Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH RESEARCH ORGANISATION| Volume 9, Issue 1
Published online: August 04, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

The New Generalized Difference of χ^2 over $p-$ Metric Spaces Defined by Musielak Orlicz Function

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Abstract. We introduce new sequence spaces by using Musielak-Orlicz function and a generalized B^{μ}_{η}-difference operator or $p-$metric space. Some topological properties are studied.

Key words and phrases. analytic sequence, double sequences, χ^2 space, difference sequence space,Musielak - Orlicz function, $p-$ metric space, Lacunary sequence, ideal.

2010 *Mathematics Subject Classification.* 40A05,40C05,40D05.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on it was investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy et al.,[6-10], Turkmenoglu [11], Raj [12-14] and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) .$$

A double sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

(1.1) $$d(x,y) = sup_{m,n}\left\{|x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m,n : 1,2,3,...\right\},$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}\, in\, \Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m,n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2.

Let M and Φ be mutually complementary Orlicz functions. Then, we have

(i) For all $u, y \geq 0$,

(1.2) $$uy \leq M(u) + \Phi(y),\, (Young's\, inequality)[See[Kampthan et al., [15]]$$

(ii) For all $u \geq 0$,

(1.3) $$u\eta(u) = M(u) + \Phi(\eta(u)).$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

(1.4) $$M(\lambda u) \leq \lambda M(u).$$

Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\, (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - f_{mn}(u) : u \geq 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak Orlicz function f, the Musielak-Orlicz sequence space t_f is defined by

$$t_f = \left\{x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m, n \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = sup_{mn}\left\{inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1\right\}.$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [16] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1}x_{mn} - \Delta^{m-1}x_{mn+1} - \Delta^{m-1}x_{m+1n} + \Delta^{m-1}x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation: $\Delta^m x_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}x_{m+i,n+j}$.
Let $\eta = (\eta_{mn})$ be a sequence of nonzero scalars. Then, for a sequence space E, the multiplier sequence space E_η, associated with the multiplier sequence η, is defined as

$$E_\eta = \left\{x = (x_{mn}) \in w^2 : (\eta_{mn}x_{mn}) \in E\right\}.$$

2. Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,

(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = |\alpha| \, \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)) = (d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p)^{1/p} \; for 1 \leq p < \infty$; (or)

(v) $d((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)) := \sup\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\}$,

for $x_1,x_2,\cdots x_n \in X, y_1,y_2,\cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}(x_{mn}))|\right) = sup\left(\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{1n},0) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \ldots & d_{nn}(x_{nn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1},\cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\cdots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Let X be a linear metric space. A function $w : X \to \mathbb{R}$ is called paranorm, if

(1) $w(x) \geq 0$, for all $x \in X$;

(2) $w(-x) = w(x)$, for all $x \in X$;

(3) $w(x+y) \leq w(x) + w(y)$, for all $x, y \in X$;

(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m,n \to \infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \to 0$ as $m,n \to \infty$, then $w(\sigma_{mn}x_{mn} - \sigma x) \to 0$ as $m,n \to \infty$.

A paranorm w for which $w(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X,w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm by Willansky [17].

$\eta = (\varphi_{rs})$ a nondecreasing sequence of positive reals tending to infinity and $\varphi_{11} = 1$ and $\varphi_{r+1,s+1} \leq \varphi_{rs} + 1$.

The generalized de la Vallee-Pussin means is defined by :

$$t_{rs}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} x_{mn},$$

where $I_{rs} = [rs - \lambda_{rs} + 1, rs]$. For the set of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallee-Poussin method.

The notion of $\lambda-$ double gai and double analytic sequences as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{00} < \lambda_{11} < \cdots \, and \, \lambda_{mn} \to \infty \, as \, m, n \to \infty$$

and said that a sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0, called a the $\lambda-$ limit of x, if $B_{\eta}^{\mu}(x) \to 0 \, as \, m, n \to \infty$, where

$$B_{\eta}^{\mu}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left(\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1} \right) |x_{mn}|^{1/m+n}.$$

The sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ double analytic if $sup \left| B_{\eta}^{\mu}(x) \right| < \infty$. If $lim_{mn} x_{mn} = 0$ in the ordinary sense of convergence, then

$lim_{rs} \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left(\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1} \right)$
$\left((m+n)! \, |x_{mn} - 0| \right)^{1/m+n} = 0$. This implies that
$lim_{rs} \left| B_{\eta}^{\mu}(x) - 0 \right| = \lim_{rs} \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}}$
$\left(\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1} \right) \left((m+n)! \, |x_{mn} - 0| \right)^{1/m+n} = 0.$
which yields that $lim_{uv} \mu_{mn}(x) = 0$ and hence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0.

Let $f = (f_{mn})$ be a Musielak-Orlicz function and $\left(X, \| (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)) \|_p \right)$ be a $p-$metric space, $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. By $w^2(p - X)$ we denote the space of all sequences defined over $\left(X, \| (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)) \|_p \right)$. The following inequality will be used throughout the paper. If $0 \le q_{mn} \le sup q_{mn} = H, K = max\left(1, 2^{H-1}\right)$ then

$$|a_{mn} + b_{mn}|^{q_{mn}} \le K \left\{ |a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}} \right\} \tag{2.1}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \le max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$.

