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A Novel Proof based on the method of infinite descent for Fermat's Last Theorem

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Abstract

Fermat's Last Theorem is that for each $k\geq 3$ (k is an integer), the equation $x^k = y^k + z^k$ involving x, y and z has no positive integer solution. This paper proposed a novel proof for the Fermat's last theorem by the methods of infinite descent and complex variable analysis.

Keywords: Fermat's Last Theorem; elementary proof; the method of infinite descent.

1. Introduction

P.de Fermat(1601─1665) wrote the famous theorem in the margin of a book in 1637 and said that he had found a very ingenious proof but the margin is too small to write it. There are some other statements included in [1].

For three and half centuries, searching for the proof, especially, a succinct proof of the theorem has puzzled and encouraged numerous mathematicians. After a century, L.Euler (1707─1783) and C.F.Gauss (1777─1855) gave the proof successively for k=3. Recently, A. Wiles [2,3] proved the theorem based on advanced algebra and cyclic groups. The present paper devotes to propose an ingenious and succinct proof for Fermat's last theorem by complex analysis[4] and the method of infinite descent that had been interested by Fermat.

The remaining part of this paper is organized as follows. In section 2 the principle of infinite descent for the Fermat's last theorem is set up by complex variable analysis. In section 3 we prove the case of k=3 for Fermat's last theorem directly by the ideas in section 2. The complete proof of Fermat's last theorem is integrated in section 4.

2. The principle of infinite descent for the Fermat's last theorem

Lemma 1. For the equation wherein *x*, *y*, *z*, *k* are positive integers

$$x^k = y^k + z^k \tag{1}$$

If for a integer $k \geq 3$, there exists a positive integer solution (x,y,z), then the equation (1) with k replaced by k-1 has also a positive integer solution.

Lemma 1 is called the principle of infinite descent for the Fermat's last theorem.

Proof

We set

$$\begin{cases} y = x - n_1 \\ z = x - n_2 \end{cases} \tag{2}$$

wherein n and n are positive integers less than x.

Substituting (2) into (1), we have

$$x^k + \sum_{j=1}^{k} (-1)^j C_k^j (n_1^j + n_2^j) x^{k-j} = 0 \tag{3}$$

in which $C_k^j = k!/[j!(k-j)!]$.

The k roots of (3) correspond with the k points in complex plane (The roots may be multiple).

Denote the roots as $x_l = a_l e^{i\varphi_l}$ or

$$x_l = a_l(\cos\varphi_l + i\sin\varphi_l), \qquad l = 1,2,\cdots,k. \tag{4}$$

in which a_l and φ_l are real numbers $(a_l \geq 0, \quad -\pi \leq \phi_l < \pi)$.

Substituting (4) into (3) and separating the real and imaginary part, we obtain a pair of equations for each l as follows:

$$\begin{cases} a_l^k \cos k\varphi_l + \sum_{j=1}^{k} (-1)^j C_k^j (n_1^j + n_2^j) a_l^{k-j} \cos[(k-j)\varphi_l] = 0 \\ a_l^k \sin k\varphi_l + \sum_{j=1}^{k-1} (-1)^j C_k^j (n_1^j + n_2^j) a_l^{k-j} \sin[(k-j)\varphi_l] = 0 \end{cases} \quad l = 1,2,\cdots,k. \tag{5}$$

It is well known that just these k points in complex plane can make the (5) hold. Note that in the second equation of (5), summation for j is from 1 up to k-1 as the k-th term is zero.

The sine and cosine functions in (5) can be expanded as power series, thus we have

$$\begin{cases} a_l^k + \sum_{j=1}^{k}(-1)^j C_k^j (n_1^j + n_2^j) a_l^{k-j} \\ = a_l^k [\frac{k^2\varphi_l^2}{2!} - \frac{k^4\varphi_l^4}{4!} + \cdots] + \sum_{j=1}^{k}(-1)^j C_k^j (n_1^j + n_2^j) a_l^{k-j} [\frac{(k-j)^2\varphi_l^2}{2!} - \frac{(k-j)^4\varphi_l^4}{4!} + \cdots] \\ a_l^k k + \sum_{j=1}^{k-1}(-1)^j C_k^j (n_1^j + n_2^j) a_l^{k-j}(k-j) \\ = a_l^k [\frac{k^3\varphi_l^2}{3!} - \frac{k^5\varphi_l^4}{5!} + \cdots] + \sum_{j=1}^{k-1}(-1)^j C_k^j (n_1^j + n_2^j) a_l^{k-j} [\frac{(k-j)^3\varphi_l^2}{3!} - \frac{(k-j)^5\varphi_l^4}{5!} + \cdots] \end{cases} \quad (6)$$

