Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 9, Issue 3

Published online: October 05, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

A computational solution of the multi-term nonlinear ODEs with variable coefficients using the integral-collocation-approach based on Legendre polynomials

Khadijah M. Abualnaja[1], M. M. Khader[2,3]

[1] Department of Mathematics and Statistics, College of Science, Taif University, Taif, KSA
[2] Department of Mathematics and Statistics, College of Science, Al-Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh: 11566, Saudi Arabia
[3] Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

Abstract

The main aim of this work is devoted to propose and analyze some schemes of the integral collocation formulation dependent on Legendre polynomials. We introduce these formulae to solve the multi-term ODEs with variable coefficients. The proposed technique is used to reduce the given problem to solve a system of algebraic equations. Numerical results are given to satisfy the accuracy and the applicability of the implemented approach.

Keywords: Integral collocation formulation; Legendre spectral method; Multi-term ODEs.

1. Introduction

Many authors have dealt with linear and nonlinear ODEs which have gained popularity because it has emerged in many fields like physics, finance, engineering and biology etc.([1], [2]). Nonlinear equations are much more difficult to solve than linear ones, especially analytically. There are lots of methods to find out solutions of these equations ([3]-[7]).

In this work, we consider the multi-term nonlinear ODE with variable coefficients in the following form:

$$y^{(n)}(t)+\sum_{i=0}^{n-1}\Lambda_i(t)y^{(i)}(t)+f(t)y^r(t)=g(t), \qquad 0<t<1, \tag{1}$$

where $\Lambda_i(t)$, $f(t)$ and $g(t)$ are known functions and $r\in\mathrm{N}$. The initial conditions are given by:

$$y^{(k)}(0)=y_k, \qquad k=0,1,\ldots,n-1. \tag{2}$$

In [8], Mai-Duy, et al. derived an integral collocation approach based on Chebyshev polynomials and used it for solving numerically the bi-harmonic equations. In this work, we derive some schemes of the integral collocation formulation based on Legendre polynomials and implement these formulae to solve numerically multi-term nonlinear ODEs with variable coefficients ([9], [10]).

2. Integration collocation formulations

The Legendre polynomials can be defined on [−1, 1] using the following recurrence formula [11]

$$L_{k+1}(z)=\frac{2k+1}{k+1}zL_k(z)-\frac{k}{k+1}L_{k-1}(z), \qquad L_0(z)=1, \qquad L_1(z)=z, \qquad k=1,2,...$$

In order to use these polynomials on [0, 1] we define the so called shifted Legendre polynomials by introducing the change of variable $z=2t-1$. In this paper, we will use the symbol $P_k(t)$ of the shifted Legendre polynomials.

In this work, to solve the equation (1), we will construct the integration collocation approach based on the truncated Legendre series of degree N to represent $y^{(n)}(t)$ as follows:

$$\frac{d^n y(t)}{dt^n}\cong\sum_{k=0}^{N}a_k P_k(t)=\sum_{k=0}^{N}a_k\Delta_k^{(n)}(t). \tag{3}$$

We will integrate Eq.(3) to obtain the $y^{(n-1)}(t), y^{(n-2)}(t),...y'(t), y(t)$ which are defined in equations (3)-(6) in [12].

If we collocate Eqs.(3)-(6) [in [12]] at (N + 1) points $t_p,\ p=0,1,...,N$ we can obtain:

$$\frac{d^n y(t_p)}{dt^n}=\Omega^{(n)}\hat{S}, \quad \frac{d^{n-1}y(t_p)}{dt^{n-1}}=\Omega^{(n-1)}\hat{S}, \quad \frac{dy(t_p)}{dt}=\Omega^{(1)}\hat{S}, \quad y(t_p)=\Omega^{(0)}\hat{S}, \tag{4}$$

where $\hat{S}=[a_0,a_1,...,a_m,c_1,c_2,...,c_n]$ and $\Omega^{(n)},\Omega^{(n-1)},...,\Omega^{(0)}$ are integrated matrices.

Theorem 1.
For any continuous and square integrable function in [0, 1], $y(t)$ with bounded second derivative (i.e $|y''(t)|\le\delta$ for some constant δ). If this function $y(t)$ is expressed and approximated in terms in the following series form [12]

$$y(t)\cong\sum_{k=0}^{N}d_k P_k(t). \tag{5}$$

Then the coefficients of the shifted Legendre expansion (5) is bounded by the term [12]:

$$|d_i|\le\frac{\sqrt{6}\delta}{\sqrt{2i-3}(2i-1)}. \tag{6}$$

Also, its shifted Legendre approximation $y_N(t)$ defined in (5) converges uniformly. Moreover, we have the following accuracy estimation

$$\|y(t)-y_N(t)\|_{L^2[0,1]}\le\sqrt{6}\delta\left(\sum_{i=N+1}^{\infty}\frac{1}{(2i-3)^4}\right)^{0.5}. \tag{7}$$

2. Solution procedure for multi-term nonlinear ODEs

In this section, the integral collocation approach based on Legendre expansion for solving the multi-term nonlinear ODEs with variable coefficients of the form (1) is introduced through the following two examples.

