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Runge-Kutta and Block by Block Methods to Solve Non-LinearVolterra Integral Equation Of The Second Kind

A. M. Al-Bugami[1], S. S. Al-Juaid[2]

[1] Department of Mathematics, Faculty of Sciences,Taif University, KSA
[2] Department of Mathematics, Faculty of Sciences,Taif University, KSA

Abstract. In this paper, we discussed Runge-Kutta method (R.KM) and Block-by-Block method (B by BM) for used to solve (NVIE) of the second kind with continuous kernel. Numerical examples are presented and results are compared with the analytical solution to demonstrate the validity and applicability of this methods.

Keywords: Non-LinearVolterra Integral Equation; Runge-Kutta method; Block- by –block method.

1.Introduction:

The integral equation methods are widely used for solving many problems in mathematical physics, engineering and basic science. There are many numerical methods to solve the linear and nonlinear integral equations(seeBaker [1],Delves and Mohamed [2], Atkinson [3, 4] and Golberg [5]).In [6], Badr solved nonlinear Volterra- Fredholm integral equation by using Block-by-Block method. Katani and Shahmorad in [7], studied the new Block-by-Block method for solving Two- dimensional linear and nonlinear Volterra integral equations of the first and second kind. In [8], EL-Kalla and AL-Bugami used Adomian and Block-by-Block methods to solve nonlinear Two- dimensional Volterra integral equation. In [10], Markroglou studied the convergence of Block- by- Block method for nonlinear Volterraintegro- differential equations.

In this paper, we use R. KM and B by BM to discuss numerically the solution of the (NVIE) of the second kind with continuous kernel of the form

$$\mu\phi(x) = f(x) + \lambda\int_0^x k(x,t)\gamma(t,\phi(t))dt \qquad (1)$$

where μ is a constant defines the kind of the integral equation. $\phi(x)$ is an unknown function, the function $f(x)$ and $k(x,t)$ are given analytical functions defined respectively on $J=[0,X]$. μ and λ are constants that have many physical meanings.

2. Existence and unique solution of NVIE:

the existence of a unique solution of equation (1) under certain conditions will be discussed and proved using Picard method.

In order to prove the existence of a unique solution of equation (1) we assume the following conditions:

1) The given continuous function $f(x)$ in $0 \le x \le X < \infty$, such that $\|f(x)\| = \max_{x \in J} |f(x)| \le A^*$
2) The kernel $k(x,t)$ satisfies the continuity condition $|k(x,t)| \le N^*$, (N^* is a constant).
3) The known continuous function $\gamma(t,\phi(t))$ in $0 \le t \le X$, satisfies for the constant $B > B_1, B > B_2$, the following conditions:

   i- $\|\gamma(t,\phi(t))\| \le B_1 \|\phi(x)\|$

   ii- $\|\gamma(x,\phi_1(x)) - \gamma(x,\phi_2(x))\| \le B_2 \|\phi_1(x) - \phi_2(x)\|$,

   where $\|\phi(x)\| = \max_{0 \le t \le T} |\phi(x)|$

Now, we prove the existence of a unique solution of equation (1), under the conditions (1-3) by using successive approximation method (Picard method).

Theorem 1:

The solution of nonlinear Volterra integral equation (1) with continuous kernel is exist and a unique under the condition:

$$B|\lambda| < \frac{|\mu|}{N^*} \tag{2}$$

To proof this theorem we must state the following lemmas

lemma 1:

Beside the conditions (1-3), the infinite series $\sum_{i=0}^{\infty} \theta_i(x)$ is uniformly converge to a continuous solution $\phi(x)$.

