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A New Homotopy Perturbation Method for Solving Systems of Nonlinear Equations of Emden-Fowler Type

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Abstract:

In this paper, we apply the new homotopy perturbation method (NHPM) to get accurate results for solving systems of nonlinear equations of Emden–Fowler type, we indicate that our method (NHPM) is equivalent to the variational iteration method (VIM) with a specific convex. Four examples are given to illustrate our proposed methods. The method is easy to carry out and gives very accurate solutions for solving linear and nonlinear differential equations.

Keywords: New Homotopy Perturbation Method; Adomian's polynomials; Variational Iteration Method; Emden–Fowler type equations; systems of nonlinear partial differential equations.

(1) Introduction:

Since the governing equations in many experiments in technology as well as in the sciences, differential equations and integral equations. These equations are much overly complicated to be solved exactly and even if an exact result is obtained, the required calculations may be excessively perplexed. In this work a novel technique of version (HPM), called the New Homotopy Perturbation Method (NHPM) has been introduced for solving the linear and non-linear differential equations and integral equations, and it is shown that the new technique performs a good deal more serious.

Many problems in the fields of mathematical physics and astrophysics are modelled by the equation

$$y''+\frac{\alpha}{x}y'+f(x)g(y)=h(x), \quad x>0, \tag{1}$$

where $\alpha>0$ is a constant.

For $f(x)=1, g(y)=y^m$, and $h(x)=0$, Eq. (1), is the standard Lane–Emden equation that has been used to model several phenomena in mathematical physics. For the steady-state case and $\alpha=2$ and $h(x)=0$, Eq. (1), becomes the Emden–Fowler type equation

$$y''+\frac{2}{x}y'+f(x)g(y)=0, \quad x>0, \tag{2}$$

subject to

$$y(0)=1, \quad y'(0)=0. \tag{3}$$

The Emden–Fowler equation arises in the study of fluid mechanics, relativistic mechanics and in the study of chemically reacting systems. The singularity behavior that occurs at $x=0$ is the main difficulty of Eq. (1), and Eq. (2).

Very lately, many powerful methods have been introduced, such as some notes on using the variational iteration method for solving systems of equations of Emden–Fowler type [1], the new homotopy perturbation method (NHPM) was proposed by Biazar an Eslami [2], for solving two dimensional wave equation, the difference between the method (NHPM) and standard (HPM) [3-7], numerical solutions of the homotopy perturbation method [8-14], the Emden–Fowler Eq. (3), and Eq. (4) was handled by using the Adomian decomposition method and the modified decomposition method and Adomian's decomposition method [15-20], the same equation was handled by using the homotopy perturbation method and the variational iteration method (VIM) was applied to study this problem [21-24], and many others [25-29], the aim of this paper is to employ (NHPM) to obtain the exact solution of systems of nonlinear equations of Emden–Fowler type.

In this work, we study systems of nonlinear equations of Emden–Fowler type subject with initial conditions given by the form:

$$u''+\frac{\alpha}{x}u'+f(u(x),v(x))=h_1(x), \quad x>0,\ \alpha>0, \tag{4}$$

$$v''-\frac{\beta}{x}v'+g(u(x),v(x))=h_2(x), \quad x>0,\ \beta>0, \tag{5}$$

subject to

$$u(0)=v(0)=1 \quad , \quad u'(0)=v'(0)=0 \tag{6}$$

In the next section we discuss the analysis of our method (NHPM).

(2) Analysis of New Homotopy Perturbation Method (NHPM):

The main aim of this section is to know how to solve the systems of nonlinear singular equations of Edmen-Fowler type using new homotopy perturbation method (NHPM).

$$u''+\frac{\alpha}{x}u'+f(u(x),v(x))=h_1(x), \tag{7}$$

$$v''-\frac{\beta}{x}v'+g(u(x),v(x))=h_2(x), \tag{8}$$

subject to

$$u(0)=A \quad , \quad u'(0)=B, \tag{9}$$

$$v(0)=C \quad , \quad v'(0)=D, \tag{10}$$

where $f(u(x),v(x)), g(u(x),v(x)), h_1(x)$, and $h_2(x)$, are known functions, $\alpha, \beta, A, B, C,$ and D, are constants.

When $\alpha=\beta=A=C=1$ and $B=D=0$, we obtain the following systems:

$$u''+\frac{1}{x}u'+f(u(x),v(x))=h_1(x), \tag{11}$$

$$v''+\frac{1}{x}v'+g(u(x),v(x))=h_2(x), \tag{12}$$

subject to

$$u(0)=v(0)=1 \quad , \quad u'(0)=v'(0)=0. \tag{13}$$

In order to solve Eq. (11), and Eq. (12), we apply homotopy method as follow:

$$(1-p)\left(U''(x)-u_0(x)\right)+p\left(U''(x)+\frac{1}{x}U'(x)+f(U(x),V(x))-h_1(x)\right)=0, \tag{14}$$

$$(1-p)\left(V''(x)-v_0(x)\right)+p\left(V''(x)+\frac{1}{x}V'(x)+g(U(x),V(x))-h_2(x)\right)=0, \tag{15}$$

or

$$U''(x)=u_0(x)-p\left(u_0(x)+\frac{1}{x}U'(x)+f(U(x),V(x))-h_1(x)\right)=0, \tag{16}$$

$$V''(x)=v_0(x)-p\left(v_0(x)+\frac{1}{x}V'(x)+g(U(x),V(x))-h_2(x)\right)=0, \tag{17}$$

By using the inverse operator, $L^{-1}=\int_0^x\int_0^x(.)\,dxdx$, for both sides of Eq. (16), and Eq. (17), we yield

$$L^{-1}\left(U''(x)\right)=L^{-1}\left(u_0(x)\right)-pL^{-1}\left(u_0(x)+\frac{1}{x}U'(x)+f(U(x),V(x))-h_1(x)\right)=0, \tag{18}$$

$$L^{-1}\left(V''(x)\right)=L^{-1}\left(v_0(x)\right)-pL^{-1}\left(v_0(x)+\frac{1}{x}V'(x)+g(U(x),V(x))-h_2(x)\right)=0, \tag{19}$$

and then,

$$U(x)=A+x\,B+\int_0^x\int_0^x u_0(x)\,dxdx-p\int_0^x\int_0^x\left(u_0(x)+\frac{1}{x}U'(x)+f(U(x),V(x))-h_1(x)\right)dxdx, \tag{20}$$

$$V(x)=C+x\,D+\int_0^x\int_0^x v_0(x)\,dxdx-p\int_0^x\int_0^x\left(v_0(x)+\frac{1}{x}V'(x)+g(U(x),V(x))-h_2(x)\right)dxdx, \tag{21}$$

where $U(0)=u(0)$, $U'(0)=u'(0)$, and $V(0)=v(0), V'(0)=v'(0)$.

So the solution of Eq. (14-17) can be written in the following form

$$U(x)=U_0(x)+pU_1(x)+p^2U_2(x)+p^3U_3(x)+\cdots \tag{22}$$

$$V(x)=V_0(x)+pV_1(x)+p^2V_2(x)+p^3V_3(x)+\cdots \tag{23}$$

Let $p=1$, in Eq. (22), and Eq. (23), we obtain the approximate solution of Eq. (11), and Eq. (12), as follow

$$U(x) = \lim_{p\to 1} U = U_0(x) + U_1(x) + U_2(x) + U_3(x) + U_4(x) + \cdots \quad (24)$$

$$V(x) = \lim_{p\to 1} V = V_0(x) + V_1(x) + V_2(x) + V_3(x) + V_4(x) + \cdots \quad (25)$$

The convergence of the above series see [17] for more details.

Substituting Eq. (22-23), and Eq. (13), into Eq. (20), and Eq. (21), comparing coefficients of p, with the same power leads to

$$p^0 : \begin{cases} U_0(x) = A + x\,B + \int_0^x\int_0^x u_0(x)\,dxdx, \\ V_0(x) = C + x\,D + \int_0^x\int_0^x v_0(x)\,dxdx, \end{cases} \quad (26)$$

$$p^1 : \begin{cases} U_1(x) = -p\int_0^x\int_0^x \left(u_0(x) + \frac{1}{x}U_0'(x) + f\,(U_0(x), V_0(x)) - h_1(x) \right) dxdx, \\ V_1(x) = -p\int_0^x\int_0^x \left(v_0(x) + \frac{1}{x}V_0'(x) + g\,(U_0(x), V_0(x)) - h_2(x) \right) dxdx, \end{cases} \quad (27)$$

$$p^2 : \begin{cases} U_2(x) = -p\int_0^x\int_0^x \left(\frac{1}{x}U_1'(x) + f\,(U_1(x), V_1(x)) \right) dxdx, \\ V_2(x) = -p\int_0^x\int_0^x \left(\frac{1}{x}V_1'(x) + g\,(U_1(x), V_1(x)) \right) dxdx, \end{cases} \quad (28)$$

$$\vdots$$

In general we have

$$u_0(x) = \sum_{n=0}^{\infty} \alpha_n x^n,\ n \ge 0, \qquad U(x) = u(x), \qquad U'(x) = u'(x), \quad (29)$$

$$v_0(x) = \sum_{n=0}^{\infty} \beta_n x^n,\ n \ge 0, \qquad V(x) = v(x), \qquad V'(x) = v'(x), \quad (30)$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots$ and $\beta_0, \beta_1, \beta_2, \ldots$ are unknown coefficients which should be determined.

