Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION

Volume 11, Issue 3
Published online: March 24, 2017|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Wa-module

Montaha Abdul-Razaq, Abeer Jabbar

Abstract.

M.A.Hassin and A.B.Hussien introduced thefollowing concept :(a,H) is called aringoverG if H is asubgroup of groupG and aϵG and a has finite order or infinite order , we called the ring $(w_a,+,.)$; (a,H) Ring over G where

$w_a = \{a^m Ha^n, m,n \epsilon Z, a\epsilon G \backslash H\}$

with two binary operation + and. Such that $\forall\ a^{m1} Ha^{n1}, a^{m2}Ha^{n2} \epsilon\ wa$, $m_1,m_2,n_1,n_2\ \epsilon Z$

1-$a^{m1}Ha^{n1}+a^{m2}Ha^{n2} = a^{m1+m2}Ha^{n1+n2}$.

2-$a^{m1}Ha^{n1} . a^{m2}Ha^{n2} = a^{m1+m2}Ha^{n1+n2}$.

In [4] , $(w_a,+,.)$ is commutative ring with unity aHa these lead us to give the definition of w_a-module define on the ring ℝ which is for any commutative ring with unity element.

The main purpose of this work is to give definition of wa-module and some properties of W_a module many new and useful results are

Obtain about this concept, and we illustrate that by examples

Key words: injective module; Homomorphism; R-balanced.

Introduction :

The concept of module is very important in algebra [2] ,[3] . We give a definition of wa-module, where R is a commutative ring with unity. In this work, we define a quotient module of this module, homomorphism and injective module of this module.

1- **Wa-module**

In this section we introduce the concept of w_a-module where R is an abelian group ring with identity, and G is an abelian group and H is a subgroupof G ,then we can give a new definition .

Definition:(1-1)

Let R be a commutativering with unity 1R. An abelian group $(w_{a,+})$ is called a left R-module of there exist a map φ**:** R $\times w_a \longrightarrow w_a$ with

φ $(r,a^mH\,a^n)=a^m\,r\,Ha^n$ satisfies the following:

1- $(r_1+r_2)(a^m\,Ha^n)=a^m\,r_1\,Ha^n+a^m\,r_2\,Ha^n$
2- $r(a^m{}_1\,Ha^n{}_1+a^m{}_2Ha^n{}_2)=a^m{}_1rHa^n{}_1+a^m{}_2rHa^n{}_2$
3- $(r_1\,r_2)a^m\,Ha^n=r_1\,(r_2\,a^m\,Ha^n)$
4- $1_R.a^m\,Ha^n=a^m\,Ha^n\,\forall\, r,r_1,r_2\in R$ $,\forall\, a^m\,Ha^n,a^m{}_1Ha^n{}_1,a^m{}_2Ha^n{}_2\in w_a$

To satisfy that.

$1\text{-}(r_1+r_2)(a^mHa^n)=a^m(r_1+r_2)Ha^n$

$=a^m(r_1H+r_2H)a^n$

$=a^m\,r_1Ha^n+a^mr_2Ha^n$

$2\text{-}r(a^m{}_1Ha^n{}_1+a^m{}_2Ha^n{}_2)=r(a^{m}{}_1{}^{+m}{}_2Ha^{n}{}_1{}^{+n}{}_2)$

$=a^{m}{}_1{}^{+m}{}_2rHa^{n}{}_1{}^{+n}{}_2$

$=a^m{}_1rHa^n{}_1+a^m{}_2rHa^n{}_2$

$3\text{-}(r_1r_2)a^mHa^n=a^m(r_1r_2)Ha^n$

$=r_1(a^mr_2\,Ha^n)$

$=r_1(r_2a^m\,Ha^n)$

$4\text{-}1.a^mHa^n=a^m1Ha^n=a^m\,H\,a^n$

Note (1-2)

If R is a commutative ring then every left R-module can be considered as a right R-module

Example (1-3) :