Proof:

We construct the sequence of the function $\phi_n(x)$ such as

$$\mu\phi_n(x) = f(x) + \lambda \int_0^x k(x,t)\gamma\left(t,\phi_{n-1}(t)\right)dt, n = 1, 2, \ldots \tag{3}$$

With

$$\phi_0(x) = f(x) \tag{4}$$

Here, it is convenient to introduce

$$\theta_n(x) = \phi_n(x) - \phi_{n-1}(x) \tag{5}$$

Where

$$\phi_n(x) = \sum_{i=0}^{n} \theta_i(x) \quad \text{and} \quad \theta_0(x) = f(x) \tag{6}$$

Using the properties of the modules, the relation (5) takes the from

$$|\theta_n(x)| \le \left|\frac{\lambda}{\mu}\right| \int_0^x |k(x,t)| \left|\gamma\left(t,\phi_{n-1}(t)\right) - \gamma\left(t,\phi_{n-2}(t)\right)\right| dt \tag{7}$$

Using the condition (3-ii), we obtain:

$$|\theta_n(x)| \le \left|\frac{\lambda}{\mu}\right| B \int_0^x |k(x,t)| \left|\phi_{n-1}(t) - \phi_{n-2}(t)\right| dt \tag{8}$$

With the aid of (5) and take the maximum over x we get

$$\max_{0\le x\le X} |\theta_n(x)| \le \left|\frac{\lambda}{\mu}\right| B \int_0^x |k(x,t)| \max_{0\le t\le T} |\theta_{n-1}(t)| dt$$

Then we have

$$\|\theta_n(x)\| \le \frac{1}{|\mu|} \|\theta_{n-1}(t)\| \left(|\lambda| B \int_0^x |k(x,t)| dt \right)$$

By using the condition (2), we obtain

$$\|\theta_n(x)\| \le \frac{1}{|\mu|} N \left(|\lambda| B \right) \|\theta_{n-1}(t)\| \tag{9}$$

Inequality (9) takes the from

$$\|\theta_n\| \le \alpha_1 \|\theta_{n-1}\| \tag{10}$$

Where

$$\alpha_1 = \frac{1}{|\mu|} N^* \left(|\lambda| B \right) < 1$$

If we let $n = 1$ in (7) and using the condition (1) we get, $\|\theta_1\| \le \alpha_1 A^*$, then, by using the mathematical induction, we obtain

$$\|\theta_n\| \le \alpha_1^{\,n} A^*, \qquad n = 0,1,2,\ldots \tag{11}$$

This bound makes the sequence $\{\theta_n\}$ converges under the condition (2), and hence the sequence $\{\phi_n(t)\}$ converges to:

$$\phi(x)=\sum_{i=0}^{\infty}\theta_i(x) \tag{12}$$

The infinite series (12) is uniformly convergent series the terms $\theta_i(x)$ are dominated by (α_1).

Lemma 2:

A continuous function $\phi(x)$ represents a unique solution of equation (1).

Proof:

to proof that $\phi(x)$ represents a unique solution of equation (1), we prove that $\phi(x)$ defined by(12), satisfies equation (1), set $\phi(x)=\phi_n(x)+g_n(x)$, where $g_n(x)\to 0$ as $n\to\infty$ then we get:

$$\phi(x)-g_n(x)=\frac{1}{\mu}f(x)+\frac{\lambda}{\mu}\int_0^x k(x,t)\big(\gamma(t,\phi(t))-g_{n-1}(t)\big)dt$$

Therefore, usingthe condition (3-i) we have:

$$\max_{0\le x\le X}\left|\phi(x)-\frac{1}{\mu}f(x)-\frac{\lambda}{\mu}\int_0^x k(x,t)\gamma(t,\phi(t))dt\right|$$

$$\le \max_{0\le x\le X}|g_n(x)|-\left|\frac{\lambda}{\mu}\right|B\int_0^x|k(x,t)|\max_{0\le x\le X}|g_{n-1}(t)|dt \tag{13}$$

In view of the condition (2), the previous inequality takes the from:

$$\left\|\phi(x)-\frac{1}{\mu}f(x)-\frac{\lambda}{\mu}\int_0^x k(x,t)\gamma(t,\phi(t))dt\right\|\le\|g_n(x)\|-\alpha_1\|g_{n-1}(t)\| \tag{14}$$

Where $\alpha_1=\frac{1}{|\mu|}N^*\{|\lambda|B\}$.