$f\,(u(x), v(x))$, and $g\,(u(x), v(x))$, are nonlinear operator. We assume that the nonlinear function are defined by infinite series of polynomials

$$f\,(u(x), v(x)), = \sum_{n=0}^{\infty} A_n\,(y_0, y_1, y_2, \ldots, y_n),\ \ g\,(u(x), v(x)), = \sum_{n=0}^{\infty} B_n\,(y_0, y_1, y_2, \ldots, y_n), \quad (31)$$

where A_n and B_n are called the Adomain's polynomials, defined by:

$$A_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^2}\left[f\left(\sum_{n=0}^{\infty} \lambda^n y_n \right) \right] \right)_{\lambda=0}, B_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^2}\left[g\left(\sum_{n=0}^{\infty} \lambda^n y_n \right) \right] \right)_{\lambda=0}, \quad (32)$$

To show the capability of the method (NHPM) applied to some examples in the next section.

(3) Applications of the method (NHPM):

In this section, to illustrate the method and to show the ability of the method four examples are presented.

Example (3.1): Consider the systems of nonlinear equations of Emden-Fowler type see [1],

$$u''(x)+\frac{1}{x}u'(x)+u^2(x)v(x)-\left(4x^2+5\right)u(x)=0, \tag{33}$$

$$v''(x)+\frac{2}{x}v'(x)+u(x)v^2(x)-\left(4x^2-5\right)v(x)=0, \tag{34}$$

subject to

$$u(0)=1, \quad u'(0)=0, \tag{35}$$

$$v(0)=1, \quad v'(0)=0, \tag{36}$$

To solve Eq. (33), and Eq. (34), by the method (NHPM), we construct the following homotopy

$$U''(x)=u_0(x)-p\left(u_0(x)+U''(x)+\frac{1}{x}U'(x)+U^2(x)V(x)-\left(4x^2+5\right)U(x)\right), \tag{37}$$

$$V''(x)=v_0(x)-p\left(v_0(x)+V''(x)+\frac{2}{x}V'(x)+U(x)V^2(x)-\left(4x^2-5\right)V(x)\right), \tag{38}$$

By taking the inverse operator, $L^{-1}=\int_0^x\int_0^x(.)dxdx$ to the both sides of the Eq. (37), and Eq. (38), we obtain

$$\begin{aligned}U(x)&=U(0)+xU'(0)+\int_0^x\int_0^x u_0(x)dxdx\\&\quad-p\int_0^x\int_0^x\left(u_0(x)+\frac{1}{x}U'(x)+U^2(x)V(x)-\left(4x^2+5\right)U(x)\right)dxdx,\end{aligned} \tag{39}$$

$$\begin{aligned}V(x)&=V(0)+xV'(0)+\int_0^x\int_0^x v_0(x)dxdx\\&\quad-p\int_0^x\int_0^x\left(v_0(x)+\frac{2}{x}V'(x)+U(x)V^2(x)-\left(4x^2-5\right)V(x)\right)dxdx,\end{aligned} \tag{40}$$

Substituting Eq. (22-23), into Eq. (39-40), and comparing coefficients of terms with identical powers of p, we get

$$p^0:\begin{cases}U_0(x)=U(0)+xU'(0)+\int_0^x\int_0^x u_0(x)dxdx\\[2ex] V_0(x)=V(0)+xV'(0)+\int_0^x\int_0^x v_0(x)dxdx\end{cases} \tag{41}$$

$$p^1:\begin{cases}U_1(x)=-\int_0^x\int_0^x\left(u_0(x)+\frac{1}{x}U_0'(x)+\left(U_0(x)\right)^2V_0(x)-\left(4x^2+5\right)U_0(x)\right)dxdx\\[2ex] V_1(x)=-\int_0^x\int_0^x\left(v_0(x)+\frac{2}{x}V_0'(x)+U_0(x)\left(V_0(x)\right)^2-\left(4x^2-5\right)V_0(x)\right)dxdx\end{cases} \tag{42}$$

$$p^2:\begin{cases} U_2(x)=-\int_0^x\int_0^x\left(\frac{1}{x}U_1'(x)+(U_0(x))^2V_1(x)+(U_1(x))^2V_0(x)-(4x^2+5)U_1(x)\right)dxdx \\ V_2(x)=-\int_0^x\int_0^x\left(\frac{2}{x}V_1'(x)+U_0(x)(V_1(x))^2+U_1(x)(V_0(x))^2-(4x^2-5)V_1(x)\right)dxdx \end{cases} \quad (43)$$

$$p^3:\begin{cases} U_3(x)=-\int_0^x\int_0^x\begin{pmatrix}\frac{1}{x}U_2'(x)+(U_0(x))^2V_2(x)+(U_2(x))^2V_0(x)\\+(U_1(x))^2V_1(x)-(4x^2+5)U_2(x)\end{pmatrix}dxdx \\ V_3(x)=-\int_0^x\int_0^x\begin{pmatrix}\frac{2}{x}V_2'(x)+U_0(x)(V_2(x))^2+U_2(x)(V_0(x))^2\\+U_1(x)(V_1(x))^2-(4x^2-5)V_2(x)\end{pmatrix}dxdx \end{cases} \quad (44)$$

$$p^{k+1}:\begin{cases} U_{k+1}(x)=-\int_0^x\int_0^x\left(\frac{1}{x}U_k'(x)+(U_k(x))^2V_k(x)-(4x^2+5)U_k(x)\right)dxdx, \quad k=0,1,2,\dots \\ V_{k+1}(x)=-\int_0^x\int_0^x\left(\frac{2}{x}V_k'(x)+U_k(x)(V_k(x))^2-(4x^2-5)V_k(x)\right)dxdx, \quad k=0,1,2,\dots \end{cases}$$

According to Eq. (31), and Eq. (32), we can gives the first few Adomian's polynomials for the nonlinear terms $U^2(x)V(x)$, in Eq. (41), and $U(x)V^2(x)$, in Eq. (42), respectively

$$\begin{aligned} A_0&=U_0^2V_0\\ A_1&=U_0^2V_1+2U_0V_0U_1\\ A_2&=U_0^2V_2+2U_0U_1V_1+2U_0V_0U_2+U_1^2V_0\\ A_3&=U_0^2V_3+2U_0U_1V_2+2U_1U_2V_0+U_1^2V_1+2U_0U_2V_1+2U_0U_3V_0\\ &\vdots \end{aligned} \quad (45)$$

and

$$\begin{aligned} B_0&=V_0^2U_0\\ B_1&=V_0^2U_1+2V_0U_0V_1\\ B_2&=V_0^2U_2+2V_0V_1U_1+2V_0U_0V_2+V_1^2U_0\\ B_3&=V_0^2U_3+2V_0V_1U_2+2V_1V_2U_0+V_1^2U_1+2V_0V_2U_1+2V_0V_3U_0\\ &\vdots \end{aligned} \quad (46)$$

To solve Eq. (41) and Eq. (42), for $U_0(x), V_0(x)$, and $U_1(x), V_1(x)$, we get

$$U_0(x)\simeq 1+\frac{1}{2}\alpha_0x^2+\frac{1}{6}\alpha_1x^3+\frac{1}{12}\alpha_2x^4+\frac{1}{20}\alpha_3x^5+\frac{1}{30}\alpha_4x^6+\frac{1}{42}\alpha_5x^7+\frac{1}{56}\alpha_6x^8+\frac{1}{72}\alpha_7x^9+\cdots$$
$$V_0(x)\simeq 1+\frac{1}{2}\beta_0x^2+\frac{1}{6}\beta_1x^3+\frac{1}{12}\beta_2x^4+\frac{1}{20}\beta_3x^5+\frac{1}{30}\beta_4x^6+\frac{1}{42}\beta_5x^7+\frac{1}{56}\beta_6x^8+\frac{1}{72}\beta_7x^9+\cdots$$

