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Some Numerical Methods for Solving Linear Two-Dimensional Volterra Integral Equation

A. M. Al-Bugami[1], M. M. Al-Wagdani[2]

[1] Department of Mathematics, Faculty of Sciences Taif University, KSA
[2] Department of Mathematics, Faculty of Sciences Taif University, KSA

Abstract: In this paper, the existence and uniqueness of solution of the linear two- dimensional Volterra integral equation of the second kind(LT-DVIE) with Continuous Kernel are discussed and proved. Trapezoidal rule and Simpson's rule are used to solve this type of two dimensional Volterra integral equation of the second kind. Numerical examples are considered to illustrate the effectiveness of the proposed methods and the error is estimated.

Keywords: Two-Dimensional Volterra Integral Equation; trapezoidal rule; Simpson's rule.

1.Introduction:

Two-dimensional integral equations provide an important tool for modeling many problems in engineering and science [1]. There are many well-written texts on the theory and applications of integral equations in different sciences. From 1960to the present day, many new numerical methods have been developed for the solution of many types of integral equations, such as the Toeplitz matrix method, the product Nyström method, the Galerkin method; trapezoidal rule and Simpson's rule (see Linz [2], Baker et al. [3], and Delves and Mohamed [4]). More information for some numerical methods can be found especially in Delves and Mohamed [4], Atkinson [5, 6] and Golberg [7]. In the references [1-3], the trapezoidal rule and Simpson's rule was used to solve the integral equation in one dimensional.In [8],the author solved L-VIE's using the trapezoidal rule and Simpson's rule. In [9], the author's solved NL-VIE's using the trapezoidal rule and Simpson's rule. In the reference [10], the authors solved the T-DVIE of the second kind using spectral Galerkin method. In [11], the authors solved the two dimensional nonlinear integral equation of the second kind using degenerate kernel method. Guoqiang et al., in [12], obtained numerically the solution of two-dimensional nonlinear Volterra integral equation by collocation and iteration collocation methods. In [13], Guoqiang and Jiong analyzed the existence of asymptotic error expansion of the Nyström solution for two-dimensional nonlinear Fredholm integral equation of the second kind. In this paper, we use trapezoidal rule and Simpson's rule to discuss numerically the solution of the LT-DVIE of the second kind with continuous kernel of the form

$$\mu u(x,y) = f(x,y) + \lambda \int_0^x \int_0^y k(x,y,t,s) u(t,s) dtds$$

(1)

where μ is a constant defines the kind of the integral equation, $u(x,y)$ is an unknown function, will be determined, the functions $f(x,y)$ and $k(x,y,t,s)$ are given analytical functions defined, respectively, $J = [0,\mathrm{X}]\times[0,\mathrm{Y}], E = \{(x,y,t,s) : 0 \le t \le x \le a, 0 \le s \le y \le b\}, \mu$ and λ are constants that have many physical meanings.

2.The existence and uniqueness of the solution:

To guarantee the existence of a unique solution of equation (1), we assume the following conditions:

(i) The kernel $k(x,y,t,s)$ is continuous function in E satisfies:

$|k(x,y,t,s)| \le K$, K is a constant

(ii) The given function $f(x,y)$ is continuous with its derivatives and belongs to $J = [0,\mathrm{X}]\times[0,\mathrm{Y}]$, and its norm is defined as:

$\|f(x,y)\| = \max_{x,y \in J} |f(x,y)| \le M \quad \forall (x,y) \in J$, M is a constant.

(iii) the unknown function $u(x,y)$ is satisfies the Lipschits condition with respect to its argument and its normal is defined in $L_2[0,X]\times L_2[0,Y]$ as:

$$\|u(x,y)\| = [\int_0^x \int_0^y |u(x,y)|^2 \, \mathrm{dxdy}]^{1/2} \le C \qquad \text{,where } C \text{ is a constant} \qquad (2)$$

To prove the existence a unique solution of (1) using Banach fixed point theorem. Rewrite equation (3.1) in the integral operator form:

$$\bar{T}u(x,y) = \frac{1}{\mu} f(x,y) + Tu(x,y) \qquad (3)$$

where

$$Tu(x,y) = \frac{\lambda}{\mu} \int_0^x \int_0^y k(x,y,t,s) u(t,s) dtds \qquad (4)$$

Theorem 1:

If the conditions (i), (ii) and (iii) are verified, then equation (1) has auneque solution in the Banach space $C([0,\mathrm{X}]\times[0,\mathrm{Y}])$, the proof of tis theorem depends on the following two lemmas.