So that, by taking n large enough, the right hand side for relation(14) can be made as small as desired, thus, the function $\phi(x)$ satisfies:

$$\mu\phi(x)-\lambda\int_0^x k(x,t)\gamma(t,\phi(t))dt=f(x) \tag{15}$$

and therefore it is a solution of equation (1).

Now, to show that $\phi(x)$ is the only solution, let $\bar{\phi}(x)$ is also a continuous solution of (1), hence:

$$\left|\phi(x)-\bar{\phi}(x)\right| \le \left|\frac{\lambda}{\mu}\right| \int_0^x |k(x,t)| \left|\gamma(t,\phi(t))-\gamma\left(t,\bar{\phi}(t)\right)\right| dt \tag{16}$$

With the aid of conditions(3-ii), the equation (15) then

$$\left\|\phi(x)-\bar{\phi}(x)\right\| \le \alpha_1 \left\|\phi(t)-\bar{\phi}(t)\right\|, \qquad \alpha_1 = \frac{N^*}{|\mu|}\{|\lambda| B\} < 1 \tag{17}$$

Since $\alpha_1 < 1$, then the inequality (17) is true only if $\phi(x) = \bar{\phi}(x)$ which is the solution of (1).

3. Runge- Kutta method

Consider the nonlinear Volterra integral equation of the second kind

$$\phi(x) = f(x) + \int_a^x k\left(x,t,\phi(t)\right) dt, \qquad x \ge 0 \tag{18}$$

with the continuous kernel, and the solution $k\left(x,t,\phi(t)\right)$ exist uniquely and satisfies $\left|k\left(x,t,\phi(t_1)\right)-k\left(x,t,\phi(t_2)\right)\right| \le M\,\left|\phi(t_1)-\phi(t_2)\right|$ let $x_n = a+nh,\ n=0,1,\ldots,N$, with $h=\frac{b-a}{N}, (N \ge 1).$

$$F_n(x) = f(x) + \int_a^{x_n} k\left(x,t,\phi(t)\right) dt, \qquad x \ge x_n, (n=0,1,\ldots,N) \tag{19}$$

And let $\tilde{F}_n(x)$ be approximation to $F_n(x)$.

$$\tilde{F}_n(x_n+\phi_i h) = h\sum_{j=1}^{m} A_{ij} k\left(x_n+\phi_j h, t_n+\phi_j h, \tilde{F}_n(x_n+\phi_j h)\right), F(0)=0 \tag{20}$$

$(i=1,\ldots,m)$.

Where $\{\phi_i\}$ satisfied $0=\phi_1 \le \phi_2 \le \ldots \le \phi_m \le 1$ and we will assume that

$$\phi_i = \sum_{j=1}^{m} A_{ij}, \qquad i=1,2,\ldots,m \tag{21}$$

Where A_{ij} are the weights.

The number $\tilde{F}_n(\phi_i h)$ is the required $O(h^{m+1})$ approximation to $F(\phi_i h)$ for $m \le 4$.

For $i=4$ we get:

$$\phi_1 = \phi_2 = \frac{1}{2}, \phi_3 = \phi_4 = 1$$

$$A_{10} = A_{21} = \frac{1}{2}, A_{20} = A_{30} = A_{31} = 0, A_{32} = 1$$

$$A_{40} = A_{43} = \frac{1}{6}, A_{41} = A_{42} = \frac{1}{3}$$

Suppose that

$$k(x,t,\phi(t)) = \sum_s \left(u_s(x)v_s(t), \phi(t)\right) \quad (22)$$

Then compensation in the integral equation

$$\phi(x) = f(x) + \int_a^x \left(\sum_s \left(u_s(x)v_s(t), \phi(t)\right) \right) dt$$