Example 1: Consider the following initial value problem

$$y'''(t) + y^2(t) = t^2, \qquad 0 < t < 1, \tag{8}$$

the initial conditions are:

$$y(0) = y'(0) = 0, \quad y''(0) = 2. \tag{9}$$

The procedure of the implementation is given by the following steps [12]:

1. Approximate the function $y(t)$ and its relevant derivatives with $N = 3$, as follows [12]

$$\begin{aligned}
\frac{d^3 y(t)}{dt^3} &\cong \sum_{k=0}^{3} a_k P_k(t) = \sum_{k=0}^{3} a_k \Delta_k^{(3)}(t), \\
\frac{d^2 y(t)}{dt^2} &\cong \sum_{k=0}^{3} a_k \Delta_k^{(2)}(t) + c_1, \\
\frac{dy(t)}{dt} &\cong \sum_{k=0}^{3} a_k \Delta_k^{(1)}(t) + c_1 t + c_2, \\
y(t) &\cong \sum_{k=0}^{3} a_k \Delta_k^{(0)}(t) + \frac{t^2}{2!} c_1 + \frac{t}{1!} c_2 + c_3,
\end{aligned} \tag{10}$$

where $\Delta_k^{(0)}(t), \Delta_k^{(1)}(t)$ and $\Delta_k^{(2)}(t)$ are defined as follows [12]

$$\begin{aligned}
\Delta_k^{(0)}(t) &= \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!(i!)^2 (i+1)(i+2)(i+3)} t^{i+3}, \\
\Delta_k^{(1)}(t) &= \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!(i!)^2 (i+1)(i+2)} t^{i+2}, \\
\Delta_k^{(2)}(t) &= \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!(i!)^2 (i+1)} t^{i+1}.
\end{aligned}$$

Then the multi-term ODE (8) can be written in the following approximated form

$$\sum_{k=0}^{3} a_k P_k(t) + \left(\sum_{k=0}^{3} a_k \Delta_k^{(0)}(t) + \frac{t^2}{2!} c_1 + \frac{t}{1!} c_2 + c_3 \right)^2 = t^4. \tag{11}$$

We now collocate Eq.(11) at $(N+1)$ points $t_p,\ p = 0,1,2,3$ as

$$\sum_{k=0}^{3} a_k P_k(t_p) + \left(\sum_{k=0}^{3} a_k \Delta_k^{(0)}(t_p) + \frac{t_p^2}{2!} c_1 + \frac{t_p}{1!} c_2 + c_3 \right)^2 = t_p^4. \tag{12}$$

For suitable collocation points we use roots of shifted Legendre polynomial $P_4(t)$ [2].

2. Also, by substituting from the initial conditions (9) in (10) we can obtain n=3 of equations which give the values of the constants c_1, c_2 and c_3 as follows

$$c_1 = 2, \qquad c_2 = 0, \qquad c_3 = 0. \tag{13}$$

The equations (12) and (13) construct system of non-linear algebraic equations which contains seven equations for the unknowns a_k, k=0,1,2,3 and c_i, i=1,2,3.

3. Solve the resulting system using the Newton iteration method to obtain the unknowns a_k, k=0,1,2,3, as follows

$$a_0 = a_1 = a_2 = a_3.$$

Therefore, using the formula (10) we can find the required approximate solution in the following form:

$$y(t) \cong \sum_{k=0}^{3} a_k \Delta_k^{(0)}(t) + \frac{t^2}{2!}c_1 + \frac{t}{1!}c_2 + c_3 = t^2,$$

which is the exact solution of the proposed problem (8).