and

$$U_1(x)=\left(-\frac{1}{2}\alpha_0-\frac{1}{2}\alpha_0-\frac{1}{2}+\frac{5}{2}\right)x^2+\left(-\frac{1}{6}\alpha_1-\frac{1}{12}\alpha_1\right)x^3$$
$$+\left(-\frac{1}{12}\alpha_2-\frac{1}{36}\alpha_2-\frac{1}{12}\alpha_0-\frac{1}{24}\beta_0+\frac{1}{3}-\frac{5}{24}\alpha_0\right)x^4$$
$$+\left(-\frac{1}{20}\alpha_3-\frac{1}{80}\alpha_3-\frac{1}{60}\alpha_1-\frac{1}{120}\beta_1+\frac{1}{24}\alpha_1\right)x^5$$
$$+\left(-\frac{1}{30}\alpha_4-\frac{1}{150}\alpha_4-\frac{1}{120}\alpha_0^2-\frac{1}{60}\alpha_0\beta_0-\frac{1}{180}\alpha_2-\frac{1}{360}\beta_2+\frac{1}{15}\alpha_0+\frac{1}{72}\alpha_2\right)x^6$$
$$+\left(-\frac{1}{42}\alpha_5-\frac{1}{252}\alpha_5-\frac{1}{252}\alpha_0\alpha_1-\frac{1}{252}\alpha_1\beta_0-\frac{1}{252}\alpha_0\beta_1+\frac{1}{63}\alpha_1+\frac{1}{168}\alpha_3\right)x^7$$
$$+\left(\begin{array}{l}-\frac{1}{56}\alpha_6-\frac{1}{392}\alpha_6+\frac{1}{168}\alpha_2+\frac{1}{336}\alpha_4-\frac{1}{448}\alpha_0{}^2\beta_0-\frac{1}{672}\alpha_0\beta_2-\frac{1}{672}\alpha_2\beta_0\\-\frac{1}{1008}\alpha_1\beta_1-\frac{1}{672}\alpha_0\alpha_2-\frac{1}{2016}\alpha_1^2\end{array}\right)x^8$$
$$+\left(\begin{array}{l}-\frac{1}{72}\alpha_7-\frac{1}{576}\alpha_7+\frac{1}{360}\alpha_3+\frac{5}{3024}\alpha_5-\frac{1}{1440}\alpha_0\beta_3-\frac{1}{2592}\alpha_1\beta_2-\frac{1}{1440}\alpha_3\beta_1\\-\frac{1}{2592}\alpha_2\beta_1-\frac{1}{1440}\alpha_0\alpha_3-\frac{1}{2592}\alpha_1\alpha_2\end{array}\right)x^9$$
$$+\cdots=0$$

and

$$V_1(x)=\left(-\frac{1}{2}\beta_0-\beta_0-\frac{1}{2}-\frac{5}{2}\right)x^2+\left(-\frac{1}{6}\beta_1-\frac{1}{6}\beta_1\right)x^3$$
$$+\left(-\frac{1}{12}\beta_2-\frac{1}{18}\beta_2-\frac{1}{12}\beta_0-\frac{1}{24}\alpha_0+\frac{1}{3}-\frac{5}{24}\beta_0\right)x^4$$
$$+\left(-\frac{1}{20}\beta_3-\frac{1}{40}\beta_3-\frac{1}{60}\beta_1-\frac{1}{24}\beta_1\right)x^5$$
$$+\left(-\frac{1}{30}\beta_4-\frac{1}{75}\beta_4-\frac{1}{72}\beta_2--\frac{1}{60}\alpha_0\beta_0-\frac{1}{120}\beta_0^2+\frac{1}{15}\beta_0-\frac{1}{50}\beta_0-\frac{1}{360}\alpha_2-\frac{1}{180}\beta_2\right)x^6$$
$$+\left(\begin{array}{l}-\frac{1}{42}\beta_5-\frac{1}{126}\beta_5-\frac{1}{252}\beta_0\beta_1-\frac{1}{252}\alpha_1\beta_0-\frac{1}{252}\alpha_0\beta_1-\frac{1}{420}\beta_3\\-\frac{1}{840}\alpha_3+\frac{1}{63}\beta_1-\frac{1}{168}\beta_3\end{array}\right)x^7$$
$$+\left(\begin{array}{l}-\frac{1}{56}\beta_6-\frac{1}{196}\beta_6+\frac{1}{168}\beta_2-\frac{1}{336}\beta_4-\frac{1}{672}\beta_0\alpha_2-\frac{1}{672}\beta_2\alpha_0\\-\frac{1}{2016}\beta_1\alpha_1-\frac{1}{672}\beta_0\beta_2-\frac{1}{2016}\beta_1^2-\frac{1}{1680}\alpha_4-\frac{1}{840}\beta_4-\frac{1}{448}\beta_0^2\alpha_0\end{array}\right)x^8$$
$$+\left(\begin{array}{l}-\frac{1}{72}\beta_7-\frac{1}{288}\beta_7-\frac{1}{360}\beta_3-\frac{5}{3024}\beta_5-\frac{1}{1440}\beta_0\alpha_3-\frac{1}{2592}\beta_1\alpha_2\\-\frac{1}{1440}\beta_3\alpha_1-\frac{1}{2592}\beta_2\alpha_1-\frac{1}{1440}\beta_0\beta_3-\frac{1}{2592}\beta_1\beta_2\end{array}\right)x^9$$
$$+\cdots=0$$

Considering the hypothesis of $U_1(x)=0, V_1(x)=0,$ the coefficients of α_n, β_n $, n=1,2,3,\ldots$ will be obtained as the following

$$\alpha_0 = 2, \alpha_1 = 0, \alpha_2 = 6, \alpha_3 = 0, \alpha_4 = 5, \alpha_5 = 0, \alpha_6 = \frac{7}{3}, \alpha_7 = 0, \alpha_8 = \frac{3}{4}, \alpha_9 = 0, \ldots$$

$$\beta_0 = -2, \beta_1 = 0, \beta_2 = 6, \beta_3 = 0, \beta_4 = -5, \beta_5 = 0, \beta_6 = \frac{7}{3}, \beta_7 = 0, \beta_8 = -\frac{3}{4}, \beta_9 = 0, \ldots$$

Hence, the approximate series solution is

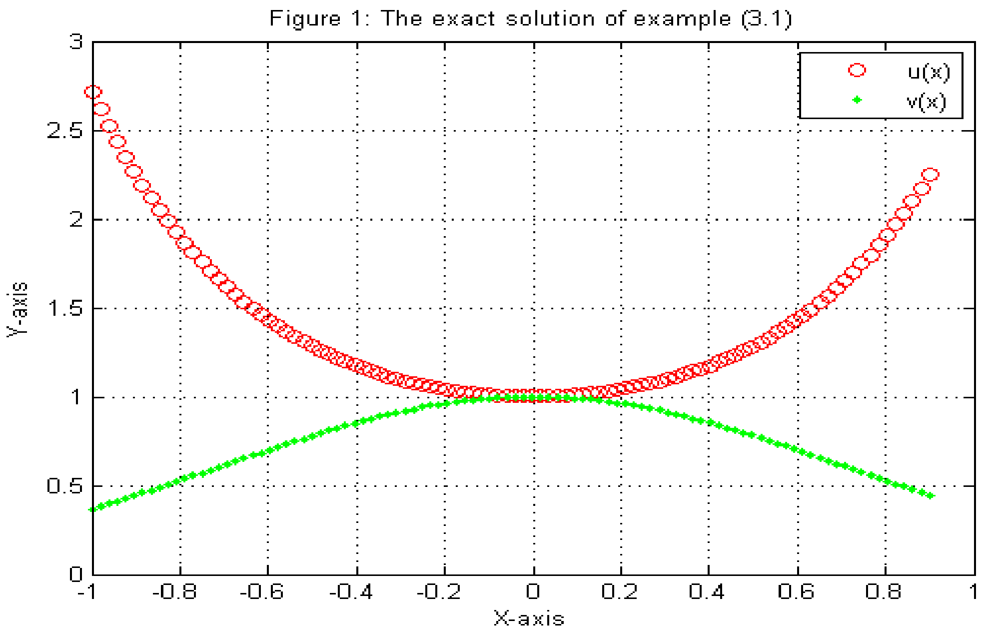
$$U_0(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \frac{1}{5!}x^{10} + \cdots$$

$$V_0(x) = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \frac{1}{5!}x^{10} + \cdots$$

Therefore, the exact solution of Eq. (33) and Eq. (34) becomes as:

$$u(x) = e^{x^2} \quad , \quad v(x) = e^{-x^2},$$

which is the same as the closed form solution obtained by Abdul-Majid Wazwaz [1] using (VIM).



Example (3.2): Consider the systems of nonlinear equations of Emden-Fowler type see [1],

$$u''(x) + \frac{2}{x}u'(x) + v^2(x) - u^2(x) + 6v(x) = 6 + 6x^2, \tag{47}$$

$$v''(x) + \frac{2}{x}v'(x) + u^2(x) - v^2(x) - 6v(x) = 6 - 6x^2, \tag{48}$$

subject to

$$u(0) = 1, \quad u'(0) = 0, \tag{49}$$

$$v(0) = -1, \quad v'(0) = 0, \tag{50}$$

To solve Eq. (47), and Eq. (48), by the method (NHPM), we construct the following homotopy

$$U''(x) = u_0(x) - p\left(u_0(x) + U''(x) + \frac{2}{x}U'(x) + V^2(x) - U^2(x) + 6V(x) - 6 - 6x^2\right), \qquad (51)$$

$$V''(x) = v_0(x) - p\left(v_0(x) + V''(x) + \frac{2}{x}V'(x) + U^2(x) - V^2(x) - 6V(x) - 6 + 6x^2\right), \qquad (52)$$