$$= f(x) + \sum_s u_s(x) \int_a^x \left(v_s(t), \phi(t)\right) dt$$

$$\phi(x) = f(x) + \sum_s u_s(x) F_s(x)$$

Where $F_s(x) = \int_a^x \left(v_s(t), \phi(t)\right) dt$ (23)

$$F_s'(x) = \left(v_s(t), \phi(t)\right), F_s(0) = 0 \quad (24)$$

Now, we apply the Runge- Kutta to (20) we get:

$$\tilde{F}_s(x_n + \phi_i h) = h \sum_{j=1}^m A_{ij} \left(v_s(x_n + \phi_j h), \tilde{\phi}(x_n, \phi_j h)\right), \qquad i = 1, \ldots, m$$

$$\tilde{\phi}(x_n + \phi_i h) = f(x_n + \phi_i h) + \sum_s u_s(x_n + \phi_i h) \tilde{F}_s(x_n + \phi_i h)$$

$$= f(x_n + \phi_i h) + h \sum_{j=1}^m A_{ij} k\left(x_n + \phi_i h, x_n + \phi_j h, \tilde{\phi}(x_n, \phi_j h)\right), \qquad i = 1, \ldots, m$$

In this way we obtain $\tilde{\phi}(\phi_i h)$ as the approximation to $\phi(\phi_i h)$.

We now state in full Pouzet'sversinon in the case $m = 4$ for the general nonlinear equation.

$$p_j(x_j) = f_j$$

$$q_j(x) = F_j\left(x_{j+\frac{1}{2}}\right) + \frac{1}{2}hk\left(x_{j+\frac{1}{2}}, x_j, p_j\right)$$

$$r_j(x) = F_j\left(x_{j+\frac{1}{2}}\right) + \frac{1}{2}hk\left(x_{j+\frac{1}{2}}, x_{j+\frac{1}{2}}, q_j\right)$$

$$s_j(x) = F_j\left(x_{j+1}\right) + hk\left(x_{j+1}, x_{j+1}, r_j\right)$$

$$\phi_j(x) = F_j\left(x_{j+1}\right) + \frac{h}{6}\left\{k\left(x_{j+1}, x_j, p_j\right) + 2k\left(x_{j+1}, x_{j+\frac{1}{2}}, q_j\right) + \right.$$

$$\left. +2k\left(x_{j+1}, x_{j+\frac{1}{2}}, r_j\right) + k\left(x_{j+1}, x_{j+1}, s_j\right)\right\}, (F_0(x) = f(x))$$

4. Block by block method:

Consider the nonlinear Volterra integral equation of the second kind.

$$\phi(x) = f(x) + \lambda\int_0^x k(x,t,\phi(t))dt \qquad (25)$$

Where the function $f(x)$ and $k(x,t,\phi(t))$ are given, we shall assume that $f(x)$ is continuous and satisfies $|f(x)| < M$ and $k(x,t,\phi(t))$ satisfies a uniform Lipschitz condition.

The idea behind the block-by-block method is to divide the interval $[0,x]$ into a mesh $0 = x_0 < x_1 < x_2 < \ldots < x_n < \ldots < x_N = x$, and then we try to evaluate the value of the unknown function $\phi(x)$ at these points except at $x = 0$, where we have that $\phi(0) = f(0)$.

Using any known rule, say Simpson's rule, we have:

$$\phi(x_2) = f(x_2) + \lambda\frac{h}{3}\left\{k\left(x_2, x_0, \phi(x_0)\right) + \right.$$

$$\left. +4k\left(x_2, x_1, \phi(x_1)\right) + k\left(x_2, x_2, \phi(x_2)\right)\right\} \qquad (26)$$

To obtain a value for $\phi(x_1)$ we introduce the point $x_{1/2} = h/2$, and then we use Simpson's rule again to obtain

$$\phi(x_1) = f(x_1) + \lambda\frac{h}{3}\left\{k\left(x_1, x_0, \phi(x_0)\right) + \right.$$

$$\left. +4k\left(x_1, x_{1/2}, \phi(x_{1/2})\right) + k\left(x_1, x_1, \phi(x_1)\right)\right\} \qquad (27)$$

