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Analysis of stochastic model of tuberculosis transmission

Jean Luc DIMI[†,‡],*Texance MBAYA[‡]

[†]Département de Mathématiques, Université Marien Ngouabi,
BP 69 Brazzaville Congo

[‡]Institut National de la Recherche en Sciences Exactes et Naturelles
Avenue de l'Auberge de Gascogne
BP 2400 Brazzaville Congo

Abstract.

In this paper, we consider a stochastic model of tuberculosis imbedded in environmental noise. We prove the existence of global solution , and we establish the stochastic stability of solutions.

Key words: Nonlinear epidemic model; Lyapounov function; stochastic asymptotic stability; *Itô* 's formula.

1 Introduction

In this paper, we consider the following model of tuberculosis transmission :

$$\begin{cases} \dot{S} = \Lambda - \beta \frac{SI}{N} - \mu S \\ \dot{E} = \beta(1-p)\frac{SI}{N} + r_2 I - (\mu + k(1-r_1))E \\ \dot{I} = \beta p \frac{SI}{N} + k(1-r_1)E - (\mu + d + \delta + r_2)I \end{cases} \tag{1}$$

Where S(t),E(t) and I(t) denote the numbers of susceptible, exposed and infected individuals at time t, respectively ,with the following parameters :
Λ is the recruitment into the population ; β, the probability that a susceptible individual will be infected by infectious ; μ is the probability that an individual in the population died from reasons not related to the disease ; d is the probability that an infectious individual dies because of the disease.An individual leaves his region to another for a new treatment with the probability δ, thus this individual goes missing of model. New infected individual

*corresponding author. Email adress: ojldimi@gmail.com

may develop the disease directly with probability p. To account for treatment, we define $r_1 E$ as the fraction of population receiving effective chemoprophylaxis and r_2 as the rate of effective per capita therapy. We assume that chemoprophylaxis of latently infected individuals E reduces their reactivation rate r_1 and that the initiation of of therapeutics immediately removes individuals from active status I and places them into state E,the time before latently infected individuals who does not received effective chemoprophylaxis become infectious is assumed to satisfy an exponential distribution, with time $\frac{1}{k}$. Thus, individuals leave the class E to I at rate $k(1-r_1)$. Also, after receiving a therapeutic treatment, individuals leave the class I to E at rate r_2 I.

System (1) has two equilibrium points : the desease equilibrium $(\frac{\Lambda}{\mu}, 0, 0)$ and the endemic equilibrium :

$$\begin{cases} S^* = \frac{\Lambda[\beta-(d+\delta)\mathcal{R}_0]}{\mu(\beta-d-\delta)]\mathcal{R}_0} \\ E^* = \frac{\Lambda(\mathcal{R}_0-1)[\beta(1-p)+r_2\mathcal{R}_0]}{[\mu+k(1-r_1)](\beta-d-\delta)\mathcal{R}_0} \\ I^* = \frac{\Lambda(\mathcal{R}_0-1)}{\beta-d-\delta} \end{cases} \tag{2}$$

satisfying the system :

$$\begin{cases} \Lambda - \beta\frac{SI}{N} - \mu S = 0 \\ \beta(1-p)\frac{SI}{N} + r_2 I - (\mu + k(1-r_1))E = 0 \\ \beta p\frac{SI}{N} + k(1-r_1)E - (\mu + d + \delta + r_2)I = 0 \\ \Lambda - \mu N^* - (d+\delta^*)I^* = 0 \end{cases} \tag{3}$$

Where $\mathcal{R}_0$ is the basic reproduction number defined below.

This paper is the continuation of that of Mbaya and al[13]. In that paper the essential work was the application of Allen's method for evaluating the transition of the probability density for stochastic differential equations. To form the SDE model, we have calculate $E(\Delta X)$ and $E((\Delta X)(\Delta X))$. We have also calculate the basic reproduction number $\mathcal{R}_0$ by using the next generation matrix from Driessche and Woutmough , 2002 [9] and is

$$\mathcal{R}_0 = \frac{\beta[\mu p + k(1-r1)] + k(1-r_1)) + \mu r_2}{(\mu+d+\delta)(\mu+k(1-r_1)) + \mu r_2}$$

We now extend the model (1) by introducing the effect of environmental fluctuations. These fluctuations are due to the fact that epidemic models are inevitably affected by environmental noise which is an important component in real life [1,6,7,10,11]. This environmental noise is generated by a 3-dimensional standard Brownian $B_i(t)$. These $B_i(t)$ i=1, 2,3 are independent and $\sigma_i^2 \geq 0$ represent the intensities of $B_i(t)$. Following this approach by introducing stochastic perturbation term into growth of susceptible,exposed, infective individuals, our model becomes :

$$\begin{cases} \dot{S} = (\Lambda - \beta\frac{SI}{N} - \mu S)dt + \sigma_1 S dB_1(t) \\ \dot{E} = (\beta(1-p)\frac{SI}{N} + r_2 I - (\mu + k(1-r_1))E)dt + \sigma_2 E dB_2(t) \\ \dot{I} = (\beta p\frac{SI}{N} + k(1-r_1)E - (\mu + d + \delta + r_2)I)dt + \sigma_3 I dB_3(t) \end{cases} \tag{4}$$

The deterministic model and the stochastic model have the same equilibria.
The deterministic model and the stochastic model have the same equilibria.
Through this paper,let $(\Omega, F, \{F\}_{i\geq 0}, P)$ be a complete space with filtration $\{F_i\}$ satisfying the usual conditions (i.e. it is right continuous and increasing while F_0 contains all null sets). We define the differential operator L associated with 3-dimensional stochastic differential equation :

$$dx(t) = f(x,t)dt + \phi(x,t)dB(t) \tag{5}$$

If V (x,t) is a Lyapounov function, we define the action of L on V by :

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace[\phi^T V_{xx}(x,t)\phi(x,t)]$$

This paper have two sections .In the first , we prove the existence and positivity and boundedness of solution of system (2). and at the second section we present some numerical simulations to illustrate our main results.