Lemma 1:

Underthe conditions (i)-(iii), the operator $\bar{T}$ defined by (3), maps the space $C([0,\mathrm{X}]\times[0,\mathrm{Y}])$ into itself.

Proof:

In view of the formulas (3) and (4), then using the condition (ii), and applying Couchy-Shwarz inequality, we have

$$\left\|\bar{T}u(x,y)\right\| \le \frac{1}{|\mu|}\left\|f(x,y)\right\| + \left|\frac{\lambda}{\mu}\right| \left\| \int_0^x \int_0^y |k(x,y,t,s)| |u(t,s)| dtds \right\| \qquad (5)$$

Using the condition (i) and (iii), the above inequality takes the form:

$$\left\|\bar{T}u(x,y)\right\| \le \frac{M}{|\mu|} + A\left\|u(x,y)\right\| \qquad , A = \left|\frac{\lambda}{\mu}\right| KC \qquad (6)$$

Inequality (6) show that, the operator $\bar{T}$ maps the space $C([0,\mathrm{X}]\times[0,\mathrm{Y}])$ into itself.

Moreover, the inequality (6) involves that the operator T is bounded where

$$\left\|Tu(x,y)\right\| \le A\left\|u(x,y)\right\| \quad (7)$$

The inequalities (6) and (7) define that the operator $\bar{T}$ is bounded.

Lemma 2:

If the conditions (i) and (iii) are satisfied .then the operator $\bar{T}$ is contractive in the Banach space $C([0,\mathrm{X}]\times[0,\mathrm{Y}])$.

Proof:

For the two functions $u_1(x,y)$ and $u_2(x,y)$ in the space $C([0,\mathrm{X}]\times[0,\mathrm{Y}])$ the formula (3) and (4) lead to,

$$\left\|(\bar{T}u_1 - \bar{T}u_2)(x,y)\right\| \le \left|\frac{\lambda}{\mu}\right| \left\| \int_0^x \int_0^y |k(x,y,t,s)| |u_1(t,s) - u_2(t,s)| dtds \right\|$$

Using the condition (iii) and then applying Cauchy-Shwarz inequality, we have

$$\left\|(\bar{T}u_1 - \bar{T}u_2)(x,y)\right\| \le A\left\|u_1(t,s) - u_2(t,s)\right\| \qquad (8)$$

Inequality (8) show that, the operator $\bar{T}$ is continuous in the space $C([0,\mathrm{X}]\times[0,\mathrm{Y}])$.

Also, $\bar{T}$ is a contraction operator, under the condition $A<1$, in the Banach space $C([0,\mathrm{X}]\times[0,\mathrm{Y}])$. Therefore, the operator $\bar{T}$ has a unique fixed point which is the unique solution of equation (1).

3.The Trapezoidal Rule:

In this section, the trapezoidal rule is used to solve LT-DVIE of the second kind:

$$\mu u(x,y)=f(x,y)+\lambda\int_0^x\int_0^y k(x,y,t,s)u(t,s)dtds \quad (9)$$

Here, $u(x,y)$ is the unknown function, will be determined, the two analytic functions $f(x,y)$ and $k(x,y,t,s)$ are given and defined, respectively, on the following domains:

$D=[0,a]\times[0,b]$,and $E=\{(x,y,t,s):0\le t\le x\le a,0\le s\le y\le b\}$. Also, μ and λ are constants that have many physical meanings. We will divide $0=x_0<x_1<...<x_N=a,0=y_0<y_1<...<y_N=b$ be a partition of $[0,a]$, $[0,b]$ with the step size h , such that, $x_i=ih$, $y_i=ih$, $t_j=jh$ and $s_j=jh$ for $i,j=0,1,...,N$.

