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New types of generalizations of θ-closed sets

Y. Gh. Gouda [1], M. M. El-Sharkasy [2], S. M. El-Sayed[3]

[1] Aswan University, faculty of science, Mathematics department, Aswan, Egypt
[2] Tanta University, faculty of science, Mathematics department, Tanta, Egypt
[3] Aswan University, faculty of science, Mathematics department, Aswan, Egypt

Abstract

The aim of this paper is to introduce and study the class of T-closed sets as a generalization of θ-closed sets, which is properly placed between θ-closed sets and closed sets. A generalization of T-closed sets, namely, generalized T-closed sets is introduced and studied, which is properly placed between T-closed sets and g-closed sets.

Keywords: T-closed sets; generalized *T*-closed sets; θ-closed sets.

1. INTRODUCTION

In 1968, N. V. Veliˇcko [1] introduced the definition of θ-closed sets via θ-closure operator. In 1970, Norman Levine [3] introduced a generalization of closed sets and studied their basic properties. In 1982, W. Dunham [4] introduced a new closure operator based on g-closed sets. In 1999, J. Dontchev and H. Maki [2] introduced a generalization of θ-closed sets, namely, θ-generalized closed sets. In this paper, we introduce a generalization of θ-closed sets called T-closed sets via T-closure operator which is based on Dunham's closure operator, also we introduce a generalization of T-closed sets which is stronger than g-closed sets.

2. PRELIMINARIES

Throughout this paper (X, τ), (Y, σ) and (Z, η) (briefly, X, Y and Z) represent topological spaces on which no separation axioms are assumed unless otherwise stated. For a subset A of a topological space (X, τ), cl(A) and int(A) denote the closure and the interior of A, respectively.

We recall the following definitions, which are useful in the sequel.

Definition 2.1 A subset A of a space (X, τ) is called.

(1) θ-closed [1] if A = cl_θ(A), Where cl_θ(A) = {x ∈ X: cl(U) ∩ A ≠ φ, ∀ U ∈ τ, x ∈ U}.

(2) generalized closed (briefly, g-closed) [3] if cl(A) ⊆ U whenever A ⊆ U and U is open.

(3) θ-generalized closed (briefly, θ-g-closed) [2] if cl_θ(A) ⊆ U whenever A ⊆ U and U is open.

(4) semi-generalized closed (briefly, sg-closed) [7] if cl_s(A) ⊆ U whenever A ⊆ U and U is semi-open.

(5) generalized α-closed (briefly, gα-closed) [8] if cl_α(A) ⊆ U whenever A ⊆ U and U is α-open.

(6) generalized semi-closed (briefly, gs-closed) [9] if cl_s(A) ⊆ U whenever A ⊆ U and U is open.

(7) α-generalized closed (briefly, αg-closed) [10] if cl_α(A) ⊆ U whenever A ⊆ U and U is open.

(8) generalized semi-preclosed (briefly, gsp-closed) [11] if cl_{sp}(A) ⊆ U whenever A ⊆ U and U is open.

(9) regular generalized closed (rg-closed) [12] if cl(A) ⊆ U whenever A ⊆ U and U is regular-open.

(10) α-closed [13] if cl (int (cl (A))) ⊆ A. The intersection of all α-closed sets containing A is called α-closure [16] and is denoted by cl_α(A).

(11) semi-closed [14] if int (cl (A)) ⊆ A. The intersection of all semi-closed sets containing A is called semi-closure [17] and is denoted by cl_s(A).

(12) semi-preclosed [15] if int(cl(int(A))) ⊆ A. The intersection of all semi-preclosed sets containing A is called semipre-closure [15] and is denoted by cl_{sp}(A).

(13) regular open [23] if A = int(cl(A)).

Definition 2.2 [4] For a subset A of a topological space (X, τ), cl^*(A) = ∩ {F: A ⊆ F, F is g-closed}.

Lemma 2.1 [4] If A ⊆ X, then A⊆ cl^*(A) ⊆ cl(A).

Definition 2.3 [6] In a space X, A is equivalent to B (written A ≡ B) iff for each open set U, A ⊆ U iff B ⊆ U.

Definition 2.4 [3] A topological space X is called $T_{1/2}$-space iff every g-closed set is closed.

Theorem 2.1 [22] X is $T_{1/2}$-space iff for each x ∈ X, either {x} is open or {x} is closed.

