



Comparison of Sumudu and Laplace decomposition method for solving fractional Lane-Emden type differential equations

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Abstract

In this work we discuss the solution of delay differential equations of the fractional order of Lane-Emden type equation using Sumudu and Laplace transforms with decomposition method, definitions and some useful theorems are stated.

Keywords: Sumudu transform; Lane-Emden equation of fraction; Caputo fractional derivative; inverse of Sumudu transform.

1. Introduction

The Adomian decomposition method first was introduced by George Adomian [1], the method is powerful and effective in solving a many different equations in mathematical and engineering field. The Lane-Emden type equations published by Janathan Homer Lane in [6], and Emden who extended details of Lane-Emden equation used the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. The Laplace transform of $f(t)$ is defined as a function $F(s)$ by the integral:

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt, \quad 0 < t < \infty$$

Beside the Sumudu transform is defined over the set of a functions :

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

$$\text{by: } F(u) = S[f(t)] = \frac{1}{u} \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt, \quad u \in (-\tau_1, \tau_2)$$

The solution of fractional Lane-Emden type equation by Sumudu decomposition method

In this section we use the technique of Sumudu decomposition method to solve Lane-Emden equation of fraction order.

Theorem 1

Let $f \in A$, then the Sumudu transform integer order derivative of f^n is given by :

$$S [f^n(t)] = \frac{S(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{n-k}} , n \geq 0$$

For more see [3].

Theorem 2

The Sumudu transform of fraction function $t D^\alpha y$ where $n-1 < \alpha \leq n$, $n = 1, 2$.

$$S [t D^\alpha y] = u \frac{d}{du} [u S(D^\alpha y)] , n-1 < \alpha \leq n , n = 1, 2$$

Proof :

$$\begin{aligned} S [D^\alpha y] &= \frac{1}{u} \int_0^\infty e^{-\frac{t}{u}} D^\alpha y dt \\ u S [D^\alpha y] &= \int_0^\infty e^{-\frac{t}{u}} D^\alpha y dt \\ \frac{d}{du} [u S [D^\alpha y]] &= \int_0^\infty \frac{d}{du} e^{-\frac{t}{u}} D^\alpha y dt \\ &= \int_0^\infty \frac{1}{u^2} e^{-\frac{t}{u}} t D^\alpha y dt \\ &= \frac{1}{u} \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} t D^\alpha y dt \\ &= \frac{1}{u} S [t D^\alpha y] \\ u \frac{d}{du} [u S [D^\alpha y]] &= S [t D^\alpha y] . \square \end{aligned}$$

Definition 1

Sumudu transform of the Caputo fractional derivative is defined as follows:

$$S [D^\alpha f(t)] = \frac{S[f(t)]}{u^\alpha} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{\alpha-k}} , n-1 < \alpha \leq n$$

Analysis of the method

Here we consider the Lane-Emden equation of fraction of the form:

$$D^\alpha y(t) + \frac{2n}{t} y'(t) + \frac{n(n-1)}{t^2} y(t) + f(t, y) = g(t) , n = 1, 2 \tag{1}$$

With initial condition:

$$y(0) = A , y'(0) = B$$

Where f is a real function, g is a known function, A and B are constant, if $n = 1$ we obtain:

$$D^\alpha y(t) + \frac{2}{t} y'(t) + f(t, y) = g(t) \quad (2)$$

Multiplying equation (2) with t and applying the Sumudu transform we get:

$$S [t D^\alpha y(t) + 2 y'(t) + t f(t, y)] = S [t g(t)] \quad (3)$$

By using the theorem (2) we have:

$$u \frac{d}{du} [u S(D^\alpha y)] + 2 \left[\frac{Y(u) - y(0)}{u} \right] + S [t f(t, y)] = S [t g(t)]$$

$$u \frac{d}{du} \left[u \left(\frac{Y(u) - y(0)}{u^\alpha} - \frac{y'(0)}{u^{\alpha-1}} \right) \right] + 2 \left[\frac{Y(u) - y(0)}{u} \right] + S [t f(t, y)] = S [t g(t)] \quad (4)$$