2 Dynamics of Stochastic differential system

In this section, we establish the global existence and boundedness of solution of system (4).The following definitions and theorems prove that there is an unique globally positive solution of system (2) for any initial value $X_0 = (S(0), E(0), I(0)) \in \mathbb{R}^3_+$ where

$$\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / x_i > 0, i = 1, 2, 3\}$$

2.1 Stochastically ultimate boundedness

Definition 1 *([1;10]) The solution of system (4) is stochastically ultimately bounded a.s. if for any $\epsilon \in (0,1)$, there exists a positive constant $\varrho = \varrho(\epsilon)$ such that for any initial value (S(0), E(0), I(0)) $\in \mathbb{R}^3_+$, the solution of system (4) has the property :*

$$\limsup_{t\to\infty} P\{|X(t)| \geq \varrho\} < \epsilon \tag{6}$$

Lemma 1 *For any given (S(0),E(0),I(0)) $\in \mathbb{R}^3_+$, there is a unique solution (St),E(t),I(t)), on $t \geq 0$ and will remain in $\mathbb{R}^3_+$ with probability one.*

Proof Since the coefficients of model (4) satisfy the local Lipchitz condition, then there exists a unique local solution on $[0, \tau_\varepsilon)$, where τ_ε is the explosion time.
We, now, want to show that this solution is global, i.e. $\tau_\varepsilon = +\infty$ a.s. Let $n_0 > 0$ be sufficiently large for for any (S(0)E(0),I(0)) remaining in the interval $[\frac{1}{n_0}, n_0]$. For each

integer $n > n_0$, we define the stopping time :

$\tau_n = \inf\{t \in [0, \tau_\varepsilon); \min(S(t), E(t), I(t)) \leq \frac{1}{n}\, or \max(S(t), E(t), I(t)) \geq n\}$

By reduction to absurdity , we suppose that $\tau_\varepsilon = +\infty$ is false, there is a pair of constant $T > 0$ and for any $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Consequently , there is an integer $n_1 \geq n_0$ such that

$$P\{\tau_n \leq T\} \geq \varepsilon, n \geq n_1 \tag{7}$$

Define V(S,E,I)=(S-Ln(S)+(E-Ln(E))+(I-Ln(I)) and we obtain

$$LV(S,E,I) = \Lambda + 3\mu + d + r_2 - \mu S - \frac{\Lambda}{S} - \mu E - \frac{\beta(1-p)SI}{E} - \frac{r_2 I}{E}$$

$$-dI - \frac{\beta PS}{N} - \frac{k(1-r_1)E}{I} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2}$$

$$LV(S,E,I) \leq C - \mu(S+E+I) \leq C$$

where $C = \Lambda + 3\mu + d + r_2$

Since

$$dV = LVdt + \sigma_1(S-1)dB_1(t) + \sigma_2(E-1)dB_2(t) + \sigma_3(I-1)(I-1)dB_3(t)$$

we have :

$$dV \leq Cdt + \sigma_1(S-1)dB_1(t) + \sigma_2(E-1)dB_2(t) + \sigma_3(I-1)(I-1)dB_3(t)$$

Integrating both side from 0 to τ_n yields that :

$$\int_0^{t\wedge T} Cdt + \int_0^{t\wedge T} \sigma_1(S-1)dB_1(t) + \int_0^{t\wedge T} \sigma_2(E-1)dB_2(t) + \int_0^{t\wedge T} \sigma_3(I-1)dB_3(t)$$

where $\tau_n \wedge T = min\{\tau_n, T\}$. Whence taking expectations of the above inequality leads to

$$EV(S(\tau_n \wedge T), E(\tau_n \wedge T), I(\tau_n \wedge T)) \leq V(S(0), E(0), I(0)) + CT \tag{8}$$

Set $\Omega_n = \{\tau_n \leq T\}$ for $n > n_1$ by inequality (7),we obtain $P(\Omega_n) \geq \varepsilon$. Note that every $\omega \in \Omega_n$, there exists at least one of $S(\tau_n, \omega)$, $E(\tau_n, \omega)$ and $I(\tau_n, \omega)$ equals either n or $\frac{1}{n}$, hence

$$V(S(\tau_n, \omega), E(\tau_n, \omega), I(\tau_n \omega)) \geq (n - 1 - lnn) \wedge (\frac{1}{n} - 1 - ln\frac{1}{n})$$

as a consequence from (8) one has :

$$V(S(0), E(0), I(0)) + CT \geq E[1_{\Omega_{n(\omega)}} V(S\tau_n, \omega), E(\tau_n, \omega), I(\tau_n, \omega)] \geq \varepsilon(n-1-lnn) \wedge (\frac{1}{n} - 1 - ln\frac{1}{n})$$

where 1_{Ω_n} is the indicator function of Ω_n. Let $n \to +\infty$ lead to the contradiction

$$+\infty > V(S(0), E(0), I(0)) + CT = +\infty \tag{9}$$

So we must have $\tau_\infty = \infty$. Therefore , the solution (S(t),E(t),I(t)) of model will not explode at a finite time with probability one. This completes the proof of lemma 1.

