t = (FM)\int_a^c \tilde{F}(t)\Delta t + (FM)\int_c^b \tilde{F}(t)\Delta t. \tag{4.9}$$

**Proof** If $\tilde{F}(t) \in FM[a,c]_{\mathbf{T}}$ and $\tilde{F}(t) \in FM[c,b]_{\mathbf{T}}$, then the interval-valued function $F_\lambda(t) = [F_\lambda^-(t), F_\lambda^+(t)]$ is McShane delta integrable on $[a,c]_{\mathbf{T}}$ and $[c,b]_{\mathbf{T}}$ for any $\lambda \in (0,1]$ and $(FM)\int_a^c \tilde{F}(t)\Delta t = \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^c F_\lambda(t)\Delta t$ and

$(FM)\int_c^b \tilde{F}(t)\Delta t = \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_c^b F_\lambda(t)\Delta t$. From Theorem 3.3 we have $F_\lambda(t) \in IM[a,b]_{\mathbf{T}}$ and $(IM)\int_a^b F_\lambda(t)\Delta t = (IM)\int_a^c F_\lambda(t)\Delta t + (IM)\int_c^b F_\lambda(t)\Delta t$ for any $\lambda \in (0,1]$. Hence $\tilde{F}(t) \in FM[a,b]_{\mathbf{T}}$ and

$$\begin{aligned}(FM)\int_a^b \tilde{F}(t)\Delta t &= \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b F_\lambda(t)\Delta t \\ &= \bigcup_{\lambda\in(0,1]} \lambda\Big((IM)\int_a^c F_\lambda(t)\Delta t + (IM)\int_c^b F_\lambda(t)\Delta t\Big) \\ &= \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^c F_\lambda(t)\Delta t + \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_c^b F_\lambda(t)\Delta t \\ &= (FM)\int_a^c \tilde{F}(t)\Delta t + (FM)\int_c^b \tilde{F}(t)\Delta t.\end{aligned}$$

□

**Theorem 4.4** If $\tilde{F}(t) \le \tilde{G}(t)$ nearly everywhere on $[a,b]_{\mathbf{T}}$ and $\tilde{F}(t), \tilde{G}(t) \in FM[a,b]_{\mathbf{T}}$, then

$$(FM)\int_a^b \tilde{F}(t)\Delta t \le (FM)\int_a^b \tilde{G}(t)\Delta t. \tag{4.10}$$

**Proof** If $\tilde{F}(t) \le \tilde{G}(t)$ nearly everywhere on $[a,b]_{\mathbf{T}}$ and $\tilde{F}(t), \tilde{G}(t) \in FM[a,b]_{\mathbf{T}}$, then $F_\lambda(t) \le G_\lambda(t)$ nearly everywhere on $[a,b]_{\mathbf{T}}$ for any $\lambda \in (0,1]$ and $F_\lambda(t)$ and $G_\lambda(t)$ are McShane delta integrable on $[a,b]_{\mathbf{T}}$ for any $\lambda \in (0,1]$ and $(FM)\int_a^b \tilde{F}(t)\Delta t = \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b F_\lambda(t)\Delta t$ and $(FM)\int_a^b \tilde{G}(t)\Delta t = \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b G_\lambda(t)\Delta t$. From Theorem 3.4 we have $(IM)\int_a^b F_\lambda(t)\Delta t \le (IM)\int_a^b G_\lambda(t)\Delta t$ for any $\lambda \in (0,1]$. Hence

$$\begin{aligned}(FM)\int_a^b \tilde{F}(t)\Delta t &= \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b F_\lambda(t)\Delta t \\ &\le \bigcup_{\lambda\in(0,1]} \lambda(IM)\int_a^b G_\lambda(t)\Delta t \\ &= (FM)\int_a^b \tilde{G}(t)\Delta t.\end{aligned}$$

□

# 5 conclusions

In this paper, we have a tendency to introduced the concept of the McShane delta integrals of interval-valued functions and fuzzy number- valued functions and discussed some properties of those integrals.

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