1- Let G=ℝ, H= 2 Z&**R** = ℝ then w_a-module is

$\{\,a^m(2Z)\,a^n\,;\,m,n\in z\,,\,a\in 2z\}$

identity element is (2) since∀x∈ M

$x=a^m(2\,Z)\,a^n$ such that x+I = x where I= $a^0(2Z)a^0$

soℝ is w_a– module

2- Z is not Wa – module since $a^m\notin$ Zif m = Z^-

Theorem (1.4)

Let N be anon empty subset of W_a such that N={a^mH;a∉H,H subgroup of G, a ϵ G}

Then (N,+) sub-module of (W_a,+)

Proof :

Let $x = a^{m1}H \in N$, $y=a^{m2}H \in N$

$x+y = a^{m1}H + a^{m2} H$

$x+y = a^{m1+m2}H \in N$

So $x+y \in N$

Let $r \in R$,$\forall x \in N$; $x = a^{m}H$

So $rx = r(a^{m}H) = a^{m}(rH)$; $a^{m} \notin rH$

$\therefore rx \in N$Implies that N is a sub-module of W_a

Remark (1.5):

1- Every sub-module of wa-module is wa-module

2- Since (W_a,+) is abelian group in [4] , and we prove (N,+) submodule of (M,+) for each $N \neq \emptyset$ so (N,+) is normal submodule of (M,+) , by definition of normal submodule in [5]

We can give the definition

Definition (1.6) : Quotient module

Let M be wa-module and N be normal submodule of M then

$M/N = \{x+N; x \in M\} = \{a^{m}Ha^{n}+ N; m,n \in Z; a \in G/H\}$

$=\{a^{m}Ha^{n}+a^{k}H; m,n,k \in Z, a \in G / H$

$=\{a^{m+k}Ha^{n}; m,n,k \in Z, a \in G/H\}$

To prove M/N is wa-module we give the theorem .

Theorem (1.7) :

Let M be wa-module, N be normal submodule of then M/N is W_a - module .

Proof :

We must prove (M/N, $\oplus$) is abelian group

1 -$\oplus$closed

$\forall x_1, x_2 \in M/N$ to prove $x_1+x_2 \in M/N$

Let $x_1=a^{m1}Ha^{n1} + N$;m,n,k $\in$ Z

$= a^{m1+k}Ha^{n1} \in M/N$

Let $x_2=a^{m2}Ha^{n2} +N = a^{m2} Ha^{n2}+ a^{k}H$;$m_2,n_2,k \in Z$

$=a^{m2+k}Ha^{n2} \in M/N$

$x_1+x_2= a^{m1+k} Ha^{n1}+ a^{m2+k} Ha^{n2}$

$= a^{m1+k+m2+k}Ha^{n1+n2}$

$= a^{w}Ha^{n}$; $w = m1+k+m2+k \in Z$, $n = n_1+n_2 \in Z$

$x_1+x_2 \in M/N$

So ⊕ is closed

2- ⊕ is associative since + is associative on Z

3- identity of M/N is H+N =N since

$H= a^0Ha^0$ & $N = a^nH$ so $H+N = (a^0Ha^0 \oplus a^nH)$

$= a^nH = N$

So $\forall$ $(x+N)\in M/N$ where $x \in M$ implies

That $(x+N)\oplus N = x+N \quad \forall x \in M$

4- inverse : $\forall x+N \in M/N$ implies

$-x+N \in M/N$; $-x= a^{-m}Ha^{-n}$

$(x+N)\oplus(-x+N) = (x+(-x))+N$

$= (a^mHa^n+a^{-m}Ha^{-n})+N$

$= (a^{m-m}Ha^{n-n})+N$

$= a^0Ha^0+N$

$= H+N=N$

5-abelian : $\forall x_1+N$, $x_2+N \in M/N$ $\forall x_1,x_2 \in M$

$(x_1+N)\oplus (x_2+N)=(x_1 +x_2) +N$

$= (x_2+x_1) + N$

$= (x_2+N) \oplus (x_1+N)$ |

So (M/N,⊕) is abelian group

Let R be any commutative ring with the unity 1_R

Let $\emptyset$:R×M / N→M/N S.t $\emptyset(r,x+N)=a^{m+k} rH a^n$

Similarly we can prove

6- $r(x_1+x_2)+N = rx_1 + rx_2 + N \ \forall r \in \boldsymbol{R}, x_1, x_2 \in M$