Theorem 1 *The solutions of System (2) are stochastically ultimately bounded for any initial value* $(S(0), E(0), I(0)) \in \Omega$

Proof From lemma 1 we know that the solution (S(t),E(t),I(t)) will remains in $\mathbb{R}^3_+$ for all $t \geq 0$ with probability 1. Define the functions $V_1 = e^t S^\theta$, $V_2 = e^t E^\theta$ and $V_3 = e^t I^\theta$ and $0 < \theta < 1$. By Itô's formula, we have :

$$\begin{array}{l} dV_1 = LV_1 dt + \sigma_1 \theta e^t S^\theta dB_1(t) \\ dV_2 = LV_2 dt + \sigma_2 \theta e^t S^\theta dB_2(t) \\ dV_3 = LV_3 dt + \sigma_3 \theta e^t S^\theta dB_3(t) \end{array} \tag{10}$$

where

$$\begin{array}{l} LV_1 = e^t S^\theta (1 + \theta(\frac{\Lambda}{S} - \beta\frac{I}{N} - \mu)) + \frac{\sigma_1^2 \theta(\theta-1)}{2} \\ LV_2 = e^t E^\theta (1 + \theta(\beta(1-p)\frac{SI}{NE} + r_2\frac{I}{E} - (\mu + k(1-r_1))) + \frac{\sigma_2^2 \theta(\theta-1)}{2} \\ LV_3 = e^t I^\theta (1\theta(\beta p\frac{S}{N} + k(1-r_1)\frac{E}{I} - (\mu + d + \delta + r_2)) + \frac{\sigma_3^2 \theta(\theta-1)}{2} \end{array} \tag{11}$$

Thus , there exists the positive constants C_1, C_2 and C_3 such that we have $LV_1 < C_1 e^t$,$LV_2 < C_2 e^t$ and $LV_3 < C_3 e^t$. It follows that $e^t E(S^\theta(t)) - E(S^\theta(0)) \leq C_1 e^t$, $e^t E(E^\theta(t)) - E(E^\theta(0)) \leq C_2 e^t$ and$e^t E(I^\theta(t)) - E(I^\theta(0)) \leq C_3 e^t$.

Hence obtain that :

$$\begin{array}{l} \limsup_{t\to\infty} ES^\theta(t) \leq C_1 < \infty \\ \limsup_{t\to\infty} E(E)^\theta(t) \leq C_2 < \infty \\ \limsup_{t\to\infty} EI^\theta(t) \leq C_3 < \infty \end{array} \tag{12}$$

for $X(t) = (S(t), E(t), I(t)) \in \mathbb{R}^3_+$, note that

$$\begin{array}{l} |X(t)|^\theta = (S^2(t) + E^2(t) + I^2(t)) \leq 2^{\frac{\theta}{2}} max\{S^\theta(t), E^\theta(t), I^\theta(t)\} \\ \leq 2^{\frac{\theta}{2}} max\{S^\theta(t) + E^\theta(t) + I^\theta(t)\} \end{array} \tag{13}$$

consequently

$$\limsup_{t\to\infty} E|X(t)| \leq 2^{\frac{\theta}{2}}(C_1 + C_2 + C_3)$$

as a result , there exists a positive constant δ_1 such that

$$\limsup_{t\to\infty} E|\sqrt{X(t)}| < \delta_1 \tag{14}$$

now for any $\varepsilon > 0$, let $\delta = \frac{\delta_1^2}{\varepsilon^2}$, then by the Chebychev's inequality,

$$P\{|X(t)|\} \leq \frac{E|\sqrt{X(t)}|}{\sqrt{\delta}} = \varepsilon \tag{15}$$

which gives us the desired assertion

2.2 Stochastic stability of disease free equilibrium

Theorem 2 *If $\mathcal{R}_0 < 1$ then for any given initial value (S(0),E(0),I(0)) the solution of system (2) has this property :*

$$\limsup_{t\to\infty} \frac{1}{t} E \int_0^t [(S(\tau) - \frac{\Lambda}{\mu})^2 + (E(\tau))^2 + (I(\tau))^2] d\tau \leq 2\sigma_1^2 \frac{\Lambda^2}{M\mu^2}$$

where

$$M = min\{2(\mu - \sigma_1^2); (2\mu - \sigma_2^2); (2(\mu + d + \delta) - \sigma_3^2\}$$

and if $\sigma_1 = 0$ the disease free equilibrium is stochastically asymptotically stable

Proof : Set $u = (S - \frac{\Lambda}{\mu})$, v=E and w=I, and define $V(u, v, w) = (u + v + w)^2 + Av + Bw$
By *Itô* formula one has

$$LV = 2(u+v+w)(-\mu u - \mu v - (\mu + d + \delta)w) + A\{\frac{\beta(1-p)(u + \frac{\Lambda}{\mu})w}{N} + r_2 w - (\mu + k(1 - r_1))v)\}$$

$$+B\{\frac{\beta p(u + \frac{\Lambda}{\mu})w}{N} + k(1 - r_1)v - (\mu + d + \delta + r_2)w\} + \sigma_1^2(u + \frac{\Lambda}{\mu})^2 + \sigma_2^2 v^2 + \sigma_3^2 w^2$$

$$= -(2\mu - \sigma_1^2)u^2 - (2\mu - \sigma_2^2)v^2 - (2(\mu + d + \delta) - \sigma_3^2)w^2 - 4\mu uv - [(4(\mu + d + \delta) - A\frac{\beta(1-p)}{N} + \frac{B\beta p}{N}]uw$$

$$+[A(\frac{\beta(1-p)\Lambda}{\mu N} + r_2) - B((\mu + d + \delta + r_2) - \frac{\beta p\Lambda}{\mu N})]w + [Bk(1 - r_1) - A(\mu + k(1 - r_1)]v$$

$$+2\sigma_1^2 \frac{\Lambda}{\mu} u + \sigma_1^2 \frac{\Lambda}{\mu}^2$$

Let $4(\mu + d + \delta) - A\frac{\beta(1-p)}{N} + \frac{B\beta p}{N}] = 0$,$A(\frac{\beta(1-p)\Lambda}{\mu N} + r_2) - B((\mu + d + \delta + r_2) - \frac{\beta p\Lambda}{\mu N}) = 0$
and $Bk(1 - r_1) - A(\mu + k(1 - r_1) = 0$
We have :

$$LV = -(2\mu - \sigma_1^2)u^2 - (2\mu - \sigma_2^2)v^2 - (2(\mu + d + \delta) - \sigma_3^2)w^2 - 4\mu uv + 2\sigma_1^2 \frac{\Lambda}{\mu} u + \sigma_1^2 \frac{\Lambda}{\mu}^2$$

therefore

$$LV \leq -(2\mu - \sigma_1^2)u^2 - (2\mu - \sigma_2^2)v^2 - (2(\mu + d + \delta) - \sigma_3^2)w^2 + \sigma_1^2 u^2 + 2\sigma_1^2 \frac{\Lambda}{\mu}^2$$