Simplifying equation (4) we obtain:

$$\frac{d}{du} \left[\frac{Y(u) - y(0)}{u^{\alpha-1}} \right] + \frac{2Y(u)}{u^2} - \frac{2y(0)}{u^2} + \frac{1}{u} S [t f(t, y)] = \frac{1}{u} S [t g(t)] \quad (5)$$

By integrating equation (5) with respect to u we get:

$$\frac{Y(u) - y(0)}{u^{\alpha-1}} = \int_0^u \frac{1}{u} S [t g(t)] du + \int_0^u \frac{2y(0)}{u^2} du - \int_0^u \frac{2Y(u)}{u^2} du - \int_0^u \frac{1}{u} S [t f(t, y)] du,$$

we arrange the above equation, we have

$$Y(u) = y(0) + u^{\alpha-1} \int_0^u \frac{1}{u} S [t g(t)] du + u^{\alpha-1} \int_0^u \frac{2y(0)}{u^2} du - u^{\alpha-1} \int_0^u \frac{2Y(u)}{u^2} du - u^{\alpha-1} \int_0^u \frac{1}{u} S [t f(t, y)] du \quad (6)$$

Where $f(t, y)$ is defined as follow:

$$f(t, y) = K(y(t)) + M(y(t)) \quad (7)$$

We define the solution of $y(t)$ and non-linear function $f(t, y)$ by infinite series as

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \quad , \quad f(t, y) = \sum_{n=0}^{\infty} A_n(t) \quad (8)$$

Respectively, where A_n is defined by the following formula:

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} M \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \right]_{\lambda=0} \quad , \quad n = 0, 1, 2, \dots \quad (9)$$

Equation (9) is given by:

$$A_0 = M[u_0] \quad , \quad A_1 = u_1 M'[u_0]$$

$$A_2 = u_2 M'[u_0] + \frac{1}{2!} u_1^2 M''[u_0]$$

$$A_3 = u_3 M'[u_0] + u_1 u_2 M''[u_0] + \frac{1}{3!} u_1^3 M'''[u_0]$$
(10)

Substituting equation (8) in to equation (6) yield to :

$$S \left[\sum_{n=0}^{\infty} y_n(t) \right] = y(0) + u^{\alpha-1} \int_0^u \frac{1}{u} S [t g(t)] du + u^{\alpha-1} \int_0^u \frac{2y(0)}{u^2} du - u^{\alpha-1} \int_0^u \frac{2}{u^2} S \left[\sum_{n=0}^{\infty} y_n(t) \right] du - u^{\alpha-1} \int_0^u \frac{1}{u} (S [t k y_n(t) + t A_n(t)]) du \quad (11)$$

$$S [y_0(t)] = y(0) + u^{\alpha-1} \int_0^u \frac{1}{u} S [t g(t)] du \quad (12)$$

By taking inverse of Sumudu transform of (6) we have :

$$y_0(t) = S^{-1} \left[y(0) + u^{\alpha-1} \int_0^u \frac{1}{u} S [t g(t)] du \right] \quad (13)$$

$$y_{n+1}(t) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2y(0)}{u^2} du - u^{\alpha-1} \int_0^u \frac{2}{u^2} S [y_n(t)] - u^{\alpha-1} \int_0^u \frac{1}{u} (S [t k y_n(t) + t A_n(t)]) du \right] \quad (14)$$

We assume the inverse Sumudu transform exist for each terms in the right hand side of equation (14).