$$l = 1,2...,k$$

Among the *k* roots of (3), we assume there are *m* roots ($m \le k$) of which modules are positive numbers (signed as b_l) and polar angles $|\varphi_l| < \varepsilon$ (ε is an infinitesimal quantity). Thus, we can change (6) into *m* pairs of equation corresponding with the *m* roots as below:

$$\begin{cases} b_l^k + \sum_{j=1}^{k}(-1)^j C_k^j (n_1^j + n_2^j) b_l^{k-j} \\ = [k^2 b_l^k + \sum_{j=1}^{k}(-1)^j (k-j)^2 C_k^j (n_1^j + n_2^j) b_l^{(k-j)}] \frac{\varphi_l^2}{2!} - [k^4 b_l^k + \sum_{j=1}^{k}(-1)^j (k-j)^4 c_k^j (n_1^j + n_2^j) b_l^{(k-j)}] \frac{\varphi_l^4}{4!} + \cdots \\ [b_l^{k-1} + \sum_{j=1}^{k-1}(-1)^j C_{k-1}^j (n_1^j + n_2^j) b_l^{(k-1)-j}]\varphi_l \\ = [k^2 b_l^{k-1} + \sum_{j=1}^{k-1}(-1)^j (k-j)^2 c_{k-1}^j (n_1^j + n_2^j) b_l^{(k-1)-j}] \frac{\varphi_l^3}{3!} - [k^4 b_l^{k-1} + \sum_{j=1}^{k-1}(-1)^j (k-j)^4 c_{k-1}^j (n_1^j + n_2^j) b_l^{(k-1)-j}] \frac{\varphi_l^5}{5!} + \cdots \end{cases}$$

(7)

Considering the infinitesimal neighborhood ($b_l e^{i\varphi_l'}$) of the m points ($b_l e^{i\varphi_l}$),which is the *m* roots of (3) in complex plane.

We suppose $\varphi_l' \neq 0$, $\varphi_l' \neq \varphi_l$ and demand $|\varphi_l' - \varphi_l|$ be small enough. Substituted $b_l e^{i\varphi_l}$ for $b_l e^{i\varphi_l'}$, the (7) are no more exactly equations. The first and second expression of (7) becomes second and third order infinitesimal quantity respectively. Thus we have:

$$\begin{cases} b_l^k + \sum_{j=1}^{k}(-1)^j c_k^j (n_1^j + n_2^j) b_l^{k-j} = o(\varphi_l'^2) \\ [b_l^{k-1} + \sum_{j=1}^{k-1}(-1)^j c_{k-1}^j (n_1^j + n_2^j) b_l^{(k-1)-j}]\varphi_l' = o(\varphi_l'^3) \end{cases} \quad l = 1,2,\cdots,m.(m \le k) \quad (8)$$

in which $o(\varphi_l'^2)$ and $o(\varphi_l'^3)$ are respectively the second and third order infinitesimal quantity.

Among the *m* roots, we suppose there are *s* roots ($s \le m$) at the real axis. For the neighborhood of the *s* points, (8)still holds. Corresponding with the neighborhood of the *s* points, we have:

$$\begin{cases} b_l^k + \sum_{j=1}^{k}(-1)^j c_k^j (n_1^j + n_2^j) b_l^{k-j} = o(\varphi_l'^2) \\ b_l^{k-1} + \sum_{j=1}^{k-1}(-1)^j c_{k-1}^j (n_1^j + n_2^j) b_l^{(k-1)-j} = o(\varphi_l'^2) \end{cases} \quad l = 1,2,\cdots,s.(s \le m). \quad (9)$$

Furthermore, limiting the neighborhood of the *s* points to the real axis, i.e. $\varphi_l' \to 0$, we obtain *s* pairs of equation from (9):

$$\begin{cases} b_l^k + \sum_{j=1}^{k} (-1)^j c_k^j (n_1^j + n_2^j) b_l^{k-j} = 0 \\ b_l^{k-1} + \sum_{j=1}^{k-1} (-1)^j c_{k-1}^j (n_1^j + n_2^j) b_l^{(k-1)-j} = 0 \end{cases} \quad l = 1,2,\cdots,s.(s \le m). \tag{10}$$

(10) shows if b_l is a positive integer, only these *s* roots of (3) at real axis can make (10) hold.