3. T-CLOSED SETS

In this section, we introduce a new class of sets, namely, T-closed sets as a generalization of θ-closed sets and study their fundamental properties.

Definition 3.1 A subset A of a topological space X is called T-closed set if A = cl_T(A), where cl_T(A) = {x ∈ X: cl^*(U) ∩ A ≠ φ,∀ U ∈ τ and x ∈ U}. The complement of a T-closed set is called T-open set. The family of all T-closed (resp. T-open) sets is denoted by TC(X) (resp. TO(X)).

Proposition 3.1 For a subset A ⊆ X, A ⊆ cl_T(A).

Proof: Let x ∈ A. Then for every open set U containing x, we have cl^*(U) ∩ A ≠ φ which means that x ∈ cl_T(A). Hence A ⊆ cl_T(A).

Definition 3.2 A subset A of a topological space X is called T-open set if A = int_T(A), where int_T(A) = {x ∈ X: cl^*(U) ⊆ A, U ∈ τ and x ∈ U}.

Proposition 3.2 For a subset A ⊆ X, int_T(A) ⊆ A.

Proof: Let x ∈ int_T(A), then there exists an open set U containing x such that cl^*(U) ⊆ A and since U ⊆ cl^*(U), we have x ∈ A. Thus int_T(A) ⊆ A.

Proposition 3.3 int_T(A) = ∪ {U ∈ τ: cl^*(U) ⊆ A}.

Proof: Let x ∈ int_T(A), then there exists an open set U containing x such that cl^*(U) ⊆ A and then x ∈ ∪ {U ∈ τ: cl^*(U) ⊆ A}. Thus int_T(A) ⊆ ∪ {U ∈ τ: cl^*(U) ⊆ A}. Conversely, let x ∈ ∪ {U ∈ τ: cl^*(U) ⊆ A}, then there exists an open set U containing x such that cl^*(U) ⊆ A and then x ∈ int_T(A). Therefore, ∪ {U ∈ τ: cl^*(U) ⊆ A} ⊆ int_T(A) and hence int_T(A) = ∪ {U ∈ τ: cl^*(U) ⊆ A}.

We give an example of T-closed sets.

Example 3.1 X = {a, b, c}, τ = {X, φ, {a}}. TC(X) = {X, φ, {b, c}}.

Proposition 3.4 For a subset A of a topological space X, cl_T(A) ⊆ cl_θ(A).

Proof: Let x ∈ cl_T(A), then for every open U containing x cl^*(U) ∩ A ≠ φ. But cl^*(U) ⊆ cl (U). Therefore, cl (U) ∩ A ≠ φ. Thus x ∈ cl_θ(A).

Proposition 3.5 For a subset A of a topological space X, cl (A) ⊆ cl_T(A).

Proof: Let x ∈ cl (A), then for every open set U containing x U ∩ A ≠ φ. Then cl^*(U) ∩ A ≠ φ and hence x ∈ cl_T(A). Thus cl (A) ⊆ cl_T(A).

Proposition 3.6 Every θ-closed set is T-closed.

Proof: Let A be θ-closed set. Then A = cl_θ(A), and we have cl_θ(A) = A ⊆ cl_T(A). Thus A = cl_T(A) and hence A is T-closed set.

Proposition 3.7 Every T-closed set is closed.

Proof: Let A be T-closed set. Then A = cl_T(A). We want to show that cl (A) = A = cl_T(A). We know that cl (A) ⊆ cl_T(A) = A. Therefore A = cl (A) and hence A is closed.

We have the following implications.

θ-closed ⟶ T-closed ⟶ closed

Figure (1)

Implications in the previous Fig. can't be reversed as shown from the following examples.

Example 3.2 Let X = {a, b, c}, τ = {X, φ, {a}}, {b, c} is T-closed but not θ-closed.

Example 3.3 Let X = {a, b, c}, τ = {X, φ, {a, c}}, {b} is closed but not T-closed.

Proposition 3.8 For two subsets A, B of a topological space X, if A ⊆ B, then cl_T(A) ⊆ cl_T(B).

Proof: Let x ∈ cl_T(A), then for every U open containing x, cl^*(U) ∩ A ≠ φ. But, A ⊆ B then, cl^*(U) ∩ B ≠ φ. Thus x ∈ cl_T(B) and therefore, cl_T(A) ⊆ cl_T(B).

Proposition 3.9 For two subsets A, B of a topological space X, cl_T(A ∪ B) = cl_T(A) ∪ cl_T(B).

