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Towards a matrix multi-level model of quark-gluon media

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Abstract.

The key feature of the model is an infinite sequence of canonical immersions of groups: U(2) into U(3), U(2) into U(4), and so on. Let us refer to these groups as to levels: U(2) – the 0^{th} level (that is, ours common), U(3) – the 1^{st}, U(4) – the 2^{nd} and so on. Levels relate to (quarks') generations whereas flavor and color are also defined purely mathematically. According to the model, quarks can be interpreted as 'sank' protons (during the beginning of the reaction process, proton (or rather the support U(2) of its wave function) is merely pushed into a deeper level. The model seems to be compatible with detection of point-like constituents within the proton in highly inelastic electron-proton scattering (and with elastic electron-quark scattering). To introduce gluons, we deal with proton-antiproton pairs (tensor product). At each level, a gluon can be interpreted as a colored photon. Not each and every feature of the model coincides with the corresponding standard assumption about quarks and gluons. In particular, the total number of colors is level-dependent. The model predicts three new quarks (of the 4^{th} generation).

Key words: Principal U(2) into U(n) immersions; multi-level quarks and gluons.

Mathematics Subject Classification (2000): 15A30, 15A90.

1. Introduction and Background

According to Segal (Irving E. Segal, 1918-1998), the global fractional linear conformal SU(2,2)-action on U(2) is a fundamental ingredient in order to determine the list of all spin ½ elementary particles in such a space-time. There are four of those and they differ by an order in which they (or rather their representation spaces) enter the composition series: $p<\nu_m<\nu_e<e$ (see [1], [2], [3]). This is where the proton's stability comes from: proton's states belong to the SU(2,2)-invariant subspace. These four particles have been mathematically deduced in terms of a certain induced representation of G = SU(2,2). The relation to a standard list of spin ½ particles was traced on the basis of: 1) the stationary subgroup of the above G is isomorphic to the Poincare group (scaling included), and 2) the Minkowski space-time is canonically embedded into U(2). We do not provide definitions of physics-related terminology (used in this paragraph) since below we mostly deal with the mathematics of the multi-level model. In particular, **flavor** and **color** (of quarks) will be defined mathematically. This alone might be of significant interest to modern physics. However, it might take a while before other mathematical features of the model will be figured out and compared to their counterparts in physics. That is why it is probably safer (for now) to call it *a toy model* of the quark-gluon media.

2. Description of the Model and Definition of Flavor

As it has been mentioned in the summary, the key feature of the model is an infinite sequence of canonical embeddings of groups: U(2) into U(3), U(2) into U(4), and so on. Call them **levels** (of matter): U(2) – the 0^{th} level (that is, our common), U(3) – the 1^{st}, U(4) – the 2^{nd} and so on. Such a convention matches the standard quarks' generations list. It is worth mentioning that one could have started with SU(2) into SU(n) embeddings since (given a particular embedding of SU(2) into SU(n)) the corresponding embedding of U(2) into U(n) emerges when one merely introduces an extra multiple (a complex number of length one). It is exactly this extra variable which plays the role of time in the U(2) space-time model (*Segal's compact cosmos*, see [3]). U(2) is thus the support of the proton's wave function (in other words, proton is initially on the U(2) level) whereas the totality of its possible states is described by the Segal's machinery above mentioned.

The other main tenets of our multi-level model are introduced below. The model seems to be compatible with detection of point-like constituents within the proton in highly inelastic electron-proton scattering (see [4]). Here the role of a quark is played by the original proton, itself. In particular, having been pushed from the 0^{th} level to a "deeper" one, the proton there becomes **flavored** and **colored** (below each of these two notions is introduced purely mathematically).

Let us first describe embeddings under which each matrix Z from D=U(2) becomes a principal minor of the corresponding 3 by 3 matrix from U(3). Namely, let us understand by D_{12} the image of an embedding A_{12} of the original D, such that:

(1) Each matrix Z from D is an upper 2 by 2 principal minor of a 3 by 3 matrix $A_{12}(Z)$,

(2) The third diagonal entry of the matrix $A_{12}(Z)$ is 1,

(3) In the matrix $A_{12}(Z)$, all other entries vanish.