Taking the expectation yields

$$E[V(u, v, w] - E[V(u(0), v(0), w(0)]$$

$$\leq E \int_0^t [-(2\mu - 2\sigma_1^2)u^2 - (2\mu - \sigma_2^2)v^2 - (2(\mu + d + \delta) - \sigma_3^2)w^2 + 2\sigma_1^2 \frac{\Lambda}{\mu}^2] d\tau$$

therefore

$$\leq E\int_0^t [2(\mu-\sigma_1^2)u^2+(2\mu-\sigma_2^2)v^2+(2(\mu+d+\delta)-\sigma_3^2)w^2]d\tau \leq E[V(u(0),v(0),w(0)]+2\sigma_1^2 t\frac{\Lambda^2}{\mu^2}$$

Let $M = min\{2(\mu-\sigma_1^2);(2\mu-\sigma_2^2);(2(\mu+d+\delta)-\sigma_3^2\}$, $u=(S-\frac{\Lambda}{\mu})$, v=E and w=I.
We obtain

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t [(S(\tau)-\frac{\Lambda}{\mu})^2+(E(\tau))^2+(2(\mu+d+\delta)-(I(\tau))^2]d\tau \leq 2\sigma_1^2\frac{\Lambda^2}{M\mu^2}$$

.

Theorem 3 *If $\mathcal{R}_0 < 1$ and $\hat{\mathcal{R}}_0 = \frac{2\beta p-\sigma_3^2}{2(\mu+d+\delta)} < 1$, for any given value (S,E,I) (t) almost surely tends to zero exponentially*

Proof from system (1) we have

$$\dot{E}+\dot{I} = \beta\frac{SI}{N}-\mu E-(\mu+d+\delta])I$$

thus

$$\dot{I} \leq \beta\frac{SI}{N}-(\mu+d+\delta])I$$

and

$$dI = (\beta p\frac{SI}{N}+-(\mu+d+\delta)I)dt+\sigma_3 I dB_3(t)$$

Dividing this equation by I we have

$$dln(I(t)) = \frac{dI(t)}{I}$$

and by $It\hat{o}$ formula ,

$$dln(I(t)) == (\beta p\frac{S}{N}-(\mu+d+\delta)dt-\frac{\sigma_3^2}{2})dt+\sigma_3 dB_3(t)$$

which gives

$$LnI(t) = LnI_0+\int_0^t(\beta p\frac{S}{N}-(\mu+d+\delta)-\frac{\sigma_3^2}{2})dt+\int_0^T\sigma_3 dB_3(t)+\int_0^t(\mu+k(1-r_1))\frac{E}{I}dt$$

We have always $S \leq N$, hence

$$LnI(t) \leq LnI_0+(\beta p-(\mu+d+\delta)-\frac{\sigma_3^2}{2})t+G(t) \tag{16}$$

where G(t) is a martingale defined by :

$$G(t) = \int_0^T \sigma_3 dB_3(t)$$

This implies

$$< G, G >_t = \int_0^T \sigma_3^2 ds = \sigma_3^2 t.$$

by the strong law of large number for martingales [10,11] we have

$$\limsup_{t\to\infty} \frac{G(t)}{t} = 0 \qquad a.s$$

It follows from (16) by dividing t on the booth sides and letting $t \to \infty$ that :

$$\limsup_{t\to\infty} \frac{I(t)}{t} \leq (\beta p - (\mu + d + \delta) - \frac{\sigma_3^2}{2}) < 0 \qquad a.s. \tag{17}$$

2.3 stochastic stability of endemic equilibrium

We know that the endemic equilibrium is globally asymptotically stable if $\mathcal{R}_0 > 1$

The next theorem shows that the solution of (2) will oscillate around the endemic equilibrium

Theorem 4 *If $C_1\beta - \mu = 0$, then the solution of model with any value (S(0), E(0), I(0)) $\in \mathbb{R}_+^3$ has the property*

$$\limsup_{t\to\infty} \frac{1}{t} E \int_0^t [(S(\tau) - \alpha_1 S^*)^2 + (E(\tau) - \alpha_2 E^*)^2 + (I(\tau) - \alpha_3 I^*)^2] d\tau \leq \frac{H_1}{H_2}$$

where

$$\alpha_1 = \frac{\mu(r_2 + \mu)}{\mu(r_2 + \mu) - r_2\sigma_1^2}$$

$$\alpha_2 = \frac{\mu r_2(\mu + k(1 - r_1)) + p\mu^2}{\mu r_2(\mu + k(1 - r_1)) + p\mu^2 - r_2^2}$$

$$\alpha_3 = \frac{\mu(r_2 + p(\mu + d + \delta + r_2))}{\mu(r_2 + p(\mu + d + \delta + r_2)) - r_2\sigma_3^2}$$

$$H_1 = \frac{\sigma_1^2\mu(r_2 + \mu)}{\mu(r_2 + \mu) - r_2\sigma_1^2}(S^*)^2 + \frac{\sigma_2^2\mu r_2(\mu + k(1 - r_1)) + p\mu^2}{\mu r_2(\mu + k(1 - r_1)) + p\mu^2 - r_2^2}(E^*)^2 + \frac{\sigma_3^2\mu(r_2 + p(\mu + d + \delta + r_2))}{\mu(r_2 + p(\mu + d + \delta + r_2)) - r_2\sigma_3^2}(I^*)^2$$

and

$$H_2 = min\{\frac{\mu(r_2 + p) - r_2\sigma_1^2}{r_2}, \frac{\mu(\mu + k(1 - r_1)) + p\mu) - r_2\sigma_2^2}{r_2}, \frac{\mu(r_2 + p(\mu + \delta + r_2)) - r_2\sigma_3^2}{r_2}\}$$