Numerical examples:

Example 1

Consider the singular fractional Lane-Emden equation:

$$D^\alpha y + \frac{2}{t} y' = -y \quad , \quad 1 < \alpha \leq 2 \quad (15)$$

With the initial:

$$y(0) = 1 \quad , \quad y'(0) = 0 \quad (16)$$

Multiplying both side of equation (15) by t we have:

$$t D^\alpha y + 2y' = -ty \quad (17)$$

On using Sumudu transform to equation (17) :

$$S [t D^\alpha y] + S [2y'] = S [-ty]$$

Applying definition 1 and theorem 2 we get:

$$u \frac{d}{du} [u S (D^\alpha y)] + 2 \left[\frac{Y(u) - y(0)}{u} \right] = S [-ty]$$

$$u \frac{d}{du} \left[u \left(\frac{Y(u) - y(0)}{u^\alpha} - \frac{y'(0)}{u^{\alpha-1}} \right) \right] + 2 \left[\frac{Y(u) - y(0)}{u} \right] = S [-t y]$$

Apply the initial condition:

$$u \frac{d}{du} \left[\left(\frac{Y(u) - 1}{u^{\alpha-1}} \right) \right] + 2 \left[\frac{Y(u) - 1}{u} \right] = S [-t y]$$

$$\frac{d}{du} \left[\left(\frac{Y(u)-1}{u^{\alpha-1}} \right) \right] + 2 \left[\frac{Y(u)-1}{u^2} \right] = \frac{1}{u} S[-ty],$$

$$\frac{d}{du} \left[\frac{Y(u)-1}{u^{\alpha-1}} \right] = \frac{1}{u} S[-ty] + \frac{2}{u^2} - \frac{2Y(u)}{u^2}, \quad (18)$$

Integrating both side of equation (18) with respect to u we get :

$$\frac{Y(u)-1}{u^{\alpha-1}} = \int_0^u \frac{2}{u^2} du + \int_0^u \frac{1}{u} S[-ty] du - \int_0^u \frac{2Y(u)}{u^2} du$$

$$Y(u)-1 = u^{\alpha-1} \int_0^u \frac{2}{u^2} du + u^{\alpha-1} \int_0^u \frac{1}{u} S[-ty] du - u^{\alpha-1} \int_0^u \frac{2Y(u)}{u^2} du$$

$$Y(u) = 1 + u^{\alpha-1} \int_0^u \frac{2}{u^2} du + u^{\alpha-1} \int_0^u \frac{1}{u} S[-ty] du - u^{\alpha-1} \int_0^u \frac{2Y(u)}{u^2} du, \quad (19)$$

by taking inverse of Sumudu transform of (19) we have:

$$y(t) = S^{-1} [1] + S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} du \right] du + S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-ty] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} Y(u) du \right]$$

$$y_0 = S^{-1} [1] = 1$$

$$y_{n+1}(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} du \right] du + S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-ty_n] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S(y_n) du \right] \quad (20)$$

By substituting $n=0$ in equation (20) we have:

$$y_1(t) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} du \right] du + S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-ty_0] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S(y_0) du \right]$$

$$y_1(t) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} (-u) du \right] = S^{-1} [-u^\alpha] = \frac{-t^\alpha}{\Gamma(\alpha+1)}$$

By substituting $n=1$ in equation (20) we have:

$$y_2(t) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-ty_1] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S(y_1) du \right]$$

$$y_2 = S^{-1} \left[u^{\alpha-1} \left(\frac{u^{\alpha+1} \Gamma(\alpha+2)}{(\alpha+1) \Gamma(\alpha+1)} \right) \right] + S^{-1} \left[\frac{2u^{2\alpha-2}}{(\alpha-1)} \right] = \frac{t^{2\alpha} \Gamma(\alpha+2)}{(\alpha+1) \Gamma(\alpha+1) \Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1) \Gamma(2\alpha-1)}$$

Again substituting $n=2$ in equation (20) we have:

$$y_3(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-ty_2] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S(y_2) du \right]$$

$$y_3(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S \left[-t \left(\frac{t^{2\alpha} \Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)} \right) du \right] \right. \\ \left. - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S \left(\frac{t^{2\alpha} \Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)} \right) du \right] \right]$$

$$y_3 = \frac{-t^{3\alpha} \Gamma(\alpha+2) \Gamma(2\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} + \frac{-2t^{3\alpha-2} \Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)} \\ - \frac{2t^{3\alpha-2} \Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)(2\alpha-1)\Gamma(3\alpha-1)} - \frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