Proof: A ⊆ A ∪ B then, cl_T(A) ⊆ cl_T(A ∪ B). Similarly, cl_T(B) ⊆ cl_T(A ∪ B) and then, cl_T(A) ∪ cl_T(B) ⊆ cl_T(A ∪ B). Now, we want to show that cl_T(A ∪ B) ⊆ cl_T(A) ∪ cl_T(B). Let x ∉ cl_T(A) ∪ cl_T(B) which leads to

cl^*(U) ∩ A = φ, cl^*(U) ∩ B = φ for every open set U containing x and therefore, cl^*(U) ∩ (A ∪ B) = φ for every open set U containing x. Thus, cl_T(A ∪ B) ⊆ cl_T(A) ∪ cl_T(B) and hence, cl_T(A ∪ B) = cl_T(A) ∪ cl_T(B).

Proposition 3.10 For two subsets A, B of a topological space X, cl_T(A ∩ B) ⊆ cl_T(A) ∩ cl_T(B).

Proof: Let x ∈ cl_T(A ∩ B) then, cl^*(U) ∩ (A ∩ B) ≠ φ for every open set U containing x. Therefore, cl^*(U) ∩ A ≠ φ, cl^*(U) ∩ B ≠ φ for every open set U containing x. Thus, x ∈ cl_T(A) ∩ cl_T(B) and therefore, cl_T(A ∩ B) ⊆ cl_T(A) ∩ cl_T(B).

Inclusion can't be replaced by equality in the previous proposition as shown from the following example.

Example 3.4 Let X = {a, b, c, d}, τ = {X, φ, {a, b}, {b, c}, {b}, {a, b, c}}. Let A = {a, c}, B = {b, c} and we have, cl_T(A) = {a, c, d}, cl_T(B) = X. But, cl_T(A ∩ B) = cl_T({c}) = {c, d} ≠ {a, c, d} = cl_T(A) ∩ cl_T(B).

Proposition 3.11 The union of two T-closed sets, is T-closed.

Proof: Let A, B be T-closed sets. We want to show that cl_T(A ∪ B) = A ∪ B. cl_T(A ∪ B) = cl_T(A) ∪ cl_T(B) = A ∪ B and therefore, A ∪ B is T-closed set.

Proposition 3.12 The intersection of two T-closed sets, is T-closed.

Proof: Let A, B be T-closed sets. We want to show that cl_T(A ∩ B) = A ∩ B. We have, A ∩ B ⊆ cl_T(A ∩ B) and cl_T(A ∩ B) ⊆ cl_T(A) ∩ cl_T(B) = A ∩ B. Thus, cl_T(A ∩ B) = A ∩ B.

4. GENERALIZED T-CLOSED SETS

In this section, we introduce a generalization of sets which introduced in section 3 and study their basic properties.

Definition 4.1 A subset A of a topological space X is called generalized T-closed set (briefly, gT-closed) if cl_T(A) ⊆ U whenever A ⊆ U and U is open. The complement of gT-closed set is called gT-open.

The family of all gT-closed (resp. gT-open) sets is denoted by GTC(X) (resp. GTO(X)).

Lemma 4.1 For a subset A of a topological space X, $(cl_T(A))^c = int_T(A^c)$.

Proof. Let x ∈ $(cl_T(A))^c$ which means that x ∉ $cl_T(A)$. Then, there exists at least one open set U containing x such that cl^*(U) ⋂ A = ϕ which implies cl^*(U) ⊆ A^c. Thus, x ∈ $int_T(A^c)$ and therefore, $(cl_T(A))^c \subseteq int_T(A^c)$. Now, we want to show that $int_T(A^c) \subseteq (cl_T(A))^c$. Let x ∉ $(cl_T(A))^c$ which means that x ∈ $cl_T(A)$. Then, for every open set U containing x, we have cl^*(U) ⋂ A ≠ ϕ. Thus x ∉ $int_T(A^c)$ and therefore $int_T(A^c) \subseteq (cl_T(A))^c$. Hence $(cl_T(A))^c = int_T(A^c)$.

Proposition 4.1 A subset A of a topological space X is generalized T-open (briefly, gT-open) iff F ⊆ int_T(A) whenever F ⊆ A and F is closed.