Proof We define a Lyapounov function

$$V(S,E,I) = \frac{1}{2}(S-S^*+E-E^*+I-I^*)^2+\frac{c_1}{2}(S-S^*+E-E^*)^2+\frac{c_2}{2}(E-E^*+I-I^*)^2+\frac{c_3}{2}(I-I^*)^2$$

$$= V_1 + c_1V_2 + c_2V_3 + c_3V_4$$

by $It\hat{o}$ formula we have :

$$dV_1 = [(S-S^*+E-E^*+I-I^*)(\Lambda-\mu S-(\mu+d+\delta)I+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2+\sigma_3^2I^2)]dt$$

$$+(S-S^*+E-E^*+I-I^*)(\sigma_1SdB_1(t)+\sigma_2EdB(t)+\sigma_3IdB_3(t)$$

$$= [(S-S^*+E-E^*+I-I^*)(-\mu(S-S^*)-(\mu+d+\delta)(I-I^*)+[(S-S^*+E-E^*+I-I^*)(+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2+\sigma_3^2I^2)$$

$$+(S-S^*+E-E^*+I-I^*)(\sigma_1SdB_1(t)+\sigma_2EdB(t)+\sigma_3IdB_3(t)$$

$$= -\mu(S-S^*)^2-(\mu+d+\delta)(I-I^*)^2-\mu(S-S^*)(E-E^*)-\mu(S-S^*)(I-I^*)-(\mu+d+\delta)(I-I^*)(E-E^*)$$

$$+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2+\sigma_3^2I^2)]+(S-S^*+E-E^*+I-I^*)(\sigma_1SdB_1(t)+\sigma_2EdB(t)+\sigma_3IdB_3(t)$$

we have :

$$dV_2 = [(S-S^*+E-E^*)(\Lambda-\frac{\beta SI}{N}-\mu S+r_2I-(\mu+k(1-r_1))E+\frac{1}{2}(\sigma_1^2+\sigma_2^2E^2]dt$$

$$+(S-S^*+E-E^*)(\sigma_1SdB_1(t)+\sigma_2EdB_2(t))$$

from (2) we have $\beta(1-p)\frac{S^*I^*}{N^*}-(\mu+k(1-r_1))E=0$ and dV_2 becomes

$$dV_2 = [(S-S^*+E-E^*)(-\beta p(\frac{SI}{N}-\frac{S^*I^*}{N^*})+r_2(I-I^*)-(\mu+k(1-r_1))(E-E^*)+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2E^2)]dt$$

$$+(S-S^*+E-E^*)(\sigma_1SdB_1(t)+\sigma_2EdB_2(t))$$

$$dV_2 = [-\beta p(S-S^*)(\frac{SI}{N}-\frac{S^*I^*}{N^*})-\beta p(E-E^*)(\frac{SI}{N}-\frac{S^*I^*}{N^*})+r_2(S-S^*)(I-I^*)$$

$$+r_2(E-E^*)(I-I^*)-(\mu+k(1-r_1))(E-E^*)(S-S^*)-(\mu+k(1-r_1))(E-E^*)^2$$

$$+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2E^2)]dt+(S-S^*+E-E^*)(\sigma_1SdB_1(t)+\sigma_2EdB_2(t))$$

We also have

$$dV_3 = ((E-E^*+I-I^*)(\frac{\beta SI}{N}-\mu E-(\mu+d+\delta))I+\frac{1}{2}(\sigma^2E^2+\sigma_3^2I^2))dt$$

$$= ((E-E^*+I-I^*)(\frac{\beta SI}{N}-\frac{\beta S^*I^*}{N^*})+\frac{1}{2}(\sigma^2E^2+\sigma_3^2I^2))dt+(E-E^*+I-I^*)(\sigma_2EdB(t)+\sigma_3IdB_3(t)$$

$$= \beta(\frac{SI}{N}-\frac{S^*I^*}{N^*})(E-E^*)+\beta(\frac{SI}{N}-\frac{S^*I^*}{N^*})(I-I^*)-\mu(E-E^*)^2-(2\mu+d+\delta)(E-E^*)(I-I^*)-(\mu+d+\delta)(I-I^*)^2$$

$$+\frac{1}{2}(\sigma_2^2E^2+\sigma_3^2I^2))dt+(E-E^*+I-I^*)(\sigma_2EdB(t)+\sigma_3IdB_3(t)$$

with the same way, we get

$$dV_4 = [(I-I^*)(\beta p(\frac{SI}{N}-\frac{S^*I^*}{N^*})+k(1-r_1))(E-E^*)-(\mu+d+\delta+r_2)(I-I^*)+\sigma_3 I^2]dt+\sigma_3(I-I^*)dB_3(t)$$

$$= (\beta p(\frac{SI}{N}-\frac{S^*I^*}{N^*})(I-I^*)+k(1-r_1)(E-E^*)(I-I^*)-(\mu+d+\delta+r_2)(I-I^*)^2+\sigma_3 I^2]dt+\sigma_3(I-I^*)IdB_3(t)$$

and now we can write dV=LVdt +W where

$$LV dt = -\mu(S-S^*)^2-(\mu+d+\delta)(I-I^*)^2-\mu(S-S^*)(E-E^*)--\mu(S-S^*)(I-I^*)-(\mu+d+\delta)(I-I^*)(E-E^*)$$

$$+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2+\sigma_3^2I^2)]+(S-S^*+E-E^*+I-I^*)(\sigma_1 SdB_1(t)+\sigma_2 EdB(t)+\sigma_3 IdB_3(t)$$

$$+C_1[\beta(\frac{S}{N}-\frac{S^*}{N^*})(E-E^*)+\beta(\frac{S}{N}-\frac{S^*}{N^*})(I-I^*)+\frac{1}{2}(\sigma^2E^2+\sigma_3^2I^2))dt+(E-E^*+I-I^*)(\sigma_2 EdB(t)+\sigma_3 IdB_3(t])$$