The series solution is given by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \dots$$

$$y(t) = 1 + \frac{-t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha} \Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)} \\ + \frac{-t^{3\alpha} \Gamma(\alpha+2) \Gamma(2\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} + \frac{-2t^{3\alpha-2} \Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)} \\ - \frac{2t^{3\alpha-2} \Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)(2\alpha-1)\Gamma(3\alpha-1)} - \frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

In particular case $\alpha = 2$ then we have:

$$y(t) = 1 - \frac{3t^2}{2} - \frac{5t^4}{3 \times 4!} - \frac{t^6}{6!} - \dots$$

Example 2

Consider the fractional nonlinear Lane-Emden equation:

$$D^\alpha y + \frac{2}{t} y' = -e^y \quad , \quad 1 < \alpha \leq 2 \tag{21}$$

With the initial:

$$y(0) = 0 \quad , \quad y'(0) = 0 \tag{22}$$

By using the same method in example 1 we have:

$$u \frac{d}{du} \left[u \left(\frac{Y(u) - y(0)}{u^\alpha} - \frac{y'(0)}{u^{\alpha-1}} \right) \right] + 2 \left[\frac{Y(u) - y(0)}{u} \right] = S \left[-t e^y \right]$$

Apply the initial condition:

$$u \frac{d}{du} \left[\left(\frac{Y(u)}{u^{\alpha-1}} \right) \right] + 2 \left[\frac{Y(u)}{u} \right] = S \left[-t e^y \right]$$

$$\frac{d}{du} \left[\left(\frac{Y(u)}{u^{\alpha-1}} \right) \right] + 2 \left[\frac{Y(u)}{u^2} \right] = \frac{1}{u} S \left[-t e^y \right]$$

$$\frac{d}{du} \left[\frac{Y(u)}{u^{\alpha-1}} \right] = \frac{1}{u} S \left[-t e^y \right] - \frac{2Y(u)}{u^2} \tag{23}$$

Integrating both side of equation (23) with respect to u we get:

$$\frac{Y(u)}{u^{\alpha-1}} = \int_0^u \frac{1}{u} S[-t e^y] du - \int_0^u \frac{2Y(u)}{u^2} du$$

$$Y(u) = u^{\alpha-1} \int_0^u \frac{1}{u} S[-t e^y] du - u^{\alpha-1} \int_0^u \frac{2Y(u)}{u^2} du \quad (24)$$

Applying inverse of Sumudu transform of equation (24) we obtain:

$$y(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-t e^y] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} Y(u) du \right]$$

$$y_0 = 0$$

$$y_{n+1}(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-t A_n] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S(y_n) du \right] \quad (25)$$

Substituting $n = 0$ in equation (25) we have:

$$y_1(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-t A_0] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S(y_0) du \right] \quad (26)$$

By using equation (10), we have :

$$A_0 = M[y_0] = y_0^5$$

$$A_1 = y_1 M'[y_0] = y_1 y_0'$$

$$A_2 = y_1 y_2 M'[y_0] + \frac{1}{2!} y_1^2 M''[y_0] = y_1 y_0' + \frac{1}{2} y_1^2 y_0''$$

$$A_3 = y_3 M'[y_0] + y_1 y_2 M''[y_0] + \frac{1}{3!} y_1^3 M'''[y_0] = y_3 y_0' + y_1 y_2 y_0'' + \frac{1}{3!} y_1^3 y_0'''$$

Apply this to equation (26) we get :

$$y_1(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-t (e^{y_0})] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} (0) du \right]$$

$$y_1(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} (-u) du \right] = S^{-1} [-u^\alpha] = \frac{-t^\alpha}{\Gamma(\alpha+1)}$$

Substituting $n = 1$ in equation (25) we have :

$$y_2(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-t A_1] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S(y_1) du \right]$$

$$y_2(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-t (y_1 e^{y_0})] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S\left(\frac{-t^\alpha}{\Gamma(\alpha+1)}\right) du \right]$$

$$y_2(u) = \frac{t^{2\alpha} \Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$