By taking the inverse operator, $L^{-1} = \int_0^x\int_0^x (.)\,dxdx$ to the both sides of the Eq. (51), and Eq. (52), we have

$$\begin{aligned} U(x) &= U(0) + xU'(0) + \int_0^x\int_0^x u_0(x)\,dxdx \\ &\quad - p\int_0^x\int_0^x\left(u_0(x) + \frac{2}{x}U'(x) + V^2(x) - U^2(x) + 6V(x) - 6 - 6x^2\right)dxdx, \end{aligned} \qquad (53)$$

$$\begin{aligned} V(x) &= V(0) + xV'(0) + \int_0^x\int_0^x v_0(x)\,dxdx \\ &\quad - p\int_0^x\int_0^x\left(v_0(x) + \frac{2}{x}V'(x) + U^2(x) - V^2(x) - 6V(x) - 6 + 6x^2\right)dxdx, \end{aligned} \qquad (54)$$

Substituting Eq. (22-23), into Eq. (53-54), and comparing coefficients of terms with identical powers of p, yields to

$$p^0 : \begin{cases} U_0(x) = U(0) + xU'(0) + \int_0^x\int_0^x u_0(x)\,dxdx \\ V_0(x) = V(0) + xV'(0) + \int_0^x\int_0^x v_0(x)\,dxdx \end{cases} \qquad (55)$$

$$p^1 : \begin{cases} U_1(x) = -\int_0^x\int_0^x\left(u_0(x) + \frac{2}{x}U_0'(x) + V_0^2(x) - U_0^2(x) + 6V_0(x) - 6 - 6x^2\right)dxdx \\ V_1(x) = -\int_0^x\int_0^x\left(v_0(x) + \frac{2}{x}V_0'(x) + U_0^2(x) - V_0^2(x) - 6V_0(x) - 6 + 6x^2\right)dxdx \end{cases} \qquad (56)$$

$$p^2 : \begin{cases} U_2(x) = -\int_0^x\int_0^x\left(\frac{2}{x}U_1'(x) + V_1^2(x) - U_1^2(x) + 6V_1(x)\right)dxdx \\ V_2(x) = -\int_0^x\int_0^x\left(\frac{2}{x}V_1'(x) + U_1^2(x) - V_1^2(x) - 6V_1(x)\right)dxdx \end{cases} \qquad (57)$$

$$p^3 : \begin{cases} U_3(x) = -\int_0^x\int_0^x\left(\frac{2}{x}U_2'(x) + V_2^2(x) - U_2^2(x) + 6V_2(x)\right)dxdx \\ V_3(x) = -\int_0^x\int_0^x\left(\frac{2}{x}V_2'(x) + U_2^2(x) - V_2^2(x) - 6V_2(x)\right)dxdx \end{cases} \qquad (58)$$

$$p^{k+1}:\begin{cases} U_{k+1}(x) = -\int_0^x\int_0^x \left(\frac{2}{x}U_k'(x) + V_k^2(x) - U_k^2(x) + 6V_k(x)\right)dxdx, & k = 0,1,2,\dots \\ V_{k+1}(x) = -\int_0^x\int_0^x \left(\frac{2}{x}V_k'(x) + U_k^2(x) - V_k^2(x) - 6V_k(x)\right)dxdx, & k = 0,1,2,\dots \end{cases}$$

According to Eq. (31), and Eq. (32), we can gives the first few Adomian's polynomials for the nonlinear terms $V^2(x)$, in Eq. (47), and $U^2(x)$, in Eq. (48), respectively

$$\begin{aligned} A_0 &= V_0^2, \\ A_1 &= 2V_0V_1, \\ A_2 &= 2V_0V_2 + V_1^2, \\ A_3 &= 2V_0V_3 + 2V_1V_2, \\ A_4 &= 2V_0V_4 + 2V_1V_3 + V_2^2, \\ A_5 &= 2V_0V_5 + 2V_1V_4 + 2V_2V_3, \\ &\vdots \end{aligned} \tag{59}$$

Thus

$$\begin{aligned} B_0 &= U_0^2, \\ B_1 &= 2U_0U_1, \\ B_2 &= 2U_0U_2 + U_1^2, \\ B_3 &= 2U_0U_3 + 2U_1U_2, \\ B_4 &= 2U_0U_4 + 2U_1U_3 + U_2^2, \\ B_5 &= 2U_0U_5 + 2U_1U_4 + 2U_2U_3, \\ &\vdots \end{aligned} \tag{60}$$

To solve Eq. (56) and Eq. (57) for $U_0(x), V_0(x)$, and $U_1(x), V_1(x)$, we obtain

$$U_0(x) \simeq 1 + \frac{1}{2}\alpha_0 x^2 + \frac{1}{6}\alpha_1 x^3 + \frac{1}{12}\alpha_2 x^4 + \frac{1}{20}\alpha_3 x^5 + \frac{1}{30}\alpha_4 x^6 + \frac{1}{42}\alpha_5 x^7 + \frac{1}{56}\alpha_6 x^8 + \frac{1}{72}\alpha_7 x^9 + \cdots$$

$$V_0(x) \simeq -1 + \frac{1}{2}\beta_0 x^2 + \frac{1}{6}\beta_1 x^3 + \frac{1}{12}\beta_2 x^4 + \frac{1}{20}\beta_3 x^5 + \frac{1}{30}\beta_4 x^6 + \frac{1}{42}\beta_5 x^7 + \frac{1}{56}\beta_6 x^8 + \frac{1}{72}\beta_7 x^9 + \cdots$$

and

$$\begin{aligned} U_1(x) = &\left(-\frac{1}{2}\alpha_0 - \alpha_0 - \frac{1}{2} + 3 + \frac{1}{2} + 3\right)x^2 + \left(-\frac{1}{6}\alpha_1 - \frac{1}{6}\alpha_1\right)x^3 \\ &+ \left(-\frac{1}{12}\alpha_2 - \frac{1}{18}\alpha_2 + \frac{1}{12}\beta_0 + \frac{1}{12}\alpha_0 - \frac{1}{4}\beta_0 + \frac{1}{2}\right)x^4 \end{aligned}$$

$$+\left(-\frac{1}{20}\alpha_3-\frac{1}{40}\alpha_3+\frac{1}{60}\beta_1+\frac{1}{60}\alpha_1-\frac{1}{20}\beta_1\right)x^5$$
$$+\left(-\frac{1}{30}\alpha_4-\frac{1}{75}\alpha_4-\frac{1}{120}\beta_0^{\ 2}+\frac{1}{180}\beta_2+\frac{1}{120}\alpha_0^{\ 2}+\frac{1}{180}\alpha_2-\frac{1}{60}\beta_2\right)x^6$$
$$+\left(-\frac{1}{42}\alpha_5-\frac{1}{126}\alpha_5-\frac{1}{252}\beta_0\beta_1+\frac{1}{420}\beta_3+\frac{1}{252}\alpha_0\alpha_1+\frac{1}{420}\alpha_3-\frac{1}{140}\beta_3\right)x^7$$
$$+\left(\begin{array}{l}-\frac{1}{56}\alpha_6-\frac{1}{196}\alpha_6-\frac{1}{672}\beta_0\beta_2-\frac{1}{2016}\beta_1^{\ 2}+\frac{1}{840}\beta_4\\+\frac{1}{672}\alpha_0\alpha_2+\frac{1}{2016}\alpha_1^{\ 2}+\frac{1}{840}\alpha_4-\frac{1}{280}\beta_4\end{array}\right)x^8$$
$$+\left(\begin{array}{l}-\frac{1}{72}\alpha_7-\frac{1}{288}\alpha_7-\frac{1}{1440}\beta_0\beta_3-\frac{1}{2592}\beta_1\beta_2+\frac{1}{1440}\alpha_0\alpha_3\\+\frac{1}{2592}\alpha_1\alpha_2-\frac{1}{504}\beta_5-\frac{1}{1512}\beta_5+\frac{1}{1512}\alpha_5\end{array}\right)x^9$$
$$+\left(\begin{array}{l}-\frac{1}{90}\alpha_8-\frac{1}{405}\alpha_8-\frac{1}{2700}\beta_0\beta_4-\frac{1}{5400}\beta_1\beta_3-\frac{1}{12960}\beta_2^{\ 2}\\+\frac{1}{2700}\alpha_0\alpha_4+\frac{1}{5400}\alpha_1\alpha_3+\frac{1}{12960}\alpha_2^{\ 2}-\frac{1}{840}\beta_6\end{array}\right)x^{10}$$
$$+\cdots=0$$