We will refer to the value of the solution at (x_i,y_i) as $u(x_i,y_i)\equiv u_{i,i}$, $f(x_i,y_i)\equiv f_{i,i}$ is the value of function f at (x_i,y_i) and the value of the kernel at (x_i,y_i,t_j,s_j) as $k(x_i,y_i,t_j,s_j)\equiv k_{i,i,j,j}$, $k(x_i,y_i,t_j,s_j)$ clearly vanishes for $t_j>x_i$ and $s_j>y_i$ as the integration (9) ends at $t_j\le x_i$ and $s_j\le y_i$.

So, if we use the trapezoidal rule with N subintervals to approximate the integral in the two-dimensional Volterra integral equation of the second kind (9), we have

$$\int_0^x\int_0^y k(x,y,t,s)u(t,s)dtds \simeq h[\frac{1}{2}k(x,y,t_0,s_0)u(t_0,s_0)+k(x,y,t_1,s_1)u(t_1,s_1)+...$$
$$+k(x,y,t_{N-1},s_{N-1})u(t_{N-1},s_{N-1})+\frac{1}{2}k(x,y,t_N,s_N)u(t_N,s_N)] \quad (10)$$

where $t_j\le x$, $s_j\le y$, $j\ge 0$, $x=x_N=t_N$ and $y=y_N=s_N$.

The integral equation (9)is the approximated by the sum

$$u(x,y)=f(x,y)+h[\frac{1}{2}k(x,y,t_0,s_0)u(t_0,s_0)+k(x,y,t_1,s_1)u(t_1,s_1)+...$$
$$+k(x,y,t_{N-1},s_{N-1})u(t_{N-1},s_{N-1})+\frac{1}{2}k(x,y,t_N,s_N)u(t_N,s_N)] \quad (11)$$

where $t_j\le x$, $s_j\le y$, $j\ge 0$, $x=x_N=t_N$ and $y=y_N=s_N$.

The integrations in (10) is over$t$ and s , $0\le t\le x$ and $0\le s\le y$, thus for $t_j>x_i$ and $s_j>y_i$ we take $k(x_i,y_i,t_j,s_j)\equiv 0$.

Since the integrals in (9) vanishes for $x=x_0=0$ and $y=y_0=0$.

$u(x_0,y_0)=f(x_0,y_0)$

and for $i,j=0,1,...,N$,$t_j\le x_i$ and $s_j\le y_i$ we have,

$$u_{i,i} = f_{i,i} + h[\frac{1}{2}k_{1,1,0,0}u_{0,0} + k_{1,1,1,1}u_{1,1} + ... + k_{i,i,j-1,j-1}u_{j-1,j-1} + \frac{1}{2}k_{i,i,j,j}u_{j,j}] \qquad (12)$$

where $k_{i,i,j,j} \equiv k(x_i, y_i, t_j, s_j)$, $j \le i$.

Which are $N+1$ equations in $u_{i,i}$, the approximations to the solution $u(x,y)$ of (9) at (x_i, y_i) for $i = 0,1,...,N$.

If we leaving only the nonhomogeneous part $f_{i,i}$ on the right side of (12), then write all the $N+1$ equations for $u_{i,i}$, $i = 0,1,...,N$, we have the following triangular system of equations:

$$\begin{aligned}
&u_{0,0} &&= f_{0,0}\\
&-\frac{h}{2}k_{1,1,0,0}u_{0,0} + (1-\frac{h}{2}k_{1,1,1,1})u_{1,1} &&= f_{1,1}\\
&-\frac{h}{2}k_{2,2,0,0}u_{0,0} - hk_{2,2,1,1}u_{1,1} + (1-\frac{h}{2}k_{2,2,2,2})u_{2,2} &&= f_{2,2}\\
&-\frac{h}{2}k_{3,3,0,0}u_{0,0} - hk_{3,3,1,1}u_{1,1} - hk_{3,3,2,2}u_{2,2} + (1-\frac{h}{2}k_{3,3,3,3})u_{3,3} &&= f_{3,3}\\
&\vdots\\
&-\frac{h}{2}k_{N,N,0,0}u_{0,0} - hk_{N,N,1,1}u_{1,1} - hk_{N,N,2,2}u_{2,2} - ... + (1-\frac{h}{2}k_{N,N,N,N})u_{N,N} &&= f_{N,N}
\end{aligned} \qquad (13)$$