7- to prove $(r_1+r_2)(x+N) = r_1x +N + r_2 x+ N$

$\forall\ r_1, r_2 \in R$, $x \in M$

8- $r_1 r_2(x_2+N) = r_1\,(r_2(X+N))$

So M\Nis wa-module

2- Homomorphism and injective module:

In this section we give the definition of homomorphism map on wa-module .in [4] we can see the definition of homomorphism module.

<u>Definition 2.1 :</u>

Let N, M be two wa-modules then $\varnothing : M \longrightarrow N$ can define by $\varnothing(a^m H a^n) = a^{\varnothing(m)} H a^{\varnothing(n)}$; m,n ϵ Z

Definition(2.2): (Homomorphism on W_a-module)

Let M,N be two W_a-modules, then mapping $\varnothing$: M $\rightarrow$N is homomorphism if

$\varnothing(a^m H a^n + a^{m1} H a^{n1}) = \varnothing(a^m H a^n) + \varnothing(a^{m1} H a^{n1})$; $m,n,m_1,n_1 \in Z$

2- $\varnothing(r a^m H a^n) = r\,\varnothing(a^m H a^n)$; m,n ϵ Z , r ϵ R

Thus we have the following theorem.

<u>Theorem (2.3)</u>: Let $\varnothing$: M $\rightarrow$ K be homomorphism such that M ,K be two W_a-module then if N be W_a-submodule of M then $\varnothing$ (N) is wa-submodule of K .

<u>Proof:</u>

Let , y ϵ $\varnothing$ (N) ,a,b$\in$N ,m,n$\in$Z

$x = \varnothing(c)$; so $c = a^m H$

$y = \varnothing(b)$; $b = a^n H$

$x = \varnothing(a^m H)$

$y = \varnothing(a^n H)$

$x+y = \varnothing(a^m H) + \varnothing(a^n H) = a^{\varnothing(m)} H + a^{\varnothing(n)} H$

$= a^{\varnothing(m)+\varnothing(n)} H$

$= a^{\varnothing(m+n)} H \in \varnothing(N)$

2- *let* r ϵR ,$x \in \varnothing(N)$ to prove $rx \in \varnothing(N)$,$x \in \varnothing(N)$ so

$x = \varnothing(b)$; b ϵ N

so $b = a^m H$; m ϵ Z

so $x = \varnothing(a^m H) = a$

$rx = r\phi(a^m H)$

$= \phi[(a^m (rH)]$

$= a^{\phi(m)}(rH) \in \phi(N)$

$\therefore \phi(N)$ is submodule of K

Definition (2-4):

Let $f : M \to M$ be homomorphism where M be wa- module

$\text{Ker } f = \{x \in M ; \emptyset (a^m H a^n) = H \}$

$= \{a \in M ; = a^{\phi(m)} H a^{\phi(n)} ; m,n \in z \}$

$= \{a \in m ; a\emptyset(m) H = Ha \emptyset^{-n} ; m,n \in z$

.