$$C_2[(\beta p(\frac{S}{N}-(\frac{S}{N})+(\mu+k(1-r_2))(\frac{E}{I}-(\frac{S^*}{*N}))+\frac{1}{2}I^*\sigma^2]dt+(I-I^*)\sigma_3 dB_3(t)]$$

$$+C_3[-\beta(S-S^*)(\frac{SI}{N}-\frac{S^*I^*}{N^*})-\mu(S-S^*)+\frac{1}{2}\sigma_1 S^2)d(t)+(\sigma_1(S-S^*))dB_1(t)]$$

$$\leq -\mu(S-S^*)^2-(\mu+d)(I-I^*)^2-\mu(S-S^*)(E-E^*)-\mu(S-S^*)(I-I^*)-(\mu+d)(I-I^*)(E-E^*)$$

$$+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2+\sigma_3^2I^2)]+(S-S^*+E-E^*+I-I^*)(\sigma_1 SdB_1(t)+\sigma_2 EdB(t)+\sigma_3 IdB_3(t)$$

$$+C_1[\beta(S-S^*)(E-E^*)+\beta(S-S^*)(I-I^*)+\frac{1}{2}(\sigma^2E^2+\sigma_3^2I^2))dt+(E-E^*+I-I^*)(\sigma_2 EdB(t)+\sigma_3 IdB_3(t))]$$

With $C_1\beta-\mu=0$ one get :

$$\leq -\mu(S-S^*)^2-(\mu+d)(I-I^*)^2-(\mu+d)(I-I^*)(E-E^*)+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2E^2+\sigma_3^2I^2)]$$

$$+(S-S^*+E-E^*+I-I^*)(\sigma_1 SdB_1(t)+\sigma_2 EdB(t)+\sigma_3 IdB_3(t)$$

$$+C_1[\frac{1}{2}(\sigma_2^2E^2+\sigma_3^2I^2))dt+(E-E^*+I-I^*)(\sigma_2 EdB(t)+\sigma_3 IdB_3(t))]$$

$$= -(\mu--\frac{1}{2}(C_3+1)\sigma_1^2)(S-\frac{\mu}{\mu--\frac{1}{2}(C_3+1)\sigma_1^2}(S^*)^2+\frac{\sigma_1^2\mu}{\mu-\sigma_1^2}(S^*)^2$$

$$-(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)(E-\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}E^*)^2+\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}(E^*)^2$$

$$+\frac{1}{2}(C_1+1)\sigma_2^2)(I-\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}I^*)^2+\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}(I^*)^2$$

Therefore

$$dV \leq -(\mu--\frac{1}{2}(C_3+1)\sigma_1^2)(S-\frac{\mu}{\mu--\frac{1}{2}(C_3+1)\sigma_1^2}(S^*)^2+\frac{\sigma_1^2\mu}{\mu-\sigma_1^2}(S^*)^2$$

$$-(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)(E-\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}E^*)^2+\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}(E^*)^2$$

$$-(\mu-\frac{1}{2}(C_1+1)\sigma_2^2)(I-\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}I^*)^2+\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}(I^*)^2$$

Integrating both sides of inequality from 0 to ∞, taking expectations, and recalling that $B_i(t)$ is a Brownian motion yields

$$limsup\frac{1}{t}E\int_0^t(\mu--\frac{1}{2}(C_3+1)\sigma_1^2)(S-\frac{\mu}{\mu--\frac{1}{2}(C_3+1)\sigma_1^2}(S^*)^2$$

$$+(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)(E-\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}E^*)^2$$

$$+(\mu+d-\frac{1}{2}(C_1+1)\sigma_3^2)(I-\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_3^2)}I^*)^2dt\leq H_1$$

Where

$$H_1=\frac{\sigma_1^2\mu}{\mu-\sigma_1^2}(S^*)^2+\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2)}(E^*)^2+\frac{(\mu+d)}{(\mu+d-\frac{1}{2}(C_1+1)\sigma_3^2)}(I^*)^2$$

Let

$$H_2=min\{(\mu-\frac{1}{2}(C_3+1)\sigma_1^2),(\mu+d-\frac{1}{2}(C_1+1)\sigma_2^2),(\mu+d-\frac{1}{2}(C_1+1)\sigma_3^2)\}$$

Then

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t[S(\tau)-\alpha_1S^*)^2+[E(\tau)-\alpha_1E^*)^2+[I(\tau)-\alpha_1I^*)^2d\tau\leq\frac{H_1}{H_2}$$

3 Numerical simulation

In this section , we use the Milstein method mentioned in Higham [4] to show the effect of noise on the dynamics of tuberculosis model

$$\begin{cases} S_{k+1}=S_k+(\Lambda-\beta\frac{SI}{N}-\mu S)\Delta t+\sigma_1S_k\sqrt{\Delta t}\xi_k+\frac{\sigma_1^2}{2}S_k(\xi_k^2-1)\Delta t \\ E_{k+1}=E_k+(\beta(1-p)\frac{SI}{N}+r_2I-(\mu+k(1-r_1))E)\Delta t+\sigma_2E_k\sqrt{\Delta t}\eta_k+\frac{\sigma_2^2}{2}E_k(\eta_k^2-1)\Delta t \\ I_{k+1}=I_k+(\beta p\frac{SI}{N}+k(1-r_1)E-(\mu+d+r_2)I)\Delta t+\sigma_3I_k\sqrt{\Delta t}\zeta_k+\frac{\sigma_3^2}{2}I_k(\zeta_k^2-1)\Delta t \end{cases} \quad (18)$$

The following parameters are taken from [5] :

$\mu=0.101,\quad r_1=0,\quad \delta=0.16288,\quad r_2=0.81862,\quad d=0.0022727,\quad p=0.1$

for $\beta=2$ and $k=0.005$ we obtain $R_0=0.27387$, $\hat{R}_0=\frac{2\beta p-\sigma_3^2}{2(\mu+d+\delta)}=0.67630$ and

where η_k and ξ_k are the gaussian random variables N(0,1)