Replacing the $\phi(x_{1/2})$ by a quadratic interpolation using the value ϕ_0 , ϕ_1 and ϕ_2 ,we have

$$\phi\left(x_{1/2}\right)=\frac{3}{8}\phi(x_0)+\frac{3}{4}\phi(x_1)-\frac{1}{8}\phi(x_2) \tag{28}$$

So that we can compute $\phi(x_1)$ by

$$\phi(x_1)=f\,(x_1)+\lambda\frac{h}{3}\left\{k\left(x_1,x_0,\phi(x_0)\right)+\right.$$

$$\left.+4k\left(x_1,x_{1/2},\left(3/8\,\phi(x_0)+3/4\,\phi(x_1)-1/8\,\phi(x_2)\right)\right)+k\left(x_1,x_1,\phi(x_1)\right)\right\} \tag{29}$$

Equations (26) and (29) are a pair of simultaneous equations for $\phi(x_1)$ and $\phi(x_2)$. For sufficiently small h, $\phi(x_1)$ and $\phi(x_2)$ can be found uniquely using any procedure such Netwon's method.

In general, for $m=0,1,\ldots,N-1$, the approximate solution of (25) is evaluated using the following two equations

$$\phi\left(x_{2m+1}\right)=f\,\left(x_{2m+1}\right)+\lambda h\sum_{i=0}^{2m}w_i\,k\left(x_{2m+1},x_i,\phi(x_i)\right)+\frac{h}{6}\left\{k\left(x_{2m+1},x_{2m},\phi(x_{2m})\right)+\right.$$
$$\left.+4k\left(x_{2m+1},x_{2m+1/2},\left(3/8\,\phi(x_{2m})+3/4\,\phi(x_{2m+1})-1/8\,\phi(x_{2m+2})\right)\right)+k\left(x_{2m+1},x_{2m+1},\phi(x_{2m+1})\right)\right\}$$

$$\phi\left(x_{2m+2}\right)=f\,\left(x_{2m+2}\right)+\lambda h\sum_{i=0}^{2m}w_i\,k\left(x_{2m+2},x_i,\phi(x_i)\right)+\frac{h}{3}\left\{k\left(x_{2m+2},x_{2m},\phi(x_{2m})\right)+\right.$$
$$\left.+4k\left(x_{2m+2},x_{2m+1},\phi(x_{2m+2})\right)+k\left(x_{2m+2},x_{2m+2},\phi(x_{2m+2})\right)\right\}$$

Where

$$w_i=\frac{1}{3}\{1,4,2,\ldots,2,4,1\},\qquad i=0,1,\ldots,m$$

$$x_{2m+1/2}=x_{2m}+\frac{h}{2}$$

5. Numerical Experiments and Discussions:

Example 1:

Consider the Non- linear Volterra integral equation:

$$\phi(x)=x+\frac{1}{5}x^5-\int_0^x t\left(\phi(t)\right)^3dt \tag{30}$$

where the exact solution is $\phi(x) = x$ and $0 \le x \le 1$, here $\lambda = -1$, $\mu = 1$. In table (5.1)-(5.2) we present the exact solution, the approximate numerical solutions andtheir corresponding errors for some points, we suppose that $N = 50, 80$.

In tables (5.1)-(5.4):

$\phi^{R.K} \rightarrow$ approximate solution of R. KM, $E^{R.K} \rightarrow$ the error of R. KM, $\phi^{B.B} \rightarrow$ approximate solution of B by BM and $E^{B.B} \rightarrow$ the error of B by BM .