and

$$V_1(x)=\left(-\frac{1}{2}\beta_0-\beta_0-\frac{1}{2}+\frac{1}{2}-3+3\right)x^2$$
$$+\left(-\frac{1}{6}\beta_1-\frac{1}{6}\beta_1\right)x^3$$
$$+\left(-\frac{1}{12}\beta_2-\frac{1}{18}\beta_2-\frac{1}{12}\alpha_0-\frac{1}{12}\beta_0+\frac{1}{4}\beta_0-\frac{1}{2}\right)x^4$$
$$+\left(-\frac{1}{20}\beta_3-\frac{1}{40}\beta_3+\frac{1}{60}\beta_1-\frac{1}{60}\alpha_1-\frac{1}{60}\beta_1+\frac{1}{20}\beta_1\right)x^5$$
$$+\left(-\frac{1}{30}\beta_4-\frac{1}{75}\beta_4-\frac{1}{120}\alpha_0^{\ 2}+\frac{1}{120}\beta_0^{\ 2}-\frac{1}{180}\beta_2+\frac{1}{60}\beta_2-\frac{1}{180}\alpha_2\right)x^6$$
$$+\left(-\frac{1}{42}\beta_5-\frac{1}{126}\beta_5-\frac{1}{252}\alpha_0\alpha_1-\frac{1}{420}\alpha_3+\frac{1}{252}\beta_0\beta_1-\frac{1}{420}\beta_3+\frac{1}{140}\beta_3\right)x^7$$
$$+\left(\begin{array}{l}-\frac{1}{56}\beta_6-\frac{1}{196}\beta_6-\frac{1}{672}\alpha_0\alpha_2-\frac{1}{2016}\alpha_1^{\ 2}-\frac{1}{840}\alpha_4\\+\frac{1}{672}\beta_0\beta_2+\frac{1}{2016}\beta_1^{\ 2}-\frac{1}{840}\beta_4+\frac{1}{280}\beta_4\end{array}\right)x^8$$
$$+\left(\begin{array}{l}-\frac{1}{72}\beta_7-\frac{1}{288}\beta_7-\frac{1}{1440}\alpha_0\alpha_3-\frac{1}{2592}\alpha_1\alpha_2+\frac{1}{1440}\beta_0\beta_3\\+\frac{1}{2592}\beta_1\beta_2+\frac{1}{504}\beta_5-\frac{1}{1512}\alpha_5+\frac{1}{1512}\beta_5\end{array}\right)x^9$$
$$+\left(\begin{array}{l}-\frac{1}{90}\beta_8-\frac{1}{405}\beta_8+\frac{1}{2700}\beta_0\beta_4+\frac{1}{5400}\beta_1\beta_3+\frac{1}{12960}\beta_2^{\ 2}\\-\frac{1}{2700}\alpha_0\alpha_4-\frac{1}{5400}\alpha_1\alpha_3-\frac{1}{12960}\alpha_2^{\ 2}+\frac{1}{840}\beta_6\end{array}\right)x^{10}$$
$$+\cdots=0$$

Considering the hypothesis of $U_1(x)=0, V_1(x)=0,$ the coefficients of $\alpha_n, \beta_n, n=1,2,3,\ldots$ will be obtained as the following

$$\alpha_0=4, \alpha_1=0, \alpha_2=6, \alpha_3=0, \alpha_4=5, \alpha_5=0, \alpha_6=\frac{7}{3}, \alpha_7=0, \alpha_8=\frac{3}{4}, \alpha_9=0, \cdots$$
$$\beta_0=0, \beta_1=0, \beta_2=-6, \beta_3=0, \beta_4=-5, \beta_5=0, \beta_6=-\frac{7}{3}, \beta_7=0, \beta_8=-\frac{3}{4}, \beta_9=0, \cdots$$

Hence, the approximate series solution is

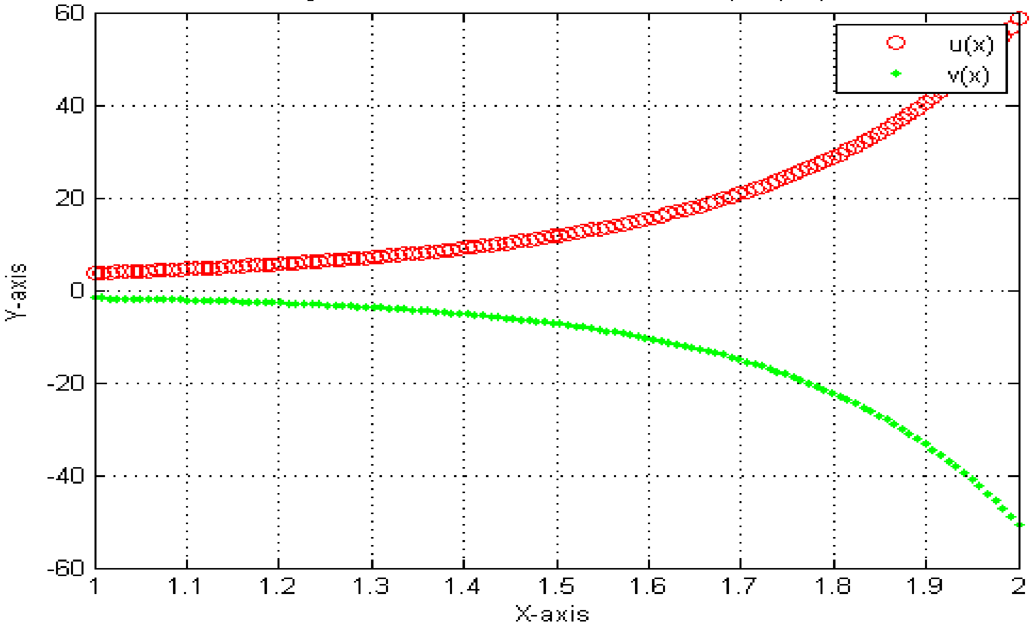
$$U_0(x)\simeq 1+2x^2+\frac{1}{2!}x^4+\frac{1}{3!}x^6+\frac{1}{4!}x^8+\frac{1}{5!}x^{10}+\cdots$$
$$V_0(x)\simeq -1-\frac{1}{2!}x^4-\frac{1}{3!}x^6+\frac{1}{4!}x^8-\frac{1}{5!}x^{10}+\cdots$$

Therefore, the exact solution of Eq. (47), and Eq. (48), becomes as:

$$u(x)=x^2+e^{x^2} \quad , \quad v(x)=x^2-e^{-x^2},$$

which is the same as the closed form solution obtained by Abdul-Majid Wazwaz [1] using (VIM).

Figure 2: The exact solution of example (3.2)



Example (3.3): Consider the systems of nonlinear equations of Emden-Fowler type see [1],

$$u''(x)+\frac{3}{x}u'(x)+u(x)v(x)+\left(x^2+4\right)u^5(x)=1, \quad (61)$$
$$v''(x)+\frac{4}{x}v'(x)+u(x)v(x)-\left(4x^2+5\right)v^3(x)=1, \quad (62)$$

subject to

$$u(0)=1, \quad u'(0)=0, \quad (63)$$
$$v(0)=1, \quad v'(0)=0, \quad (64)$$

To solve Eq. (61), and Eq. (62), by the method (NHPM), we construct the following homotopy

$$U''(x) = u_0(x) - p\left(u_0(x) + U''(x) + \frac{3}{x}U'(x) + U(x)V(x) + \left(x^2+4\right)U^5(x) - 1\right), \quad (65)$$

$$V''(x) = v_0(x) - p\left(v_0(x) + V''(x) + \frac{4}{x}V'(x) + U(x)V(x) - \left(4x^2+5\right)V^3(x) - 1\right), \quad (66)$$

By taking the inverse operator, $L^{-1} = \int_0^x \int_0^x (.)\,dxdx$ to the both sides of the Eq. (65), and Eq. (66), we obtain

$$U(x) = U(0) + xU'(0) + \int_0^x \int_0^x u_0(x)\,dxdx$$
$$- p\int_0^x \int_0^x \left(u_0(x) + \frac{3}{x}U'(x) + U(x)V(x) + \left(x^2+4\right)U^5(x) - 1\right)dxdx, \quad (67)$$

$$V(x) = V(0) + xV'(0) + \int_0^x \int_0^x v_0(x)\,dxdx$$
$$- p\int_0^x \int_0^x \left(v_0(x) + \frac{4}{x}V'(x) + U(x)V(x) - \left(4x^2+5\right)V^3(x) - 1\right)dxdx, \quad (68)$$

Substituting Eq. (22-23), into Eq. (67), and Eq. (68), and comparing coefficients of terms with identical powers of p, gives

$$P^0 : \begin{cases} U_0(x) = U(0) + xU'(0) + \int_0^x \int_0^x u_0(x)\,dxdx \\ V_0(x) = V(0) + xV'(0) + \int_0^x \int_0^x v_0(x)\,dxdx \end{cases} \quad (69)$$

$$P^1 : \begin{cases} U_1(x) = -\int_0^x \int_0^x \left(u_0(x) + \frac{3}{x}U_0'(x) + U_0(x)V_0(x) + \left(x^2+4\right)U_0^5(x) - 1\right)dxdx \\ V_1(x) = -\int_0^x \int_0^x \left(v_0(x) + \frac{4}{x}V_0'(x) + U_0(x)V_0(x) - \left(4x^2+5\right)V_0^3(x) - 1\right)dxdx \end{cases} \quad (70)$$

$$P^2 : \begin{cases} U_2(x) = -\int_0^x \int_0^x \left(\frac{3}{x}U_1'(x) + U_0(x)V_1(x) + U_1(x)V_0(x) + \left(x^2+4\right)U_1^5(x)\right)dxdx \\ V_1(x) = -\int_0^x \int_0^x \left(\frac{4}{x}V_1'(x) + U_0(x)V_1(x) + U_1(x)V_0(x) - \left(4x^2+5\right)V_1^3(x)\right)dxdx \end{cases} \quad (71)$$