We exhibit in the next three figures below some fluctuations with different values of σ_i

i=1..3. If $\beta = 20$ we have $R_0 = 2.7387$ and one has endemic equilibrium :

$$\begin{cases} S* = \frac{\Lambda[\beta-(d+\delta)R_0]}{\mu(\beta-d-\delta)R_0} = 555. \\ E* = \frac{\Lambda(R_0-1)[\beta(1-p)+r_2R_0]}{[\mu+k(1-r_1)](\beta-d-\delta)R_0} = 159 \\ I* = \frac{\Lambda(R_0-1)}{\beta-d-\delta} = 13.67 \end{cases} \tag{19}$$

and for $\mathcal{R}_0 \leq 1$ we obtain

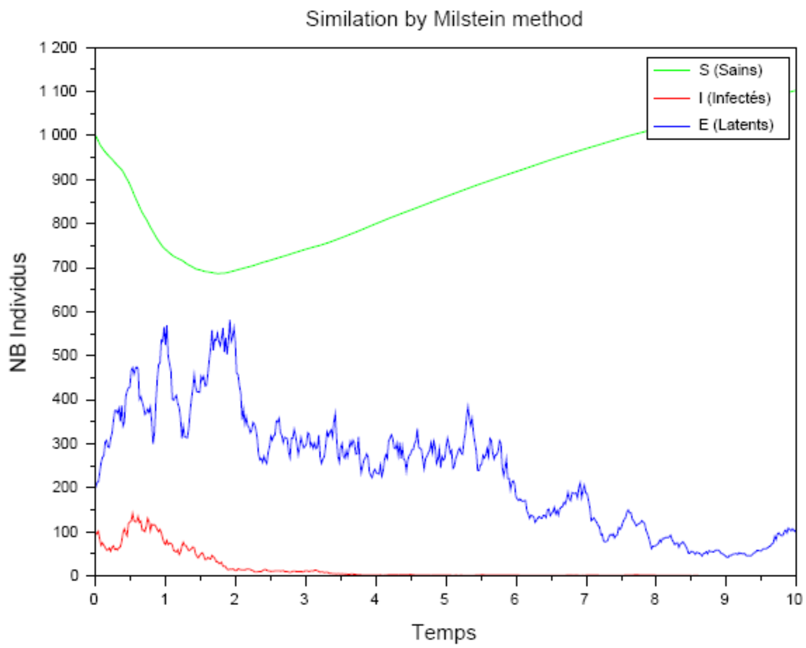


figure 1 is for $\sigma_1 = 0.0$, $\sigma_2 = 0.5$ and $\sigma_3 = 0.2$

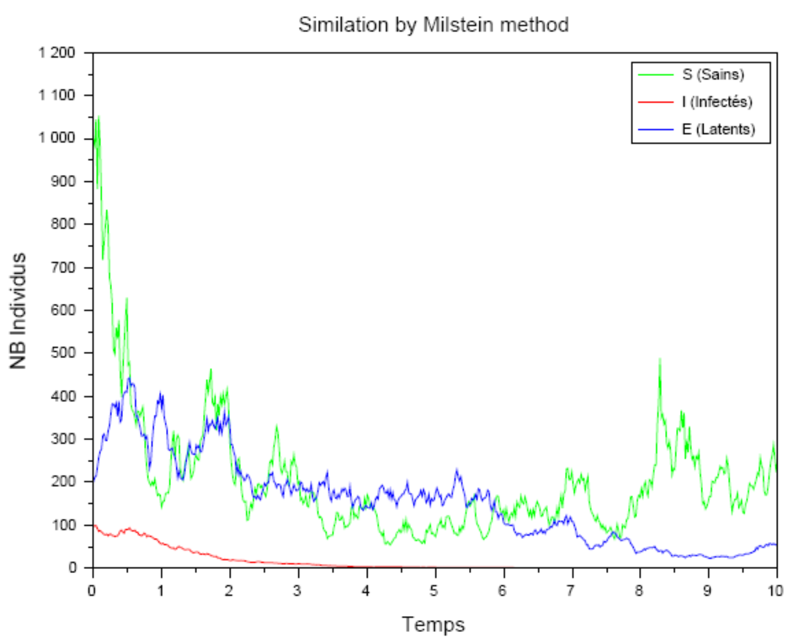
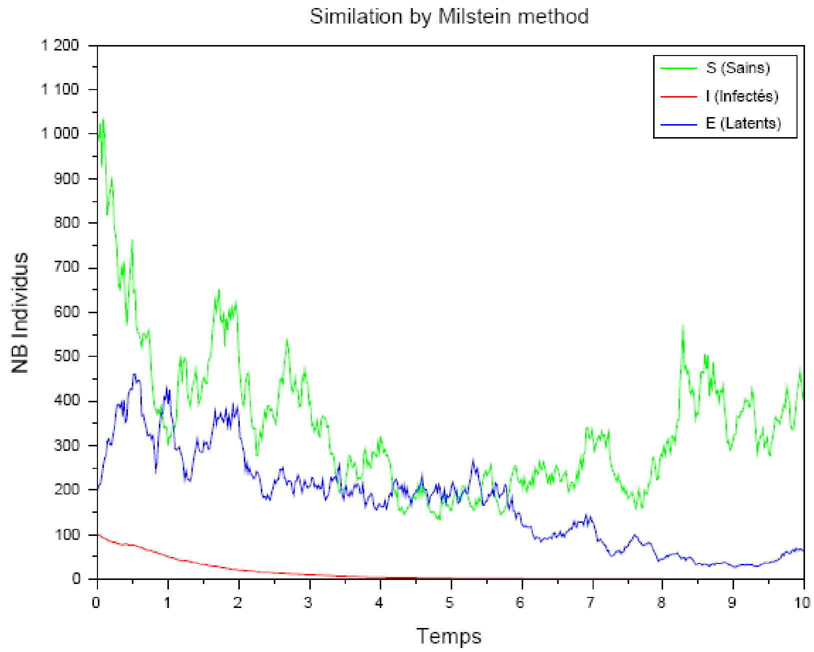