Case 1: $N = 50$,

x	Exact sol.	$\phi^{R.K}$	$E^{R.K}$	$\phi^{B.B}$	$E^{B.B}$
0	0	0	0	0	0
0.1	0.100000	0.0997267933	2.7320666×10^{-4}	0.09983750649	1.6249351×10^{-4}
0.2	0.200000	0.1992790511	7.209489×10^{-4}	0.19972679334	5.813662×10^{-4}
0.3	0.300000	0.2942789348	5.7210652×10^{-3}	0.2983235545	1.6764455×10^{-3}
0.4	0.400000	0.3903370122	9.6629878×10^{-3}	0.3969725972	3.0274028×10^{-3}
0.5	0.500000	0.4903037547	9.6962453×10^{-3}	0.4952277514	4.7722486×10^{-3}
0.6	0.600000	0.5703260241	2.9673975×10^{-2}	0.5930905809	6.9094191×10^{-3}
0.7	0.700000	0.6615191696	3.8480830×10^{-2}	0.6905626412	9.4373588×10^{-3}
0.8	0.800000	0.7396378531	6.0362146×10^{-2}	0.7876454799	1.2354520×10^{-2}
0.9	0.900000	0.8344427662	6.5557233×10^{-2}	0.8843406360	1.5659364×10^{-2}
1	1.000000	0.9055801198	9.4419880×10^{-2}	0.9806496407	1.9350359×10^{-2}

Table(5.1)

Case 2: $N = 80$,

x	Exact sol.	$\phi^{R.K}$	$E^{R.K}$	$\phi^{B.B}$	$E^{B.B}$
0	0	0	0	0	0
0.125	0.1250000	0.1248237317	1.762683×10^{-4}	0.1248588618	1.411382×10^{-4}
0.250	0.2500000	0.2492586214	7.413786×10^{-4}	0.2496358708	3.641292×10^{-4}
0.375	0.3750000	0.3777039148	2.703914×10^{-3}	0.3733052315	1.694768×10^{-3}
0.500	0.5000000	0.4950865375	4.913462×10^{-3}	0.4969647705	3.035229×10^{-3}
0.625	0.6250000	0.6200232157	4.976784×10^{-3}	0.6202384419	4.761558×10^{-3}
0.750	0.7500000	0.7428903601	7.109639×10^{-3}	0.7431274445	6.872555×10^{-3}
0.875	0.8750000	0.8651354304	9.864569×10^{-3}	0.8656329718	9.367028×10^{-3}
1	1.0000000	0.9282383052	7.176169×10^{-2}	0.9877562124	1.224378×10^{-2}

Table(5.2)

Example 2:

Consider the Non- linear Volterra integral equation:

$$\phi(x) = \sin x + \frac{x\,(1-\cos 2x)}{16} + \frac{x^2\,(x-\sin 2x)}{8} - \int_0^x \frac{xt}{2}\left(\phi(t)\right)^2 dt \qquad (31)$$

where the exact solution is $\phi(x) = \sin x$ and $0 \le x \le 1$, here $\lambda = -1$, $\mu = 1$. In table (5.3)-(5.4) we present the exact solution, the approximate numerical solutions andtheir corresponding errors for some points, we suppose that $N = 50, 80$.

Case 1: $N = 50$,

x	Exact sol	$\phi^{R.K}$	$E^{R.K}$	$\phi^{B.B}$	$E^{B.B}$
0	0	0	0	0	0
0.1	0.0998334166	0.0998470697	1.36531×10^{-5}	0.0998252949	8.12173×10^{-6}
0.2	0.1986693308	0.1987270218	5.76910×10^{-5}	0.1985975165	7.18143×10^{-5}
0.3	0.2955202067	0.2963744738	8.54267×10^{-4}	0.2952714708	2.48735×10^{-4}
0.4	0.3894183423	0.3913362225	1.91788×10^{-3}	0.3888257187	5.92623×10^{-4}
0.5	0.4794255386	0.4839544320	4.52889×10^{-3}	0.4782741785	1.15136×10^{-3}
0.6	0.5646424734	0.5733495071	8.70703×10^{-3}	0.5626772979	1.96517×10^{-3}
0.7	0.6442176972	0.6570307644	1.28130×10^{-2}	0.6411526614	3.06502×10^{-3}
0.8	0.7173560909	0.7403396458	2.29835×10^{-2}	0.7128849044	4.47118×10^{-3}
0.9	0.7833269096	0.8102229618	2.68960×10^{-2}	0.7771348129	6.19209×10^{-3}
1	0.8414709848	0.8878602256	4.63892×10^{-2}	0.8332475625	8.22348×10^{-3}