Proof. Let A be a gT-open set and F ⊆ A where F is closed. Then, $A^c \subseteq F^c$ = U, and since U is open and A^c is gT-closed and from lemma 4.1we have, $cl_T(A^c)$ ⊆ U and F ⊆ $(cl_T(A^c))^c = int_T$(A). Conversely, Let A^c ⊆ U where U is open, then F = U^c ⊆ A and from the assumption we have, F ⊆ int_T(A) and from lemma 4.1 $(int_T(A))^c = cl_T(A^c)$ ⊆ U. Thus A^c is gT-closed set and hence, A is gT-open.

Proposition 4.2 Every T-closed set is gT-closed set.

Proof: Let A be a T-closed set and A ⊆ U, U open. Then, cl_T(A) = A ⊆ U and therefore A is gT-closed set.

The converse of the previous proposition is not true in general as shown from the following example.

Example 4.1 Let X = {a, b, c}, τ = {X, φ, {b}, {c}, {b, c}, {a, c}}. We can see that {a, b} is gT-closed set but not T-closed set.

Proposition 4.3 Every gT-closed set is g-closed set.

Proof: Let A be a gT-closed set and A ⊆ U, U is open. Then cl_T(A) ⊆ U, but cl (A) ⊆ cl_T(A) and then, cl (A) ⊆ U. Thus A is g-closed set.

The converse of the previous proposition is not true in general as shown from the following example.

Example 4.2 Let X = {a, b, c, d}, τ = {X, φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}}. We have {a, d} is g-closed set but not gT-closed.

Proposition 4.4 Every θ-g-closed set is gT-closed.

Proof: Let A be a θ-g-closed set where A ⊆ U and U is open. Then cl_θ(A) ⊆ U. But cl_T(A) ⊆ cl_θ(A), hence cl_T(A) ⊆ U. Thus A is gT-closed.

The converse of the previous proposition is not true in general as shown from the following example.

Example 4.3 Let X = {a, b, c, d, e}, τ = {X, φ, {a, b, c}, {c, d, e}, {c}}. We have {a} is gT-closed but not θ-g-closed.

From propositions 4.2, 4.3, and 4.4 the diagram in [2] can be extended to the following one.

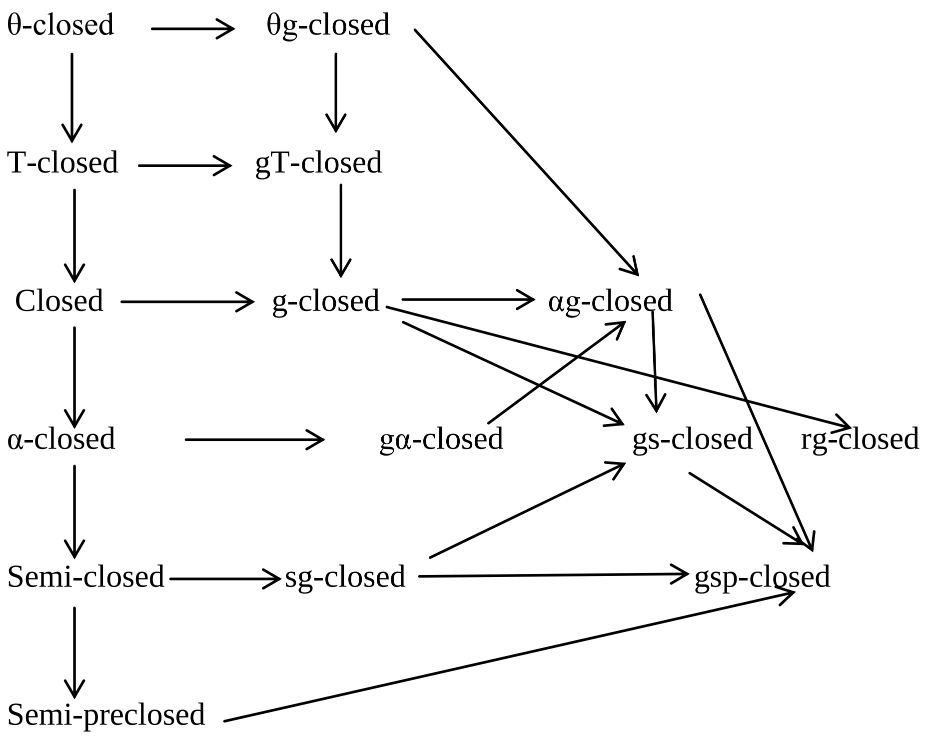

Figure (2)

Proposition 4.4 If A ⊆ X is gT-closed, then cl_T(A) – A does not contain a non-empty closed set.