The two remaining embeddings, A_{13} and A_{23} are introduced quite similarly. Clearly, D_{12}, D_{13}, and D_{23} are U(2)–subgroups in U(3). Recall that the group U(2) is closed w.r.t. the operations of complex conjugation and of matrix transposition. The transposed matrix Z^T can be viewed as a symmetric one to Z w.r.t. the principal diagonal. From where it follows that each of D_{12}, D_{13}, D_{23} is invariant w.r.t. any of the two mentioned operations in U(3). Also, to enumerate all D_{ij}, it is enough to consider cases $i < j$, only. In the totality of all *m* by *m* matrices, introduce P_m, the symmetry w.r.t. the second diagonal. Clearly, when Z is from U(2), then $P_2(Z)$ is also in U(2). From here it follows that the subgroup D_{13} is P_3-invariant in U(3) whereas $P_3(D_{12}) = D_{23}$ and $P_3(D_{23}) = D_{12}$. In other words, embeddings A_{12} and A_{23} are equivalent (one becomes the other when composed with P_3). **This relates to the presence of two u-quarks 'in' a proton whereas the embedding A_{13} relates to the presence of a d-quark in that proton**. The notion of quark's flavor has been thus introduced for quarks of level (generation) one.

3. Definition of Color

Let us now introduce the notion of quark's color. Recall (see [2]) that each matrix g_n (from G_n=SU(n,n)) has block structure (with *n* by *n* blocks A_n, B_n, C_n, D_n) and that G_n acts linear-fractionally on U(n). Let us start with the embedding A_{12}. It selects the upper principal minor in corresponding 3 by 3 matrices. In other words, rows one and two (as well as columns one and two) are selected. This choice determines an SU(2,2)-subgroup G_{12} in G_3. Namely, each g_3 in G_{12} is built (of blocks A_2, B_2, C_2, D_2, of the original g_2 from G_2 = SU(2,2)) as follows: A_2 is the upper left principal minor of the 6 by 6 matrix g_3; D_2 is the principal minor which occupies the intersection of rows and columns with numbers four and five; the (non-principal) minor B_2 occupies the intersection of rows with numbers one and two and columns with numbers four and five; another non-principal minor, C_2 occupies the intersection of rows with numbers four and five and columns with numbers one and two. Each of the remaining entries of g_3 is 1(if on the principal diagonal) or 0 (if off-diagonal). The subgroups G_{13} and G_{23} are introduced quite similarly. As a result (if no preference is given to any of A_{12}, A_{13}, A_{23}), three SU(2,2)-subgroups are thus selected in G_3. It is obvious how to define an (linear fractional) action of each of these three subgroups on any of the U(2)-subgroups D_{12}, D_{13} и D_{23}. **Flavor** (one of the three possible on level 1) **of the quark is determined by the choice of a single subgroup**: G_{12}, G_{13}, or G_{23}.

4. Description of Deeper Levels and the Number of Quarks

It seems natural to suppose that when a proton is hit hard enough, than it can be displaced (from level 0) right to level 2. To deal with such case, consider embeddings of D=U(2) into U(4). Here they are: A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}; notation mimics the one which has been already used in the U(3)-case. To determine (possible) equivalencies, consider the (earlier defined) operator P_4. Clearly, A_{12} is equivalent to A_{34}, and A_{13} is equivalent to A_{24}. Each of the subgroups D_{14}, D_{23} is P_4–invariant. It seems unavoidable to relate A_{23} to an **s-quark**, and A_{14} – to a **c-quark**. On this (second) level A_{12} (which is equivalent to A_{34}) is associated with a **u-quark** whereas A_{13} (equivalent to A_{24}) – with a **d-quark**. Hence, quarks of both generations (one and two) are present on level two. Colors can be likewise introduced and the total **number of level two colors is six**.

Here is the list of D=U(2) embeddings into U(5) таковы: A_{12}, A_{13}, A_{14}, A_{15}, A_{23}, A_{24}, A_{25}, A_{34}, A_{35}, A_{45}. Clearly, $P_5(D_{12}) = D_{45}$, **u-quark**; $P_5(D_{13}) = D_{35}$, **d-quark**; $P_5(D_{14}) = D_{25}$, **c-quark**; $P_5(D_{23}) = D_{34}$, **s-quark**. Each of the following two subgroups, D_{15} (**t-кварк**) and D_{24} (**b-кварк**), is P_5–invariant. The total **number of colors is 10**.

Conjecture (to be proved below): there are **three new flavors** (quarks of the 4th generation) on the level of U(6).