By substituting $n = 2$ in equation (25) we get :

$$y_3(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S[-t A_2] du \right] - S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S(y_2) du \right]$$

$$y_3(u) = S^{-1} \left[u^{\alpha-1} \int_0^u \frac{1}{u} S \left[-t \left(y_2 + \frac{1}{2} y_1^2 \right) e^{y_0} \right] du \right]$$

$$- S^{-1} \left[u^{\alpha-1} \int_0^u \frac{2}{u^2} S \left(\frac{t^{2\alpha} \Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)} \right) du \right]$$

$$y_3(u) = \frac{-t^{3\alpha} \Gamma(2\alpha+2)\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} - \frac{2t^{3\alpha-2} \Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$

$$- \frac{1}{2} \frac{t^{3\alpha} \Gamma(2\alpha+2)}{[\Gamma(\alpha+1)]^2 (2\alpha+1)\Gamma(3\alpha+1)} - \frac{2t^{3\alpha-2} \Gamma(\alpha+2)}{(\alpha+1)(2\alpha-1)\Gamma(\alpha+1)\Gamma(3\alpha-1)}$$

$$- \frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

The series solution is given by :

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \dots$$

$$y(t) = 1 + \frac{-t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha} \Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$

$$- \frac{t^{3\alpha} \Gamma(2\alpha+2)\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} - \frac{2t^{3\alpha-2} \Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$

$$- \frac{1}{2} \frac{t^{3\alpha} \Gamma(2\alpha+2)}{[\Gamma(\alpha+1)]^2 (2\alpha+1)\Gamma(3\alpha+1)} - \frac{2t^{3\alpha-2} \Gamma(\alpha+2)}{(\alpha+1)(2\alpha-1)\Gamma(\alpha+1)\Gamma(3\alpha-1)}$$

$$- \frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

In particular case $\alpha = 2$ then we have :

$$y(t) = \frac{-3t^2}{2} + \frac{-5t^4}{3 \times 4!} + \frac{-4t^6}{6!} + \dots$$

Solution Of fractional Lane-Emden type equation by Laplace decomposition method

In this section we discuss the solution of fractional Lane-Emden differential equation type by Laplace decomposition method and compare the solution with that by Sumudu transform.

In the following theorem we discuss the Laplace transform of non-constant fractional derivative.

Theorem 3

$$L[t D^\alpha y] = -\frac{d}{ds} L[D^\alpha y] \quad , \quad n-1 < \alpha \leq n \quad , \quad n = 1, 2, 3, \dots$$

Proof:

$$L[D^\alpha y] = \frac{1}{u} \int_0^\infty e^{-st} D^\alpha y dt$$

$$\frac{d}{ds} L[D^\alpha y] = \frac{d}{ds} \int_0^\infty e^{-st} D^\alpha y dt$$

$$\frac{d}{ds} L[D^\alpha y] = \int_0^\infty \frac{d}{ds} e^{-st} D^\alpha y dt$$

$$= \int_0^\infty -t e^{-st} D^\alpha y dt$$

$$\frac{d}{ds} L[D^\alpha y] = - \int_0^\infty e^{-st} t D^\alpha y dt$$

$$= -L[t D^\alpha y]$$

$$L[t D^\alpha y] = -\frac{d}{ds} L[D^\alpha y].$$

Definition 2

The Laplace transform of Caputo fractional derivative for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ can be obtained in the form of :

$$L[D^\alpha f(t)] = \frac{s^m F(s) - s^{m-1}f(0) - s^{m-2}f'(0) - \dots - f^{(m-1)}(0)}{s^{m-\alpha}},$$