as a system of $N+1$ equations in the $N+1$ desired unknowns $u_{0,0}, u_{1,1}, ..., u_{N,N}$.

Now, we must recognize that the set of equations (13) can be written in a matrix notation form:

$$AU = F \qquad (14)$$

where A is the $(N+1)\times(N+1)$ matrix of the coefficients of the system of equations (13), $U = (u_{i,i})$ is the column matrix of the sample solutions, and $F = (f_{i,i})$ is the column matrix of the nonhomogeneous parts $f_{i,i}$ in (13).

4.The Simpson's Rule:

For LT-DVIE of the second kind:

$$u(x,y) = f(x,y) + \int_0^x\int_0^y k(x,y,t,s)u(t,s)dtds \qquad (15)$$

the functions $f(x,y)$ and $k(x,y,t,s)$ are given functions. We shall assume that $f(x,y)$ is continuous on $[0,a]\times[0,b]$ and $k(x,y,t,s)$ is continuous on $E = \{(x,y,t,s): 0 \le t \le x \le a, 0 \le s \le y \le b\}$ and that it satisfies a uniform Lipschitz condition in u .

In equation (15), for $x \in [0,a]$ and $y \in [0,b]$ we divide $0 = x_0 < x_1 < ... < x_N = a$, $0 = y_0 < y_1 < ... < y_N = b$ be a partition of $[0,a]$, $[0,b]$ with the step size h, such that, $x_i = ih$, $y_i = ih$, $t_j = jh$ and $s_j = jh$ for $i, j = 0,1,...,N$; $h = \frac{x_N - x_0}{N}$.

If $(x, y) = (x_0, y_0)$, equation (15) becomes:

$$u_{0,0} \simeq u(x_0, y_0) = f(x_0, y_0) \tag{16}$$

Now, we can apply Simpson's rule by setting $(x, y) = (x_r, y_r)$; $r = 2,3,...,N$, Then equation (15) take the following form :

$$u_{r,r} \simeq u(x_r, y_r) = f(x_r, y_r) + \frac{h}{3}\sum_{j=0}^{r} w_{rj} k(x_r, y_r, t_j, s_j) u(x_j, y_j) \tag{17}$$

The weights w_{rj} are given by:

$$w_{r0} = w_{rr} = 1, \ w_{rj} = 3 - (-1)^j, \ 1 \le j \le r-1.$$

where $u_{1,1} \simeq u(x_1, y_1)$ have unknown value can be computed from Day's starting procedure as the following manner:

Define,

$$u_{11} = f(x_1, y_1) + hk(h,h,0,0) f(x_0, y_0)$$

$$u_{12} = f(x_1, y_1) + \frac{h}{2}[k(h,h,0,0) f(x_0, y_0) + k(h,h,h,h) u_{11}]$$

$$u_{13} = f(x_{1/2}, y_{1/2}) + \frac{h}{4}[k(\frac{h}{2},\frac{h}{2},0,0) f(x_0, y_0) + k(\frac{h}{2},\frac{h}{2},\frac{h}{2},\frac{h}{2})[\frac{1}{2} f(x_0, y_0) + \frac{1}{2} u_{12}]]$$

Then,

$$u_{1,1} = f(x_1, y_1) + \frac{h}{6}[k(h,h,0,0) f(x_0, y_0) + 4k(h,h,\frac{h}{2},\frac{h}{2}) u_{13} + k(h,h,h,h) u_{12}] \tag{18}$$