It follows from the above that there are n(n-1)/2 colors on level U(n). Introduce m_n – the number of possible quarks' flavors on level U(n). By [*x*], the value (at a real number *x*) of the *greatest integer function* is understood.

Theorem. On the level U(n), let an U(2)-subgroup D_{ij} be not P_n-invariant. Then D_{ij} corresponds to a quark from a lower level. The recurrent (1) and explicit (2) formulas hold:

$m_2 = 1,\ m_n = m_{n-1} + [n/2],$ (1)

$m_n = \{n(n-1)/2 + [n/2]\}/2.$ (2)

Remark. The term [n/2] in (1) is the number of P_n-invariant U(2)-subgroups on that level. Clearly, $m_3 = 2$, $m_4 = 4$, $m_5 = 6$, $m_6 = 9$: the first three equalities in (2) are thus in compliance with the standard convention to have two flavors at each quark generation (currently, the search for quarks of the 4th generation is in progress).

The above theorem would be a corollary of the following

Lemma. If D_{ij} is not P_n-invariant, then D_{ij} is equivalent to D_{pq} with $p+q \le n$. In other words, D_{pq} (and D_{ij}) corresponds to a quark which 'emerged' on a certain level U(m) with m<n (such a quark will be thus present on any level U(s) with s > m).

The **proof** is straightforward, and it is omitted.

5. On Gluons

Recall that Segal models photon on the basis of a tensor product of proton and anti-proton spaces (see [5], p. 37 and p. 56). On each level U(n) with *n* > 2 we can introduce **gluons** similarly (to how Segal introduced photon). Namely, in the multi-level model, color of a quark has been already defined. Hence, each anti-quark has an anti-color (see Section 6 for more details) which allows the interpretation of gluons as **colored photons**. Clearly, there are eight gluons on U(3) level: each gluon is specified by a pair (color, anti-color) – in exact compliance with standard chromo-dynamics. It follows from the above that (when trying to apply the model) one has to use new reactions cross sections formulas. In particular, the (standard) notion of mass of a quark should be interpreted in terms of the energy needed to embed the original proton into the corresponding 'cell' of the quark-gluon media.

6. Results

For each level U(n), n>2, a quark (having a certain flavor and a certain color) can be defined as a composition $((D_{pq},f), (G_{ij},c))$ of two ordered pairs. By D_{pq} the image of U(2) under a certain principal embedding A_{pq} into U(n) is denoted, *f* is either 1 or negative 1 (depending on whether we deal with a particle or with its antiparticle), G_{ij} is a certain SU(2,2)-subgroup in SU(n,n), *c* is either 1 or negative 1. Here $((D_{pq},f)$

(respectively, (G_{ij},*c*)) is called *flavor* (respectively, *color*). A 'hidden part' of this definition is a particular representation space (the *p*-space) with the corresponding action of G_{ij} in it (see our Section 1, top of). Each anti-quark is (formally) the pair ((D_{pq},-*f*), (G_{ij},-*c*)) where the subgroup G_{ij} in (G_{ij},-*c*) acts *differently* on the *p*-space. This last action is the complex-conjugate to the action of G_{ij} from (G_{ij},*c*) - according to the way how one gets an anti-proton when the original proton is specified (see [1]). The notion of both an *anti-color* and of an *anti-flavor* (of an anti-quark) is thus defined. Color-anticolor pair (being a characteristic of a gluon) is formally defined as ((G_{ij},*c*), (G_{sk},-*c*)).

7. Conclusions and Discussions

As far as the author is informed, the multi-level model is the only known construct within which such notions from theoretical physics as flavor and color are rigorously (mathematically) defined. Seemingly, the infinite number of generations should not be a huge problem: each generation relates to certain energy (range of), and it is of utmost interest whether quarks of the 4th generation will soon be discovered (see the Conjecture in the above Section 4). A more challenging discrepancy with standard physics is the number of colors: in the most natural option (above) this number is level-dependent. However, one can ("by hand") reduce the number of colors to just 3 (namely, to the three colors which appear on U(3)-level). Recall that the number of colors being 3 is the current standard assumption about quarks. On the other hand, it seems reasonable to try to test the original multi-level model, first (which can be done by developing new reactions cross sections formulas and by carrying out the corresponding *theory vs experiment* investigation). The number of flavors (up to the 3^{d} generation, included) in the multi-level model is in compliance with the standard convention to have two flavors for each quark generation.

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