Example 2:

Consider the following initial value problem

$$y'''(t) + \Lambda_2(t)y''(t) + \Lambda_1(t)y'(t) + \Lambda_0(t)y(t) + f(t)y^2(t) = g(t), \qquad 0 < t < 1, \tag{14}$$

where

$$\Lambda_2(t) = \Lambda_1(t) = \Lambda_0(t) = e^{-t}, \quad f(t) = e^{-2t}, \quad g(t) = 4 + e^t,$$

with the following initial conditions

$$y(0) = y'(0) = 0, \qquad y''(0) = 2. \tag{15}$$

We can follow the same procedure of the implementation as in the previous example to obtain:

The values of the constants c_1, c_2 and c_3 as follows

$$c_1 = 1, \qquad c_2 = 1, \qquad c_3 = 1. \tag{16}$$

The values of the unknowns a_k, k=0,1,2,3,4,5 as follows

$$a_0 = 1.71880, \quad a_1 = 0.845155, \quad a_2 = 0.139864, \quad a_3 = 0.013931, \quad a_4 = 0.000993, \quad a_5 = 0.000056.$$

Therefore, using the formula (10) we can find the required approximate solution in the following form:

$$\begin{aligned} y(t) &\cong \sum_{k=0}^{5} a_k \Delta_k^{(0)}(t) + \frac{t^2}{2!}c_1 + \frac{t}{1!}c_2 + c_3 \\ &= 1 + t + 0.5t^2 + 0.16667t^3 + 0.04167t^4 + 0.00832t^5 + 0.0.0142t^6 + 0.00017t^7 + 0.00004t^8. \end{aligned}$$

In Figure 1, we give the numerical results of the proposed problem (14) with different values of (N =3,5) in the interval [0, 1]. From this figure, since the obtained numerical solutions are in excellent agreement with the exact solution $y(t) = e^t$, so, we can conclude that the proposed approach is well for solving such class of ODEs.

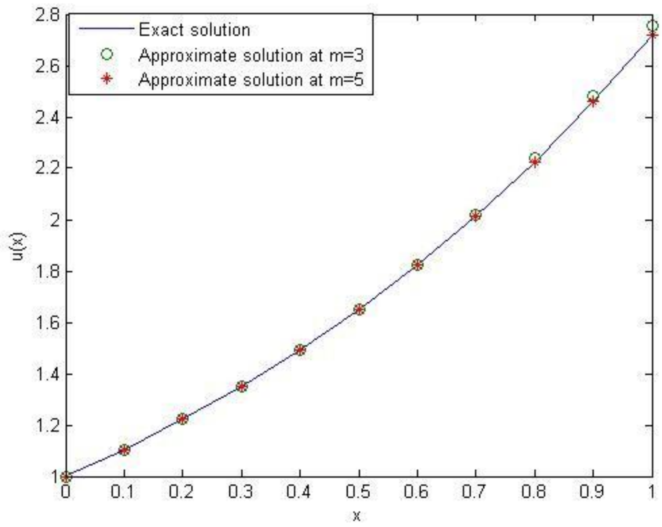

Figure 1. A comparison between the exact solution and the approximate solution at N = 3 and N = 5.

References

[1] Khader, MM (2012). Introducing an efficient modification of the variational iteration method by using Chebyshev polynomials. *Application and Applied Mathematics: An International Journal*, 7(1), 283-299.

[2] Khader, MM (2014). On the numerical solution and convergence study for system of non-linear fractional diffusion equations. *Canadian Journal of Physics*, 92(12), 1658-1666.

[3] Abualnaja, KM (2015). A block procedure with linear multi-step methods using Legendre polynomials for solving ODEs. *Applied Mathematics*, 6, 717-723.

[4] Burden, RL & Faires, JD (1993). Numerical Analysis. PWS, *Boston*.

[5] Khader, MM (2011). On the numerical solutions for the fractional diffusion equation. *Communications in Nonlinear Science and Numerical Simulation*, 16, 2535-2542.

[6] Momani, AS & Al-Khaled, K (2005). Numerical solutions for systems of fractional differential equations by the decomposition method. *Applied Mathematics and Computation*, 162, 1351-1365.

[7] Sweilam, NH, Khader, MM, Mahdy, AMS (2012). Numerical studies for fractional-order Logistic differential equation with two different delays. *Journal of Applied Mathematics*, 2012, 1-14.

[8] Mai-Duy, N, See, H, Tran-Cong, T (2009). A spectral collocation technique based on integrated Chebyshev polynomials for biharmonic problems in irregular domains. *Applied Mathematical Modelling*, 33, 284-299.

[9] Bhrawy, AH & Alofi, AS (2013). The operational matrix of fractional integration for shifted Chebyshev polynomials, *Applied Mathematics Letters*, 26, 25-31.

[10] Khader, MM & Hendy, AS (2012). The approximate and exact solutions of the fractional-order delay differential equations using Legendre pseudo-spectral method. *International Journal of Pure and Applied Mathematics*, 74(3), 287-297.

[11] Bell, WW (1998). Special Functions for Scientists and Engineers, Great Britain, Butler and Tanner Ltd, *Frome and London*.

[12] Khader, MM, Mahdy, AMS, Shehata, MM (2014). An integral collocation approach based on Legendre polynomials for solving Riccati, Logistic and delay differential equations. *Applied Mathematics*, 5, 2360-2369.