Proof: Let F ⊆ cl_T(A) – A be a closed set. Then, A ⊆ F^c and since A is gT-closed, we have cl_T(A) ⊆ F^c. Thus, F ⊆ cl_T(A) ∩ $(cl_T(A))^c$ = φ which means that F is empty.

Proposition 4.5 A subset A of a topological space X is gT-closed iff A ≡ cl_T(A).

Proof. Let A be a gT-closed set. Then, A ⊆ U iff cl_T(A) ⊆ U where U is open. Thus, A ≡ cl_T(A). Conversely, Let A ≡ cl_T(A). Then, if A ⊆ U and U is open implies cl_T(A) ⊆ U and then A is gT-closed.

Proposition 4.6 The union of two gT-closed sets is gT-closed.

Proof: Let A, B be gT-closed sets and suppose that A ∪ B ⊆ U and U is open. Then A ⊆ U and hence cl_T(A) ⊆ U since A is gT-closed set. Similarly, cl_T(B) ⊆ U and therefore, cl_T(A ∪ B) = cl_T(A) ∪ cl_T(B) ⊆ U. Thus A ∪ B is also gT-closed set.

The intersection of two gT-closed sets is not gT-closed in general as shown from the following example.

Example 4.3 Let X = {a, b, c}, τ = {X, φ, {a}}. The sets {a, b} and {a, c} are gT-closed sets but its intersection {a} is not gT-closed set.

Proposition 4.6 The intersection of a gT-closed and a T-closed is always a gT-closed.

Proof: Let A be a gT-closed and let F be a T-closed. Suppose that A ∩ F ⊆ U where U is an open set. Putting G = F^c, then A ⊆ U ∪ G. But since G is T-open and U is open, then G ∪ U is open set and hence cl_T(A) ⊆ G ∪ U. Now, we can write cl_T(A ∩ F) ⊆ cl_T(A) ∩ cl_T(F) = cl_T(A) ∩ F ⊆ (G ∪ U) ∩ F = (G ∩ F) ∪ (U ∩ F) = φ ∪ (U ∩ F) ⊆ U. Thus A ∩ F is a gT-closed set.

Proposition 4.7 A topological space X is a $T_{1/2}$-space iff every gT-closed is closed.

Proof: Let X be a $T_{1/2}$-space, and suppose that A ⊆ X is a gT-closed set, then, A is g-closed and since X is $T_{1/2}$-space hence, A is closed. Conversely, Let x ∈ X. If {x} is not closed, then $\{x\}^c$ is not open and hence the only superset of $\{x\}^c$ is X. Thus, $\{x\}^c$ is gT-closed and hence closed from the assumption which means that {x} is open. Thus, every singleton set is either open or closed and therefore, X is a $T_{1/2}$-space.

5. APPLICATION OF T-CLOSED SETS.

In this section, we introduce new separation axioms called T_c-space, T_θ-space and we study its properties and its relation with $T_{1/2}$-space which is considered as a main tool in digital Topology.

Definition 5.1 A topological space X is called T_c-space if every g-closed set is T-closed.

Definition 5.2 A topological space X is called T_θ-space if every T-closed set is θ-closed.

Proposition 5.1 Every T_c-space is $T_{1/2}$-space.

Proof. Let X be T_c-space. Suppose that A is g-closed set and since X is T_c-space then, A is T-closed and hence closed. Therefore X is $T_{1/2}$-space.

The converse of the previous proposition is not true in general as shown from the following example.

Example 5.1 Let X = {a, b, c, d} and τ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}}, then (X, τ) is a $T_{1/2}$-space. But (X, τ) is not T_c-space since {c} is g-closed set but not T-closed.

Proposition 5.2 Every $T_{1/2}$-space is T_θ-space.

Proof. Let X be $T_{1/2}$-space, then g-closed sets and closed sets coincide. Hence for any subset A ⊆ X, cl (A) = cl^*(A). Therefore, θ-closed sets and T-closed sets coincide which means that X is T_θ-space.

The converse of the previous proposition is not true in general as shown from the following example.

Example 5.2 Let X = {a, b, c} and τ = {X, ϕ, {a, c}}. (X, τ) is T_θ-space but not $T_{1/2}$-space, since {b} is g-closed but not closed.

From the above propositions, we have the following implications

T_c-space ⟶ $T_{1/2}$-space ⟶ T_θ-space

Figure (3)

These implications can't be reversed as we shown earlier.

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