Table(5.3)

Case 2: $N = 80$,

x	Exact sol	$\phi^{R.K}$	$E^{R.K}$	$\phi^{B.B}$	$E^{B.B}$
0	0	0	0	0	0
0.125	0.1246747334	0.1246835292	8.7958×10^{-6}	0.1246737330	1.0004×10^{-6}
0.250	0.2474039593	0.2476985837	2.9462×10^{-4}	0.2473120443	9.1915×10^{-5}
0.375	0.3662725291	0.3676282777	1.3557×10^{-3}	0.3659611541	3.1137×10^{-4}
0.500	0.4794255385	0.4830800698	3.6545×10^{-3}	0.4786949777	7.3056×10^{-4}
0.625	0.5850972724	0.5926912852	7.5940×10^{-3}	0.5836975569	1.3997×10^{-3}
0.750	0.6816387600	0.6951347139	1.3495×10^{-2}	0.6792836827	2.3550×10^{-3}
0.875	0.7675434022	0.7891248129	2.1581×10^{-2}	0.7639272915	3.6162×10^{-3}
1	0.8414709848	0.8734243449	3.1953×10^{-2}	0.8362869287	5.1840×10^{-3}

Table(5.4)

6. The Conclusion:

From the previous discussions we conclude the following:

1) As x is increasing in interval $[0,1]$, the errors due to Runge-Kutta and Block-by-block methods are also increasing.
2) As N is increasing, the errors are decreasing in the Runge-Kutta and Block-by-block methods.
3) The error in the evaluation of the approximate solution, using the Block-by-block method, is less than the error in the evaluation of the approximate solution, using the Runge-Kutta method, in all cases of the two examples.
4) The stability of the Block-by-block method more than the Runge-Kutta method.

References:

[1] C. T. H. Baker, H. Geoffrey, F. Miller, Treatment of Integral Equations by Numerical Methods, Acad. Press, 1982.

[2] L. M. Delves and J. L. Mohamed, Computational Methods for Integral Equations, Cambridge, 1985.

[3] K. E. Atkinson, A Survey of Numerical Method for the Solution of Fredholm Integral Equation of the Second Kind, Philadelphia, 1976.

[4] K. E. Atkinson, The Numerical Solution of Integral Equation of the Second Kind, Cambridge, 1997.

[5] M. A. Golberg. ed, Numerical Solution of Integral Equations, Boston, 1990.

[6] A.A. Badr, Block-by-Block Method for Solving NonLinearVolterra-Fredholm Integral Equation, Mathematical Problems in Engineering, 537909 (2010) 8.

[7] R. Katani and S. Shahmorad, A New Block-by-Block Method for Solving Two-Dimensional Linear and NonLinearVolterra Integral Equations of The First and Second Kinds, Bulletin of the Iranian Mathematical Society, Vol. 39, No. 4 (2013), pp 707-724.

[8] I. L. El-Kalla and A. M. Al-Bugami, Adomian and Block-by-Block Methods to Solve Nonlinear Two-Dimensional Volterra Integral Equation , AJBAS, 6(3) (2011) 335-340.

[9] F. Mirzaee and Z. Rafei, The Block-by-Block Method for the Numerical Solution of The Nonlinear Two-Dimensional Volterra Integral Equations, Journal of King Saud University-Science, (2011) 23, 191-195.

[10] A. Makroglou, Convergence of Block-by-Block Method for Nonlinear VolterraIntegro-Differential Equations, MATHEMATICS OF COMOUTATION, 151 (1980) 783-796.