The equations (16), (17) and (18) (which are $N+1$ equations in $u_{i,i} \simeq u(x_i, y_i)$, $0 \le i \le N$)represented the approximation to the solution $u(x, y)$ of (15), and can be written as the following system:

$$
\begin{aligned}
u_{0,0} &= f(x_0,y_0) \\
u_{1,1} &= f(x_1,y_1)+\frac{h}{6}[k(h,h,0,0)f(x_0,y_0)+4k(h,h,\frac{h}{2},\frac{h}{2})u_{13}+k(h,h,h,h)u_{12}] \\
u_{2,2} &= f(x_2,y_2)+\frac{h}{3}[k(2h,2h,0,0)u_{0,0}+4k(2h,2h,h,h)u_{1,1}+k(2h,2h,2h,2h)u_{2,2}] \\
u_{3,3} &= f(x_3,y_3)+\frac{h}{3}[k(3h,3h,0,0)u_{0,0}+4k(3h,3h,h,h)u_{1,1}+2k(3h,3h,2h,2h)u_{2,2} \\
&\qquad +k(3h,3h,3h,3h)u_{3,3}] \\
&\vdots \\
u_{n,n} &= f(x_n,y_n)+\frac{h}{3}[k(nh,nh,0,0)u_{0,0}+4k(nh,nh,h,h)u_{1,1}+\cdots \\
&\qquad +4k(nh,nh,(n-1)h,(n-1)h)u_{(n-1),(n-1)}+k(nh,nh,nh,nh)u_{n,n}]
\end{aligned} \tag{19}
$$

5. Numerical Experiments and Discussions:

Example 1:

Consider the linear two-dimensional Volterra integral equation:

$$u(x,y)=xy-0.125x^5y^3+\int_0^x\int_0^y xys^2u(t,s)dtds \tag{20}$$

where the exact solution is $u(x,y)=xy$ and $0\le x,y\le 1$, here $\lambda=1$, $\mu=1$. In table (5.1)-(5.3) we present the exact solution, the approximate numerical solutions and their corresponding errors for some points,we suppose that $N=20,50,80$.

In tables (5.1)-(5.6):

$u^T\rightarrow$ approximate solution of trapezoidal rule, $E^T\rightarrow$ the error of trapezoidal rule, $u^S\rightarrow$ approximate solution of Simpson's rule, $E^S\rightarrow$the error of Simpsons rule.

Case 1: $N=20$,

x	y	Exact sol.	u^T	E^T	u^S	E^S
0	0	0	0	0	0	0
0.1	0.1	0.010000000	0.0100000268	2.688×10^{-8}	0.0100000195	1.959×10^{-8}
0.2	0.2	0.040000000	0.0400025051	2.50514×10^{-6}	0.0400022467	2.24672×10^{-6}
0.3	0.3	0.090000000	0.0900375693	3.75693×10^{-5}	0.0900355692	3.55692×10^{-4}
0.4	0.4	0.160000000	0.1602545793	2.54579×10^{-4}	0.1602460440	2.46044×10^{-4}
0.5	0.5	0.250000000	0.2511040831	1.10408×10^{-3}	0.2510774212	1.07742×10^{-3}
0.6	0.6	0.360000000	0.3635947993	3.59479×10^{-3}	0.3635248916	2.48916×10^{-3}
0.7	0.7	0.490000000	0.4995822509	9.58225×10^{-3}	0.4994142993	9.41429×10^{-3}
0.8	0.8	0.640000000	0.6620428387	2.20428×10^{-2}	0.6616490814	2.14690×10^{-2}
0.9	0.9	0.810000000	0.8553186227	4.53186×10^{-2}	0.8543821368	4.43821×10^{-2}
1.0	1.0	1.000000000	1.085393572	8.53935×10^{-2}	1.083106343	8.31063×10^{-2}

Table(5.1)