$\vdots$

According to Eq. (31), and Eq. (32), we can gives the first few Adomian's polynomials for the nonlinear terms $U(x)V(x)$, and $u^5(x)$, in Eq. (61), and $V^3(x)$, in Eq. (62), respectively

$$\begin{aligned}
A_0 &= U_0V_0, \\
A_1 &= U_0V_1 + U_1V_0, \\
A_2 &= U_0V_2 + U_2V_0 + U_1V_1, \\
A_3 &= U_0V_3 + U_3V_0 + U_1V_2 + U_2V_1, \\
A_4 &= U_0V_4 + U_4V_0 + U_1V_3 + U_3V_1 + U_2V_2, \\
A_5 &= U_0V_5 + U_5V_0 + U_1V_4 + U_4V_1 + U_2V_3 + U_3V_2, \\
&\vdots
\end{aligned} \tag{72}$$

while,

$$\begin{aligned}
B_0 &= U_0^5, \\
B_1 &= 5U_0^4U_1, \\
B_2 &= 5U_0^4U_2 + 10U_1^2U_0^3, \\
B_3 &= 5U_0^4U_3 + 20U_0^3U_1U_2 + 10U_0^2U_1^3, \\
B_4 &= 5U_0^4U_4 + 10U_0^3U_2^2 + 20U_0^3U_1U_3 + 30U_0^2U_1^2U_2 + 5U_0U_1^4, \\
B_5 &= 5U_0^4U_5 + 20U_0^3U_1U_4 + 20U_0^3U_2U_3 + 30U_0^2U_2^2U_1 + 30U_0^2U_1^2U_3 + 20U_0U_1^3U_2 + U_1^5, \\
&\vdots
\end{aligned} \tag{73}$$

and

$$\begin{aligned}
C_0 &= V_0^3, \\
C_1 &= 3V_0^2V_1, \\
C_2 &= 3V_0^2V_2 + 3V_1^2V_0, \\
C_3 &= 3V_0^2V_3 + 6V_0V_1V_2 + V_1^3, \\
C_4 &= 3V_0^2V_4 + 3V_1^2V_2 + 3V_2^2V_0 + 6V_0V_1V_3, \\
C_5 &= 3V_0^2V_5 + 6V_0V_1V_4 + 3V_1^2V_3 + 3V_2^2V_1 + 6V_0V_2V_3, \\
&\vdots
\end{aligned} \tag{74}$$

To solve Eq. (69), and Eq. (70), for $U_0(x), V_0(x)$, and $U_1(x), V_1(x)$, we get

$$U_0(x) = 1 + \frac{1}{2}\alpha_0 x^2 + \frac{1}{6}\alpha_1 x^3 + \frac{1}{12}\alpha_2 x^4 + \frac{1}{20}\alpha_3 x^5 + \frac{1}{30}\alpha_4 x^6 + \frac{1}{42}\alpha_5 x^7 + \frac{1}{56}\alpha_6 x^8 + \frac{1}{72}\alpha_7 x^9 + \cdots$$
$$V_0(x) = 1 + \frac{1}{2}\beta_0 x^2 + \frac{1}{6}\beta_1 x^3 + \frac{1}{12}\beta_2 x^4 + \frac{1}{20}\beta_3 x^5 + \frac{1}{30}\beta_4 x^6 + \frac{1}{42}\beta_5 x^7 + \frac{1}{56}\beta_6 x^8 + \frac{1}{72}\beta_7 x^9 + \cdots$$

and

$$\begin{aligned}
U_1(x) = &\left(-\frac{1}{2}\alpha_0 - \frac{3}{2}\alpha_0 - 2 + \frac{1}{2} - \frac{1}{2}\right)x^2 \\
&+ \left(-\frac{1}{6}\alpha_1 - \frac{1}{4}\alpha_1\right)x^3 + \left(-\frac{1}{12}\alpha_2 - \frac{1}{12}\alpha_2 - \frac{1}{24}\beta_0 - \frac{1}{24}\alpha_0 - \frac{5}{6}\alpha_0 - \frac{1}{12}\right)x^4 \\
&+ \left(-\frac{1}{20}\alpha_3 - \frac{3}{80}\alpha_3 - \frac{1}{120}\beta_1 - \frac{1}{120}\alpha_1 + \frac{1}{6}\alpha_1 - \alpha_1\right)x^5 \\
&+ \left(-\frac{1}{30}\alpha_4 - \frac{1}{50}\alpha_4 - \frac{1}{120}\alpha_0\beta_0 - \frac{1}{360}\beta_2 - \frac{1}{360}\alpha_2 - \frac{1}{12}\alpha_0 - \frac{1}{3}\alpha_0^2 - \frac{1}{18}\alpha_2\right)x^6
\end{aligned}$$

$$+\left(\begin{array}{l}-\frac{1}{42}\alpha_5-\frac{1}{84}\alpha_5-\frac{1}{504}\alpha_0\beta_1-\frac{1}{504}\alpha_1\beta_0-\frac{1}{840}\beta_3\\-\frac{1}{840}\alpha_3-\frac{5}{252}\alpha_1-\frac{10}{63}\alpha_0\alpha_1-\frac{1}{42}\alpha_3\end{array}\right)x^7$$

$$+\left(\begin{array}{l}-\frac{1}{56}\alpha_6-\frac{3}{392}\alpha_6-\frac{1}{1344}\alpha_0\beta_2-\frac{1}{1344}\alpha_2\beta_0-\frac{1}{2016}\alpha_1\beta_1-\frac{1}{1680}\beta_4-\frac{1}{1680}\alpha_4-\frac{5}{112}\alpha_0^2\\-\frac{5}{56}\alpha_0^3-\frac{5}{672}\alpha_2-\frac{5}{252}\alpha_1^2-\frac{5}{84}\alpha_0\alpha_2-\frac{1}{84}\alpha_4\end{array}\right)x^8$$

$$+\left(\begin{array}{l}-\frac{1}{72}\alpha_7-\frac{1}{192}\alpha_7-\frac{1}{2880}\alpha_0\beta_3-\frac{1}{2880}\alpha_3\beta_0-\frac{1}{5184}\alpha_1\beta_2\\-\frac{1}{5184}\alpha_2\beta_1-\frac{1}{3024}\beta_5-\frac{1}{3024}\alpha_5-\frac{5}{216}\alpha_0\alpha_1\\-\frac{1}{288}\alpha_3-\frac{5}{72}\alpha_0^2\alpha_1-\frac{1}{36}\alpha_0\alpha_3-\frac{5}{324}\alpha_1\alpha_2\end{array}\right)x^9$$

$$+\left(\begin{array}{l}-\frac{1}{90}\alpha_8-\frac{1}{270}\alpha_8-\frac{1}{10800}\alpha_1\beta_3-\frac{1}{10800}\alpha_3\beta_1\\-\frac{1}{5400}\alpha_0\beta_4-\frac{1}{5400}\alpha_4\beta_0-\frac{1}{12960}\alpha_2\beta_2-\frac{1}{72}\alpha_0^3\\-\frac{1}{324}\alpha_1^2-\frac{1}{108}\alpha_0\alpha_2-\frac{1}{72}\alpha_0^4-\frac{1}{540}\alpha_4-\frac{1}{36}\alpha_0^2\alpha_2-\frac{1}{54}\alpha_0\alpha_1^2\end{array}\right)x^{10}$$

$$+\cdots=0$$

and

$$V_1(x)=\left(-\frac{1}{2}\beta_0-2\beta_0-\frac{1}{2}+\frac{1}{2}+\frac{5}{2}\right)x^2$$

$$+\left(-\frac{1}{6}\beta_1-\frac{1}{3}\beta_1\right)x^3+\left(-\frac{1}{12}\beta_2-\frac{1}{9}\beta_2-\frac{1}{24}\beta_0-\frac{1}{24}\alpha_0-\frac{5}{8}\beta_0+\frac{1}{3}\right)x^4$$

$$+\left(-\frac{1}{20}\beta_3-\frac{1}{20}\beta_3-\frac{1}{120}\beta_1-\frac{1}{120}\alpha_1+\frac{1}{8}\beta_1\right)x^5$$

$$+\left(-\frac{1}{30}\beta_4-\frac{2}{75}\beta_4-\frac{1}{120}\alpha_0\beta_0-\frac{1}{360}\beta_2-\frac{1}{360}\alpha_2+\frac{1}{5}\beta_0+\frac{1}{8}\beta_0^2+\frac{1}{24}\beta_2\right)x^6$$

$$+\left(\begin{array}{l}-\frac{1}{42}\beta_5-\frac{1}{63}\beta_5-\frac{1}{504}\alpha_0\beta_1-\frac{1}{504}\alpha_1\beta_0-\frac{1}{840}\beta_3\\-\frac{1}{840}\alpha_3+\frac{1}{21}\beta_1+\frac{5}{84}\beta_0\beta_1+\frac{1}{56}\beta_3\end{array}\right)x^7$$