So far, we can see if (3) has positive integer number solution, the (10) must hold.

Once more, we consider equation:

$$x^{k-1} = y^{k-1} + z^{k-1} \tag{11}$$

Analogously, we set

$$\begin{cases} y = x - n_1' \\ z = x - n_2' \end{cases} \tag{12}$$

where n_1' and n_2' are some positive integers less than *x*. Substituting (12) into (11) gives:

$$x^{k-1} + \sum_{j=1}^{k-1} (-1)^j c_{k-1}^j (n_1'^j + n_2'^j) x^{(k-1)-j} = 0 \tag{13}$$

By using of the combination identity

$$C_k^j (k - j) = k C_{k-1}^j$$

We have from (13)

$$\begin{aligned} & kx^{k-1} + \sum_{j=1}^{k-1} (-1)^j k C_{k-1}^j (n_1'^j + n_2'^j) x^{(k-1)-j} \\ & = kx^{k-1} + \sum_{j=1}^{k-1} (-1)^j (k-j) C_k^j (n_1'^j + n_2'^j) x^{(k-1)-j} = 0 \end{aligned} \tag{14}$$

Now, suppose ,on the contrary, (11) has no positive integer solution,then for arbitrary positive integer *x, n'*$_1$*, n'*$_2$, we obtain from (13):

$$x^{k-1} + \sum_{j=1}^{k-1} (-1)^j C_{k-1}^j (n_1'^j + n_2'^j) x^{(k-1)-j} \neq 0 \tag{15}$$

Because n_1, n_2, n_1', n_2' are arbitrary positive integers, the second equation of (7) is of the same pattern as the equation (14). If (15) holds, the second expression of (7) will not hold. Thus, (1) has no positive integer solution for index *k* if it has no positive integer solution for index k-1 .

3. The proof for the case of k=3 in Fermat's last theorem

Lemma 2. For the equation wherein *x*, *y*, *z* are positive integers

$$x^3 = y^3 + z^3$$

There exists not a positive integer solution (x, y, z).

Proof. According to the equations (7) obtained above, it follows

$$\begin{cases} b^3-3(n_1+n_2)b^2+3(n_1^2+n_2^2)b-(n_1^3+n_2^3)=0 \\ b^2-2b(n_1+n_2)+(n_1^2+n_2^2)=0 \end{cases} \tag{16}$$

Combine two equations in (16) results in

$$(n_1+n_2)b^2-2(n_1^2+n_2^2)b+(n_1^3+n_2^3)=0$$

Hence we obtain with (16b)

$$4b=n_1+n_2 \tag{17}$$

On the other hand, (16b) is a quadratic equation, it has roots

$$b=n_1+n_2\pm\sqrt{2n_1n_2} \tag{18}$$

b,n_1,n_2 being positive, thus we have from (17) and (18)

$$9n_1^2-14n_1n_2+9n_2^2=0 \tag{19}$$

The $\triangle$=196-324=-128<0 for (19) means that there are no positive n_1, n_2, thus b, to satisfy (16) .

4. The Proof for the Fermat's Last Theorem

Theorem (*Fermat's last theorem*) For the equation wherein x, y, z, k are positive integers

$$x^k=y^k+z^k$$

$k\geq 3$. there exists not a positive integer solution (x,y,z).

Proof. Combining the results of the principle of infinite descent for the Fermat's last theorem in §2 and the case of k=3 for Fermat's last theorem in §3 obtained above, we obtain that (1) has no positive integer solution for arbitrary positive integer $k\geq 3$, *i.e.* Fermat's last theorem holds.

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