If $1 < \alpha \leq 2$ then:

$$L[D^\alpha f(t)] = \frac{s^2 F(s) - s f(0) - f'(0)}{s^{2-\alpha}}$$

Example 3

Consider the fractional Lane-Emden equation:

$$D^\alpha y + \frac{2}{t} y' = -y, \quad n-1 < \alpha \leq n \tag{27}$$

With the initial condition:

$$y(0) = 1, \quad y'(0) = 0 \tag{28}$$

Multiplying both side of equation (27) with t , we get :

$$t D^\alpha y + 2y' = -ty \tag{29}$$

Applying Laplace transform to equation (29) we get:

$$L[t D^\alpha y] + L[2y'] = L[-ty] \tag{30}$$

On using theorem 3 and definition 2 we have :

$$-\frac{d}{ds} [L(D^\alpha y)] + 2[sY(s) - y(0)] = L[-ty]$$

$$-\frac{d}{ds} \left[\frac{sY(s) - 1}{s^{1-\alpha}} \right] + 2[sY(s) - 1] = L[-ty] - \frac{d}{ds} [L(D^\alpha y)] + 2[sY(s) - y(0)] = L[-ty]$$

$$-\frac{d}{ds} \left[\frac{sY(s)-1}{s^{1-s}} \right] = 2sY(s) - 2 - L[-t y] \quad (31)$$

Integrating both side of equation (31) with respect to u , then we have :

$$\begin{aligned} \frac{sY(s)-1}{s^{1-s}} &= \int_0^s 2sY(s) ds - \int_0^s 2 ds - \int_0^s L[-t y] ds \\ sY(s)-1 &= s^{1-\alpha} \int_0^s 2sY(s) ds - s^{1-\alpha} \int_0^s 2 ds - s^{1-\alpha} \int_0^s L[-t y] ds \\ sY(s) &= 1 + s^{1-\alpha} \int_0^s 2sY(s) ds - s^{1-\alpha} \int_0^s 2 ds - s^{1-\alpha} \int_0^s L[-t y] ds \\ Y(s) &= \frac{1}{s} + \frac{1}{s} s^{1-\alpha} \int_0^s 2sY(s) ds - \frac{1}{s} s^{1-\alpha} \int_0^s 2 ds - \frac{1}{s} s^{1-\alpha} \int_0^s L[-t y] ds \end{aligned} \quad (32)$$

By taking inverse of Laplace transform of (32) we

get:

$$\begin{aligned} y(s) &= L^{-1} \left[\frac{1}{s} \right] + L^{-1} \left[s^{-\alpha} \int_0^s 2sY(s) ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s 2 ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t y] ds \right] \\ y_0 &= L^{-1} \left[\frac{1}{s} \right] = 1 \\ y_{n+1} &= L^{-1} \left[s^{-\alpha} \int_0^s 2s L[y_n] ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s 2 ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t y_n] ds \right] \end{aligned} \quad (33)$$

Substituting $n=0$ in equation (33) we get :

$$\begin{aligned} y_1 &= L^{-1} \left[s^{-\alpha} \int_0^s 2s L[y_0] ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s 2 ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t y_0] ds \right] \\ y_1 &= -L^{-1} \left[s^{-\alpha} (s^{-\alpha}) \right] = -L^{-1} \left[\frac{1}{s^{\alpha+1}} \right] = -\frac{t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

Substituting $n=1$ in equation (33) we get :

$$\begin{aligned} y_2 &= L^{-1} \left[s^{-\alpha} \int_0^s 2s L[y_1] ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t y_1] ds \right] \\ y_2 &= L^{-1} \left[s^{-\alpha} \int_0^s -2s \frac{1}{s^{\alpha+1}} ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)} \frac{1}{s^{\alpha+2}} ds \right] \\ y_2 &= \frac{2}{\alpha-1} \cdot \frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)} \cdot \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{aligned}$$