$$+\left(\begin{array}{l}-\frac{1}{56}\beta_6-\frac{1}{98}\beta_6-\frac{1}{1344}\alpha_0\beta_2-\frac{1}{1344}\alpha_2\beta_0-\frac{1}{2016}\alpha_1\beta_1-\frac{1}{1680}\beta_4\\-\frac{1}{1680}\alpha_4+\frac{3}{56}\beta_0^2+\frac{1}{56}\beta_2+\frac{5}{448}\beta_0^3+\frac{5}{672}\beta_1^2+\frac{5}{224}\beta_0\beta_2+\frac{1}{112}\beta_4\end{array}\right)x^8$$

$$
\begin{aligned}
&+\begin{pmatrix} -\frac{1}{72}\beta_7-\frac{1}{144}\beta_7-\frac{1}{2880}\alpha_0\beta_3-\frac{1}{2880}\alpha_3\beta_0-\frac{1}{5184}\alpha_1\beta_2 \\ -\frac{1}{5184}\alpha_2\beta_1-\frac{1}{3024}\beta_5-\frac{1}{3024}\alpha_5+\frac{1}{36}\beta_0\beta_1+\frac{1}{120}\beta_3 \\ +\frac{5}{576}\beta_0^{\,2}+\frac{1}{96}\beta_0\beta_3+\frac{5}{864}\beta_1\beta_2+\frac{5}{1008}\beta_5+\frac{1}{192}\beta_0\beta_3 \end{pmatrix}x^9 \\
&+\begin{pmatrix} -\frac{1}{90}\beta_8-\frac{2}{405}\beta_8-\frac{1}{10800}\alpha_1\beta_3-\frac{1}{10800}\alpha_3\beta_1-\frac{1}{5400}\alpha_0\beta_4-\frac{1}{5400}\alpha_4\beta_0 \\ -\frac{1}{12960}\alpha_2\beta_2+\frac{1}{180}\beta_0^{\,3}+\frac{1}{270}\beta_1^{\,2}+\frac{1}{90}\beta_0\beta_2+\frac{1}{225}\beta_4 \\ +\frac{1}{288}\beta_0^{\,2}\beta_2+\frac{1}{432}\beta_0\beta_1^{\,2}+\frac{1}{180}\beta_0\beta_4+\frac{1}{360}\beta_1\beta_3+\frac{1}{844}\beta_2^{\,2} \end{pmatrix}x^{10} \\
&+\cdots=0
\end{aligned}
$$

Considering the hypothesis of $U_1(x)=0, V_1(x)=0$, and the coefficients of $\alpha_n, \beta_n, n=1,2,3,\ldots$ will be obtained as the following

$$\alpha_0=-1, \alpha_1=0, \alpha_2=\frac{9}{2}, \alpha_3=0, \alpha_4=-\frac{75}{8}, \alpha_5=0, \alpha_6=\frac{245}{16}, \alpha_7=0, \cdots$$

$$\beta_0=1, \beta_1=0, \beta_2=-\frac{3}{2}, \beta_3=0, \beta_4=\frac{15}{8}, \beta_5=0, \beta_6=-\frac{35}{16}, \beta_7=0, \cdots$$

Hence, the approximate series solution is

$$U_0(x)\simeq 1-\frac{1}{2}x^2+\frac{3}{8}x^4-\frac{5}{16}x^6+\frac{35}{128}x^8+\cdots$$

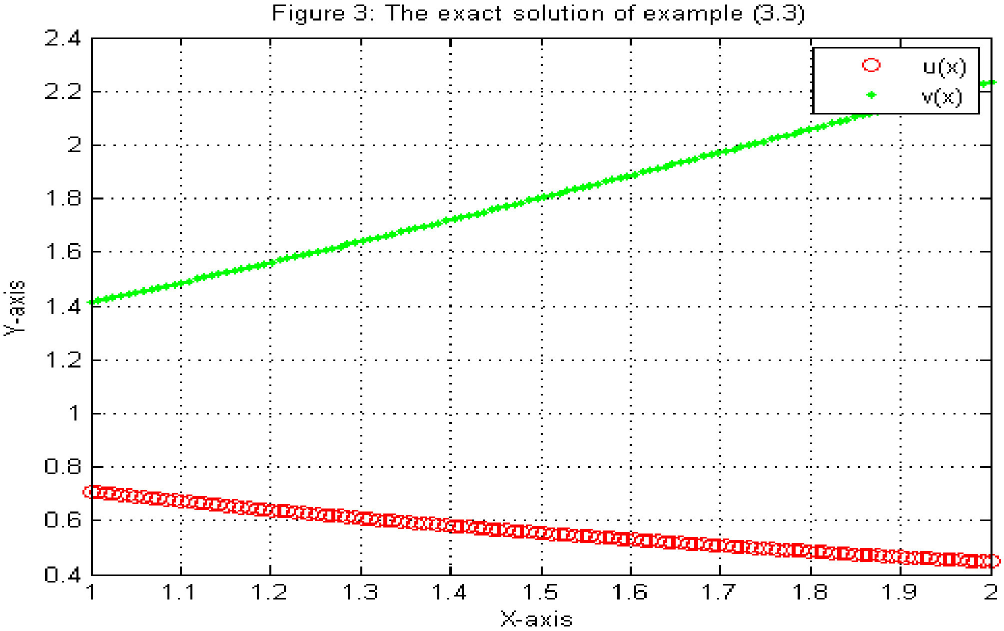
$$V_0(x)\simeq -1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6-\frac{5}{128}x^8+\cdots$$

Therefore, the exact solution of Eq. (61), and Eq. (62), becomes as:

$$u(x)=\frac{1}{\sqrt{1+x^2}} \quad , \quad v(x)=\sqrt{1+x^2}$$

which is the same as the closed form solution obtained by Abdul-Majid Wazwaz [1] using (VIM).

Figure 3: The exact solution of example (3.3)



Example (3.4): Consider the systems of linear equations of Emden-Fowler type see [1],

$$u''(x)+\frac{1}{x}u'(x)+v(x)=x^3+5, \tag{75}$$

$$v''(x)+\frac{2}{x}v'(x)+w(x)=x^4+12x+1, \tag{76}$$

$$w''(x)+\frac{3}{x}w'(x)+u(x)=25x^2+1, \tag{77}$$

subject to

$$u(0)=1, \quad u'(0)=0, \tag{78}$$

$$v(0)=1, \quad v'(0)=0, \tag{79}$$

$$w(0)=1, \quad w'(0)=0, \tag{80}$$

To solve Eq. (75), Eq. (76), and Eq. (77), by the method (NHPM), we construct the following homotopy

$$U''(x)=u_0(x)-p\left(u_0(x)+U''(x)+\frac{1}{x}U'(x)+V(x)-x^3-5\right), \tag{81}$$

$$V''(x)=v_0(x)-p\left(v_0(x)+V''(x)+\frac{2}{x}V'(x)+W(x)-x^4-12x-1\right), \tag{82}$$

$$W''(x)=w_0(x)-p\left(w_0(x)+W''(x)+\frac{3}{x}W'(x)+U(x)-25x^2-1\right), \tag{83}$$

By taking the inverse operator, $L^{-1}=\int_0^x\int_0^x(.)\,dxdx$ to the both sides of the Eq. (81), Eq. (82), and Eq.(83), we obtain

$$U(x)=U(0)+xU'(0)+\int_0^x\int_0^x u_0(x)dxdx-p\int_0^x\int_0^x\left(u_0(x)+\frac{1}{x}U'(x)+V(x)-x^3-5\right)dxdx, \tag{84}$$

$$V(x)=V(0)+xV'(0)+\int_0^x\int_0^x v_0(x)dxdx-p\int_0^x\int_0^x\left(\begin{array}{l}v_0(x)+\frac{2}{x}V'(x)\\+W(x)-x^4-12x-1\end{array}\right)dxdx, \tag{85}$$

$$W(x)=W(0)+xW'(0)+\int_0^x\int_0^x w_0(x)dxdx-p\int_0^x\int_0^x\left(\begin{array}{l}w_0(x)+\frac{3}{x}W'(x)\\+U(x)-25x^2-1\end{array}\right)dxdx, \tag{86}$$

Substituting Eq. (22-23), into Eqs. (84-86), and comparing coefficients of terms with identical powers of p, gives as the following

$$p^0:\begin{cases} U_0(x)=U(0)+xU'(0)+\int_0^x\int_0^x u_0(x)\,dxdx \\ V_0(x)=V(0)+xV'(0)+\int_0^x\int_0^x v_0(x)\,dxdx \\ W_0(x)=W(0)+xW'(0)+\int_0^x\int_0^x w_0(x)\,dxdx \end{cases}, \quad (87)$$

$$p^1:\begin{cases} U_1(x)=-\int_0^x\int_0^x\left(u_0(x)+\frac{1}{x}U_0'(x)+V_0(x)-x^3-5\right)dxdx \\ V_1(x)=-\int_0^x\int_0^x\left(v_0(x)+\frac{2}{x}V_0'(x)+W_0(x)-x^4-12x-1\right)dxdx \\ W_1(x)=-\int_0^x\int_0^x\left(w_0(x)+\frac{3}{x}W_0'(x)+U_0(x)-25x^2-1\right)dxdx \end{cases}, \quad (88)$$

$$p^2:\begin{cases} U_2(x)=-\int_0^x\int_0^x\left(\frac{1}{x}U_1'(x)+V_1(x)\right)dxdx \\ V_2(x)=-\int_0^x\int_0^x\left(\frac{2}{x}V_1'(x)+W_1(x)\right)dxdx \\ W_2(x)=-\int_0^x\int_0^x\left(\frac{3}{x}W_1'(x)+U_1(x)\right)dxdx \end{cases}, \quad (89)$$