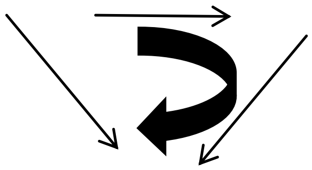
We can find the relation between wa-module and injective module where a module is called injective if for each monomorphism $f : A \to B$ where A, B are two module and for each homomorphism $g : A \to M$ there exist a homomorphism $h : B \to M$ s.t h of $= g$ see [5]

f

A B

$^g\exists\exists^h$

M



hof = g

Theorem 2.5

If M is R – wa – module then M is injective module

Proof

Let $f : A \to A X A$; A be any module such that $f(a) = (a,a)$

f is monomorphism since f is I-I

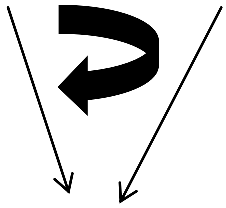
if $f(a_1) = f(a_2)$ for each $a_1, a_2 \in A$

$\to (a_1,a_1) = (a_2,a_2)$

$\to a_1 = a_2$

so f is I-I

A f AX A $\longrightarrow$



$g \exists h$

hof = M   M

1- $f(a_1+a_2) = (a_1+a_2, a_1+a_2) \forall a_1, a_2 \in A$

$= (a_1,a_1)+(a_2,a_2)$

$= f(a_1) + f(a_2)$

2- $f(ra) = (ra,ra)$

$= r(a,a) = rf(a)$

Then f is homomorphism so f is monomorphism

Let there exist homomorphism function

$g: A \rightarrow M$; $g(a) = aHa$

1- $g(a_1+a_2) = (a_1+a_2)H(a_1+a_2) \; \forall a_1,a_2 \in A$

$= a_1Ha_1 + a_2Ha_2$

$= g(a_1) + g(a_2)$

$g(ra) = a(rH)a$; $r \in R$, $a \in A$

$= r(a H a)$

$= r\, g(a)$

So g is homomorphism, there exit homomorphism $h: A x A \rightarrow M$; $h((a,a)) = aHa$

1- $h(a_1,a_1) + h(a_2,a_2) = h((a_1+a_2, a_1+a_2))$

$= (a_1+a_2) H (a_1+a_2)$

$= a_1 H a_1 + a_2Ha_2$

$= h(a_1) + h(a_2)$

2- $h(r(a_1,a_1)) = h(ra_1, ra_2)$

$= a_1(rh)\, a_2$

$= r(a_1H_{a2}) = r\, h(a_1)$

$= rh(a_1)$

∴ h is homomorphism to prove hof (a) =g(a) h of

ho f(a) = h(f(a))

= h((a,a)) = aHa

h of (a) =g(a)

then M is injective

but the converse of theorem is not true for example :

Q as z-module is injective since by theorem an abelian group if Z is injective module [5] but Q is not Z – wa-module.

3- R-balanced and wa- module

In this section we study the relation between R – balanced and wa- module

Definition (3.1) R – balanced [1] Let M,N be two module over ring R and let G be abelian group ,a function f:MXN⟶ G is said to be R-balanced in case for all $m_1,m_2 \epsilon$ M,$n_1,n_2 \epsilon$ N and r ϵR

1- $f(m_1+m_2,n)=f(m_1,n)+f(m_2,n)$

2- $f(m,n_1+n_2)= f(m,n_1)+f(m,n_2)$

3- $f(mr,n)= f(m_1rn)$

Theorem 3.2

Every function from wa-module to an abelian group is R – balanced

Proof:

Let M,N between wa-module and G be abelian group

Let the function f: MXN →G ; f(a,b)=ab

1- $f(m_1+m_2 ,n) = (m_1+m_2)\ n$; $m_1,m_2 \epsilon M$, $n\epsilon N$ let

$m_1= a^kHa^w$

$m_2= a^{k1}Ha^w$

$n=a^{k2}Ha^{w2};k_1,k_2,k,w,w_1,w_2\epsilon Z$

$= (a^kHa^w+a^{k1}Ha^{w1})(a^{k2}Ha^{w2})$

$= (a^{k+k1}Ha^{w+w1})(a^{k2}Ha^{w2})$

$= a^{(k+k1)k2}Ha^{(w+w1)w2}$

$= a^{kk2+k1k2}Ha^{ww2+w1w2}$

$= a^{kk2}Ha^{ww2}+a^{k1k2}Ha^{w1w2}$

$= (a^kHa^w)(a^{k2}Ha^{w2})+(a^{k1}Ha^{w1})(a^{k2}Ha^{w2})$

$= m_1n+m_2n$

$= f(m_1,n)+f(m_2,n)$

2- to prove $f(m,n_1+n_2) = f(m,n_1) + f(m,n_2)$ $f(m_{1,}n_1+n_2) = m_1(n_{1+}n_2)$; $m_1 = a^kHa^w, n_1 = a^{k1} H a^{w1} n_2 = a^{k2} H a^{w2}$;k,k_1, k_2; $w,w_1w_2 \in z$

$= a^k H a^w . (a^{k1}Ha^{w1} + a^{k2}Ha^{w2})$

$= a^kHa^w (a^{k1+k2} Ha^{w1+w2})$

$= a^{k(k1+k2)} H a^{w(w1+w2)}$

$= a^{kk1+kk2} Ha^{ww1+ww2}$

$= a^{kk1} Ha^{ww1} + a^{kk2} Ha^{ww2}$

$= (a^kHa^w)(a^{k1}Ha^{w1}) + (a^kHa^w) (a^{k2} Ha^{w2})$

$= mn_1+mn_2$

$= f(m_1n_1)+ f(m_1n_2)$

3- to prove $f(mr,n) = f(m,rn)$ let $m = a^kHa^w$, $n=a^{k1}Ha^{w1}$

$f(mr,n) = (mr)n$

$= (a^kHa^w) r(a^{k1}H^{aw1})$

$= (a^kHa^w) (a^{k1}(rH)a^{w1}$

$= m (rn)$

$= f(m,rn)$

So wa- module is R-balanced.

References:

[1] Anderson , frank wand fuller R. kent,1973, Ring and categories of modules, department of ma the metics, Eugene, oreg on.

[2] Dwyer, w.g and Greeuless , j.p , zool ,complete and torsion Modules

[3] Kasck .f ,1979, modules and rings, academic press, London

[4] Razak, M.Hassin and A.B. Hussien, 2011,on (a,H). Ring over G, AL Mustanseria University, department of mathematics.

[5] Sharpe, D.w, 1972, injective module, Cambridge, university, press.

المقاس من الخط wa

م . منتهى عبد الرزاق م.م عبير جبار

م. منتهى عبد الرزاق و م . أحمدباقر قدما المفهوم الأتي يقال (a,H) انه حلقة اذا كان H زمرة جزئية من الزمرة G و aϵG وله رتبة منتهيه وغير منتهية . حيث نسمي الحلقة (.,+,wa) بالحلقة (a,H) على الزمرة G

$$Wa = \{a^m Ha^n \, ; m,n \in Z \, , \, a \in G \setminus H \}$$

مع عمليتين ثنائيتين + , . بحيث ان :-

$$1\text{-}\ a^{m1}Ha^{n1} + a^{m2} Ha^{n2} = a^{m1+m2}Ha^{n1+n2}$$

$$2\text{-}\ a^{m1}Ha^{n1} \, . \, a^{m2}Ha^{n2} = a^{m1m2}Ha^{n1n2}$$

$\square$ $m1,m2 \, , \, n1, \, n2 \in Z \, , \, a^{m1}Ha^{n1}, a^{n2}Ha^{n2} \in wa$

وفي المصدر[4] تم برهان ان (.,+,wa) زمرة ابدالية مع عنصر محايد وهذه قادتنا الى تعريف wa-module على الحلقة R وهي مجموعة من الاعداد الحقيقية .

الهدف الرئيسي من هذا العمل ان نجد تعريف (wa-module) وبعض خواصه ولقد وجدنا مجموعة من النتائج المفيدة والجديدة حول هذا المفهوم واوضحنا المفاهيم بالامثله .