Case 2: $N = 50$,

x	y	Exact sol.	u^T	E^T	u^S	E^S
0	0	0	0	0	0	0
0.1	0.1	0.010000000	0.0100000200	2.008×10^{-8}	0.0100000147	1.472×10^{-8}
0.2	0.2	0.040000000	0.0400022826	2.28269×10^{-6}	0.0400022402	2.24025×10^{-6}
0.3	0.3	0.090000000	0.0900358718	3.58718×10^{-5}	0.0900313327	3.13327×10^{-5}
0.4	0.4	0.160000000	0.1602473909	2.47390×10^{-4}	0.1602460031	2.46003×10^{-4}
0.5	0.5	0.250000000	0.2510819429	1.08194×10^{-3}	0.2509813232	9.81323×10^{-4}
0.6	0.6	0.360000000	0.3635387326	3.58732×10^{-3}	0.3635264005	3.52640×10^{-3}
0.7	0.7	0.490000000	0.4994573408	9.45734×10^{-3}	0.4986723976	8.67239×10^{-3}
0.8	0.8	0.640000000	0.6617873619	2.17873×10^{-2}	0.6616972504	2.16972×10^{-2}
0.9	0.9	0.810000000	0.8548245433	4.48245×10^{-2}	0.8510224747	4.10224×10^{-2}
1.0	1.0	1.000000000	1.084469500	8.44695×10^{-2}	1.083763894	8.37638×10^{-2}

Table(5.2)

Case 3: $N = 80$,

x	y	Exact sol.	u^T	E^T	u^S	E^S
0	0	0	0	0	0	0
0.1	0.1	0.010000000	0.0100000192	1.927×10^{-8}	0.0100000187	1.876×10^{-8}
0.2	0.2	0.040000000	0.0400022567	2.25673×10^{-6}	0.0400022401	2.24011×10^{-6}
0.3	0.3	0.090000000	0.0900356742	3.56742×10^{-5}	0.0900355471	3.55471×10^{-5}
0.4	0.4	0.160000000	0.1602465553	2.46555×10^{-4}	0.1602460077	2.46007×10^{-4}
0.5	0.5	0.250000000	0.2510793710	1.07937×10^{-3}	0.2510775807	1.07758×10^{-3}
0.6	0.6	0.360000000	0.3635322220	3.53222×10^{-3}	0.3635269753	3.52697×10^{-3}
0.7	0.7	0.490000000	0.4994428404	9.44284×10^{-3}	0.4994275726	9.42757×10^{-3}
0.8	0.8	0.640000000	0.6617577157	2.17577×10^{-2}	0.6617120576	2.17120×10^{-2}
0.9	0.9	0.810000000	0.8547672347	4.47672×10^{-2}	0.8546294729	4.46294×10^{-2}
1.0	1.0	1.000000000	1.084362381	8.43623×10^{-2}	1.083954862	8.39548×10^{-2}

Table(5.3)

Example 2:

Consider the linear two-dimensional Volterra integral equation:

$$u(x,y) = x\sin(y) + \frac{1}{4}x^2(x^3\cos(y) - \sin^2(y) - x^3) + \int_0^x\int_0^y (xt^2 + \cos(s))u(t,s)dtds \quad (21)$$

where the exact solution is $u(x,y) = x\sin y$ and $0 \le x, y \le 1$, here $\lambda = 1$, $\mu = 1$. In table (5.4)-(5.6) we present the exact solution, the approximate numerical solutions and their corresponding errors for some points, we suppose that $N = 20, 50, 80$.