figure 2 is for $\sigma_1 = 0.0$, $\sigma_2 = 0.5$ and $\sigma_3 = 0.2$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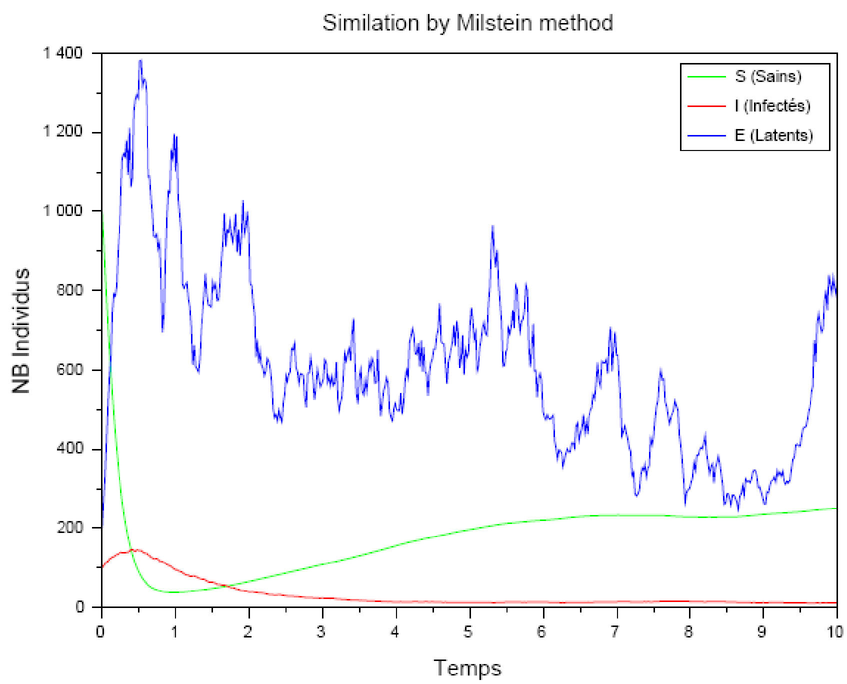
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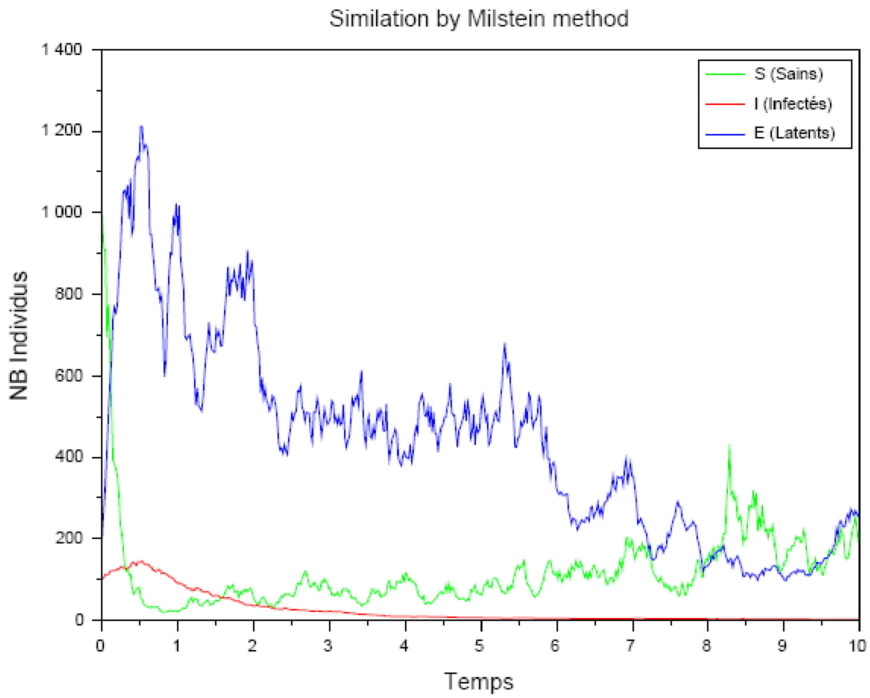
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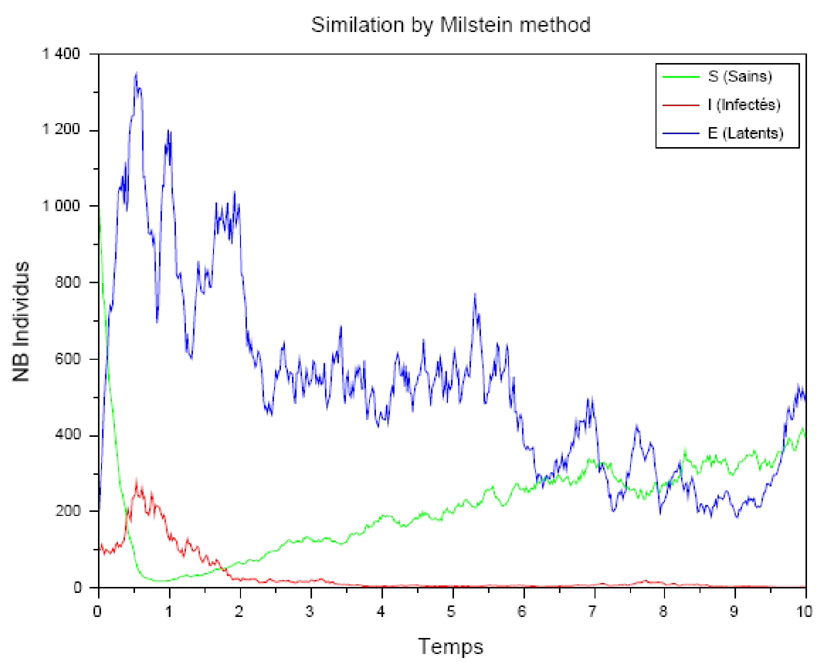
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4 Acknowledgments

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