Substituting $n=2$ in equation (33) we get :

$$y_3 = L^{-1} \left[s^{-\alpha} \int_0^s 2s L[y_2] ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t y_2] ds \right]$$

$$y_3 = \frac{4}{(\alpha-1)} \cdot \frac{-1}{(2\alpha-3)} L^{-1} \left[\frac{1}{s^{3\alpha-3}} \right] + \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)} \cdot \frac{-1}{(2\alpha-1)} L^{-1} \left[\frac{1}{s^{3\alpha-1}} \right]$$

$$+ \frac{2\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)} \cdot \frac{1}{(2\alpha-1)} L^{-1} \left[\frac{1}{s^{3\alpha-1}} \right] + \frac{\Gamma(\alpha+2)\Gamma(2\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} \cdot \frac{1}{(2\alpha+1)} L^{-1} \left[\frac{1}{s^{3\alpha+1}} \right]$$

$$y_3 = -\frac{4}{(\alpha-1)} \cdot \frac{1}{(2\alpha-3)} \frac{t^{3\alpha-4}}{\Gamma(3\alpha-3)} - \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)} \cdot \frac{1}{(2\alpha-1)} \frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}$$

$$- \frac{2\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)} \cdot \frac{1}{(2\alpha-1)} \frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{\Gamma(\alpha+2)\Gamma(2\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} \cdot \frac{1}{(2\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

The series solution is given by :

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \dots$$

$$y(t) = 1 + \frac{-t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$

$$+ \frac{-t^{3\alpha}\Gamma(\alpha+2)\Gamma(2\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} + \frac{-2t^{3\alpha-2}\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$

$$- \frac{2t^{3\alpha-2}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)(2\alpha-1)\Gamma(3\alpha-1)} - \frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

In particular case $\alpha = 2$ then we have :

$$y(t) = 1 - \frac{3t^2}{2} - \frac{5t^4}{3 \times 4!} - \frac{t^6}{6!} - \dots$$

Example 4

Consider the singular fractional Lane-Emden equation :

$$D^\alpha y + \frac{2}{t} y' = -e^y, \quad n-1 < \alpha \leq n \tag{34}$$

With the initial condition :

$$y(0) = 0, \quad y'(0) = 0 \tag{35}$$

Multiplying both side of equation (34) with t , we get :

$$t D^\alpha y + 2y' = -t e^y \tag{36}$$

Applying Laplace transform to equation (36) :

$$L[t D^\alpha y] + L[2y'] = L[-t e^y] \tag{37}$$

Using theorem 3 and definition 2 we have :

$$-\frac{d}{ds} [L(D^\alpha y)] + 2[sY(s) - y(0)] = L[-t e^y]$$

$$-\frac{d}{ds} \left[\frac{s^2 Y(s) - sy(0) - y'(0)}{s^{2-s}} \right] + 2[sY(s) - y(0)] = L[-t e^y]$$

$$-\frac{d}{ds} \left[\frac{sY(s)}{s^{1-s}} \right] + 2[sY(s)] = L[-t e^y]$$

$$-\frac{d}{ds} \left[\frac{sY(s)}{s^{1-s}} \right] = 2sY(s) - L[-t e^y] \tag{38}$$

Integrating both side equation (38) with respect to u , then we have :

$$\frac{sY(s)}{s^{1-s}} = \int_0^s 2sY(s) ds - \int_0^s L[-t e^y] ds$$

$$sY(s) = s^{1-\alpha} \int_0^s 2sY(s) ds - s^{1-\alpha} \int_0^s L[-t e^y] ds$$

$$sY(s) = s^{1-\alpha} \int_0^s 2sY(s) ds - s^{1-\alpha} \int_0^s L[-t e^y] ds$$

$$Y(s) = \frac{1}{s} \cdot s^{1-\alpha} \int_0^s 2sY(s) ds - \frac{1}{s} \cdot s^{1-\alpha} \int_0^s L[-t e^y] ds \tag{39}$$

by taking inverse of Laplace transform of equation (39) we get:

$$y(s) = L^{-1} \left[s^{-\alpha} \int_0^s 2sY(s) ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t e^y] ds \right]$$

$$y_0 = 0$$

$$y_{n+1} = L^{-1} \left[s^{-\alpha} \int_0^s 2s L[y_n] ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t A_n] ds \right] \tag{40}$$