$\vdots$

$$p^{k+1}:\begin{cases} U_{k+1}(x)=-\int_0^x\int_0^x\left(\frac{1}{x}U_k'(x)+V_k(x)\right)dxdx \\ V_{k+1}(x)=-\int_0^x\int_0^x\left(\frac{2}{x}V_k'(x)+W_k(x)\right)dxdx \\ W_{k+1}(x)=-\int_0^x\int_0^x\left(\frac{3}{x}W_k'(x)+U_k(x)\right)dxdx \end{cases}$$

To solve Eq. (87) and Eq. (88) for $U_0(x), V_0(x)$, and $U_1(x), V_1(x)$, we can obtain

$$U_0(x)=1+\frac{1}{2}\alpha_0x^2+\frac{1}{6}\alpha_1x^3+\frac{1}{12}\alpha_2x^4+\frac{1}{20}\alpha_3x^5+\frac{1}{30}\alpha_4x^6+\frac{1}{42}\alpha_5x^7+\frac{1}{56}\alpha_6x^8+\frac{1}{72}\alpha_7x^9+\cdots$$
$$V_0(x)=1+\frac{1}{2}\beta_0x^2+\frac{1}{6}\beta_1x^3+\frac{1}{12}\beta_2x^4+\frac{1}{20}\beta_3x^5+\frac{1}{30}\beta_4x^6+\frac{1}{42}\beta_5x^7+\frac{1}{56}\beta_6x^8+\frac{1}{72}\beta_7x^9+\cdots$$
$$W_0(x)=1+\frac{1}{2}\gamma_0x^2+\frac{1}{6}\gamma_1x^3+\frac{1}{12}\gamma_2x^4+\frac{1}{20}\gamma_3x^5+\frac{1}{30}\gamma_4x^6+\frac{1}{42}\gamma_5x^7+\frac{1}{56}\gamma_6x^8+\frac{1}{72}\gamma_7x^9+\cdots$$

Similarly, we obtain

$$U_1(x)=\left(-\frac{1}{2}\alpha_0-\frac{1}{2}\alpha_0-\frac{1}{2}+\frac{5}{2}\right)x^2$$
$$+\left(-\frac{1}{6}\alpha_1-\frac{1}{12}\alpha_1\right)x^3+\left(-\frac{1}{12}\alpha_2-\frac{1}{36}\alpha_2-\frac{1}{24}\beta_0\right)x^4$$
$$+\left(-\frac{1}{20}\alpha_3-\frac{1}{80}\alpha_3-\frac{1}{120}\beta_1+\frac{1}{20}\right)x^5$$
$$+\left(-\frac{1}{30}\alpha_4-\frac{1}{150}\alpha_4-\frac{1}{360}\beta_2\right)x^6$$
$$+\left(-\frac{1}{42}\alpha_5-\frac{1}{252}\alpha_5-\frac{1}{840}\beta_3\right)x^7$$
$$+\cdots=0$$

and

$$V_1(x)=\left(-\frac{1}{2}\beta_0-\beta_0-\frac{1}{2}+\frac{1}{2}\right)x^2+\left(-\frac{1}{6}\beta_1-\frac{1}{6}\beta+2\right)x^3$$
$$+\left(-\frac{1}{12}\beta_2-\frac{1}{18}\beta_2-\frac{1}{24}\gamma_0\right)x^4+\left(-\frac{1}{20}\beta_3-\frac{1}{40}\beta_3-\frac{1}{120}\gamma_1\right)x^5$$
$$+\left(-\frac{1}{30}\beta_4-\frac{1}{75}\beta_4-\frac{1}{360}\gamma_2+\frac{1}{30}\right)x^6$$
$$+\left(-\frac{1}{42}\beta_5-\frac{1}{126}\beta_5-\frac{1}{840}\gamma_3\right)x^7+\cdots=0$$

and

$$W_1(x)=\left(-\frac{1}{2}\gamma_0-\frac{3}{2}\gamma_0-\frac{1}{2}+\frac{1}{2}\right)x^2$$
$$+\left(-\frac{1}{6}\gamma_1-\frac{1}{4}\gamma_1\right)x^3$$
$$+\left(-\frac{1}{12}\gamma_2-\frac{1}{12}\gamma_2-\frac{1}{24}\alpha_0+\frac{25}{12}\right)x^4$$
$$+\left(-\frac{1}{20}\gamma_3-\frac{3}{80}\gamma_3-\frac{1}{120}\alpha_1\right)x^5$$
$$+\left(-\frac{1}{30}\gamma_4-\frac{1}{50}\gamma_4-\frac{1}{360}\alpha_2\right)x^6$$
$$+\left(-\frac{1}{42}\gamma_5-\frac{1}{84}\gamma_5-\frac{1}{840}\alpha_3\right)x^7+\cdots=0$$

Considering the hypothesis of $U_1(x)=0, V_1(x)=0,$ the coefficients of α_n, and β_n, $n=1,2,3,\ldots$ will be obtained as the following

$$\alpha_0=2, \alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=\cdots=0$$
$$\beta_0=0, \beta_1=6, \beta_2=\beta_3=\beta_4=\beta_5=\cdots=0$$
$$\gamma_0=0, \gamma_1=0, \gamma_2=12, \gamma_3=\gamma_4=\gamma_5=\cdots=0$$

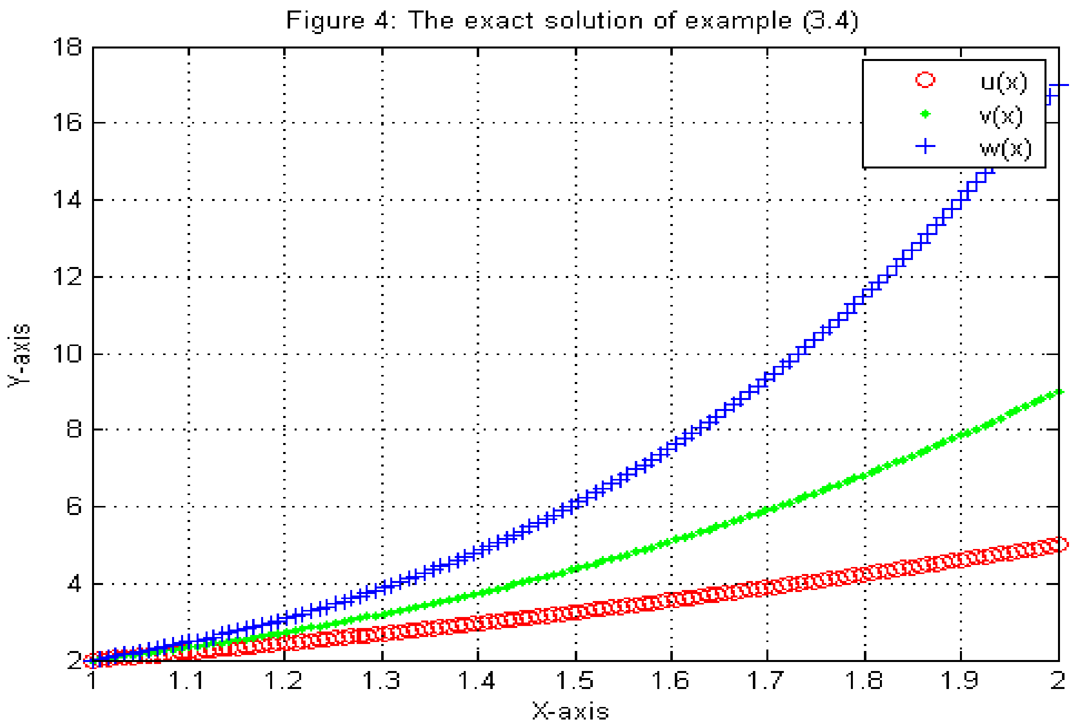
Hence, the approximate series solution is

$U_0(x) = 1 + x^2$,
$V_0(x) = 1 + x^3$,
$W_0(x) = 1 + x^4$,

Therefore, the exact solution of Eq. (75-77), becomes as:

$u(x) = 1 + x^2 \quad , \quad v(x) = 1 + x^3 \quad , \quad w(x) = 1 + x^4$

which is the same as the closed form solution obtained by Abdul-Majid Wazwaz [1] using (VIM).



(4) Conclusion:

The primary goal of this paper, the new homotopy perturbation method (NHPM) is a powerful instrument which is capable of handling linear and nonlinear partial differential equations. The method has been successfully applied to systems of nonlinear equations of Emden-Fowler type. This method can be applied to many complicated linear and non-linear equations of Emden-Fowler type and does not require linearization. The outcomes indicate that the new homotopy perturbation method is a powerful mathematical tool for noticing the exact and approximate solutions of nonlinear equations.

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