Case 1: $N = 20$,

x	y	Exact sol.	u^T	E^T	u^S	E^S
0	0	0	0	0	0	0
0.1	0.1	0.009983341	0.0103439169	3.605752×10^{-4}	0 .0102985368	3.151951×10^{-4}
0.2	0.2	0.039733866	0.0421913047	2.457438×10^{-3}	0 .0420771962	2.343330×10^{-3}
0.3	0.3	0.088656062	0.0962145972	7.558535×10^{-3}	0.0959834699	7.327407×10^{-3}
0.4	0.4	0.155767336	0.1722292770	1.646194×10^{-2}	0.1718012130	1.603387×10^{-2}
0.5	0.5	0.239712769	0.2693367370	2.962396×10^{-2}	0.2685880546	2.887528×10^{-2}
0.6	0.6	0.338785484	0.3861185815	4.733309×10^{-2}	0.3848589634	4.607347×10^{-2}
0.7	0.7	0.450952381	0.5208094830	6.985710×10^{-2}	0.5187401959	6.778781×10^{-2}
0.8	0.8	0.573884872	0.6713710356	9.748616×10^{-2}	0.6680103133	9.412544×10^{-2}
0.9	0.9	0.704994218	0.8353876323	1.303934×10^{-1}	0.8299384882	1.249442×10^{-1}
1	1	0.841470984	1.009693702	1.682227×10^{-1}	1.000809799	1.593388×10^{-1}

Table(5.4)

Case 2: $N = 50$,

x	y	Exact sol.	u^T	E^T	u^S	E^S
0	0	0	0	0	0	0
0.1	0.1	0.009983341	0.0103058199	3.224782×10^{-4}	0.0102431716	2.598299×10^{-4}
0.2	0.2	0.039733866	0.0421103174	2.376451×10^{-3}	0.0420842591	2.350393×10^{-3}
0.3	0.3	0.088656062	0.0960866148	7.430552×10^{-3}	0.0954390956	6.783033×10^{-3}
0.4	0.4	0.155767336	0.1720485410	1.628120×10^{-2}	0.1719144612	1.614712×10^{-2}
0.5	0.5	0.239712769	0.2690929677	2.938019×10^{-2}	0.2671246742	2.741190×10^{-2}
0.6	0.6	0.338785484	0.3857937622	4.700827×10^{-2}	0.3853452101	4.655972×10^{-2}
0.7	0.7	0.450952381	0.5203742741	6.942189×10^{-2}	0.5159177238	6.496534×10^{-2}
0.8	0.8	0.573884872	0.6707805384	9.689566×10^{-2}	0.6695166538	9.563178×10^{-2}
0.9	0.9	0.704994218	0.8345759902	1.295817×10^{-1}	0.8252535362	1.202593×10^{-1}
1	1	0.841470984	1.008566907	1.670959×10^{-1}	1.005117920	1.636469×10^{-1}

Table(5.5)

Case 3: $N = 80$,

x	y	Exact sol.	u^T	E^T	u^S	E^S
0	0	0	0	0	0	0
0.1	0.1	0.009983341	0.0103014008	3.180591×10^{-4}	0.0102978028	3.144611×10^{-4}
0.2	0.2	0.039733866	0.0421009221	2.367056×10^{-3}	0.0420877339	2.353867×10^{-3}
0.3	0.3	0.088656062	0.0960717667	7.415704×10^{-3}	0.0960364056	7.380343×10^{-3}
0.4	0.4	0.155767336	.1720275722	1.626023×10^{-2}	0.1719496296	1.618229×10^{-2}
0.5	0.5	0.239712769	.2690646866	2.935191×10^{-2}	0.2689132906	2.920052×10^{-2}
0.6	0.6	0.338785484	.3857560815	4.697059×10^{-2}	0.3854844300	4.669894×10^{-2}

0.7	0.7	0.450952381	.5203237946	6.937141×10^{-2}	0.5198586850	6.890630×10^{-2}
0.8	0.8	0.573884872	.6707120612	9.682718×10^{-2}	0.6699347338	9.604986×10^{-2}
0.9	0.9	0.704994218	.8344818935	1.294876×10^{-1}	0.8331936216	1.281994×10^{-1}
1	1	0.841470984	1.008436319	1.669533×10^{-1}	1.006296852	1.648258×10^{-1}

Table(5.6)

6. The Conclusion:

From the previous discussions we conclude the following:

1) As x and y is increasing in $[0,1]\times[0,1]$, the errors due to trapezoidal rule and Simpson's rule are also increasing.
2) As N is increasing, the errors are decreasing.
3) The errors due to the Simpson's rule less than the errors due to the trapezoidal rule.(i.e. Simpson's rule better than trapezoidal rule).

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