Substituting $n=0$ in equation (40) we get :

$$y_1 = L^{-1} \left[s^{-\alpha} \int_0^s 2s L[y_0] ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t A_0] ds \right]$$

$$y_1 = -L^{-1} \left[s^{-\alpha} (s^{-1}) \right] = -L^{-1} \left[\frac{1}{s^{\alpha+1}} \right] = -\frac{t^\alpha}{\Gamma(\alpha+1)}$$

Substituting $n=1$ in equation (40) we get:

$$y_2 = L^{-1} \left[s^{-\alpha} \int_0^s 2s L[y_1] ds \right] - L^{-1} \left[s^{-\alpha} \int_0^s L[-t A_1] ds \right]$$

$$y_2 = L^{-1} \left[s^{-\alpha} \left(\frac{-2}{s^{\alpha-1}} \cdot \frac{-1}{(\alpha-1)} \right) \right] - L^{-1} \left[s^{-\alpha} \int_0^s \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)} \frac{1}{s^{\alpha+2}} ds \right]$$

$$y_2 = \frac{2}{(\alpha-1)} \frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Substituting $n=2$ in equation (40) we get :

$$\begin{aligned}
 y_3 &= -L^{-1} \left[s^{-\alpha} \int_0^s L \left[-t \left(y_2 + \frac{1}{2} y_1^2 \right) \right] ds \right] \\
 &\quad + L^{-1} \left[s^{-\alpha} \int_0^s 2s L \left[\frac{2}{(\alpha-1)\Gamma(2\alpha-1)} \frac{t^{2\alpha-2}}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right] ds \right] \\
 y_3 &= -L^{-1} \left[s^{-\alpha} \int_0^s L \left[-t \left(\left(\frac{2}{(\alpha-1)\Gamma(2\alpha-1)} \frac{t^{2\alpha-2}}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right) + \frac{1}{2} \left[\frac{-t^\alpha}{\Gamma(\alpha+1)} \right]^2 \right) \right] ds \right] \\
 &\quad + L^{-1} \left[s^{-\alpha} \int_0^s 2s L \left[\frac{2}{(\alpha-1)} \cdot \frac{1}{s^{2\alpha-1}} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)} \cdot \frac{1}{s^{2\alpha+1}} \right] ds \right] \\
 y_3 &= \frac{-2\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)} \frac{t^{3\alpha-2}}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} - \frac{\Gamma(\alpha+2)\Gamma(2\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
 &\quad - \frac{1}{2} \frac{\Gamma(2\alpha+2)}{[\Gamma(\alpha+1)]^2(2\alpha+1)\Gamma(3\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)} \frac{t^{3\alpha-4}}{\Gamma(\alpha+1)(\alpha+1)(2\alpha-1)\Gamma(3\alpha-1)} - \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)(2\alpha-1)\Gamma(3\alpha-1)} \frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}
 \end{aligned}$$

The series solution is given by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \dots$$

$$\begin{aligned}
 y(t) &= 1 + \frac{-t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)} \\
 &\quad - \frac{t^{3\alpha}\Gamma(2\alpha+2)\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} - \frac{2t^{3\alpha-2}\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)} \\
 &\quad - \frac{1}{2} \frac{t^{3\alpha}\Gamma(2\alpha+2)}{[\Gamma(\alpha+1)]^2(2\alpha+1)\Gamma(3\alpha+1)} - \frac{2t^{3\alpha-2}\Gamma(\alpha+2)}{(\alpha+1)(2\alpha-1)\Gamma(\alpha+1)\Gamma(3\alpha-1)} \\
 &\quad - \frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}
 \end{aligned}$$

In particular case $\alpha = 2$ then we have:

$$y(t) = \frac{-3t^2}{2} + \frac{-5t^4}{3 \times 4!} + \frac{-4t^6}{6!} + \dots$$

Conclusion:

In this paper we studied tow method for solving fractional of Lane-Emden type equation the two method are very powerful and efficient for solving different kinds of linear and non-linear fractional differential equations in different fields of science and engineering.

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