Let $\left(X, \| (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)) \|_p \right)$ be an $p-$ metric space and let $s\left(w^2 - x\right)$ denote the space of $X-$ valued sequences. Let $q = (q_{mn})$ be any bounded sequence of positive real numbers and $f = (f_{mn})$ be a Musielak-Orlicz function. We define the following sequence spaces in this paper:

$$\left[\chi_{fB_{\eta}^{\mu}}^{2q}, \| (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)) \|_p^{\varphi} \right]^V =$$
$$\left\{ x = (x_{mn}) \in s\left(w^2 - x\right) : lim_{rs} \left[f_{mn} \left(\left\| B_{\eta}^{\mu}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p \right) \right]^{q_{mn}} = 0 \right\},$$

$$\left[\Lambda_{fB_{\eta}^{\mu}}^{2q}, \| (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)) \|_p^{\varphi} \right]^V =$$
$$\left\{ x = (x_{mn}) \in s\left(w^2 - x\right) : sup_{rs} \left[f_{mn} \left(\left\| B_{\eta}^{\mu}(x), (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} < \infty \right\},$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2q}_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|^{\varphi}_p\right]^V =$$
$$\left\{x = (x_{mn}) \in s\left(w^2 - x\right) : lim_{rs}\left[f_{mn}\left(\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|^{\varphi}_p\right]^V =$$
$$\left\{x = (x_{mn}) \in s\left(w^2 - x\right) : sup_{rs}\left[f_{mn}\left(\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} < \infty\right\},$$

If we take $q = (q_{mn}) = 1$, we get

$$\left[\chi^{2}_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|^{\varphi}_p\right]^V =$$
$$\left\{x = (x_{mn}) \in s\left(w^2 - x\right) : \left[f_{mn}\left(\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right] = 0\right\},$$

$$\left[\Lambda^{2}_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|^{\varphi}_p\right]^V =$$
$$\left\{x = (x_{mn}) \in s\left(w^2 - x\right) : \left[f_{mn}\left(\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right] < \infty\right\},$$

In the present paper we plan, some topological properties are studied in the following sequence spaces. $\left[\chi^{2q}_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|^{\varphi}_p\right]^V$ and $\left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|^{\varphi}_p\right]^V$ which we shall discuss in this paper.

3. Main Results

3.1. **Theorem.** Let $f = (f_{mn})$ be a Musielak-Orlicz function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence space $\left[\chi^{2q}_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|^{\varphi}_p\right]^V$ is a paranormed space with respect to the paranorm defined by

$g(x) = inf$
$$\left\{\left[f_{mn}\left(\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \le 1\right\} = 0.$$

Proof: Clearly $g(x) \ge 0$ for $x = (x_{mn}) \in \left[\chi^{2q}_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|^{\varphi}_p\right]^V$ Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then
$inf\left\{\left[f_{mn}\left(\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \le 1\right\} = 0$
Suppose that $B^{\mu}_{\eta}(x) \ne 0$ for each $m, n \in \mathbb{N}$. Then $\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^{\varphi}_p \to \infty$. It follows that $\left(\left[f_{mn}\left(\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \to \infty$ which is a contradiction. Therefore $B^{\mu}_{\eta}(x) = 0$. Let

$$\left(\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1$$

and

$$\left(\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(y\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1$$

Then by using Minkowski's inequality, we have

$$\left(\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(x+y\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq$$
$$\left(\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}+$$
$$\left(\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(y\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}.$$

So we have

$$g\left(x+y\right)=inf\left\{\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(x+y\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}\leq$$
$$inf\left\{\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}+$$
$$inf\left\{\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(y\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

Therefore,

$$g\left(x+y\right)\leq g\left(x\right)+g\left(y\right).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g\left(\lambda x\right)=inf\left\{\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(\lambda x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}.$$

Then

$$g\left(\lambda x\right)=inf\left\{\left(\left(|\lambda|t\right)^{q_{mn}/H}:\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(\lambda x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}}\leq max\left(1,|\lambda|^{supq_{mn}}\right)$, we have

$$g\left(\lambda x\right)\leq max\left(1,|\lambda|^{supq_{mn}}\right)inf$$
$$\left\{t^{q_{mn}/H}:\left[f_{mn}\left(\left\|B^{\mu}_{\eta}\left(\lambda x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

This completes the proof.

3.2. **Theorem.** (i) If the sequence (f_{mn}) satisfies uniform Δ_2- condition, then

$$\left[\chi^{2q}_{fB^{\mu}_{\eta}},\left\|B^{\mu}_{\eta}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|^{\varphi}_p\right]^{V\alpha}=$$
$$\left[\chi^{2qB^{\mu}_{\eta}}_{g},\left\|B^{\mu}_{\eta}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|^{\varphi}_p\right]^{V}.$$

(ii) If the sequence (g_{mn}) satisfies uniform Δ_2- condition, then

$$\left[\chi^{2qB^{\mu}_{\eta}}_{g},\left\|B^{\mu}_{\eta}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|^{\varphi}_p\right]^{V\alpha}=$$
$$\left[\chi^{2q}_{fB^{\mu}_{\eta}},\left\|B^{\mu}_{\eta}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|^{\varphi}_p\right]^{V}$$

Proof: Let the sequence (f_{mn}) satisfies uniform Δ_2- condition, we get

(3.1)
$$\left[\chi_g^{2qB_\eta^\mu}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V \subset \left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^{V\alpha}$$

To prove the inclusion
$$\left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^{V\alpha} \subset$$

$$\left[\chi_g^{2qB_\eta^\mu}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V,$$

let $a \in \left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V$. Then for all $\{x_{mn}\}$ with $(x_{mn}) \in$ $\left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}a_{mn}| < \infty. \tag{3.2}$$

Since the sequence (f_{mn}) satisfies uniform Δ_2- condition, then
$(y_{mn}) \in \left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V$,
we get $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\varphi_{rs}y_{mn}a_{mn}}{\Delta^m\lambda_{mn}(m+n)!}\right| < \infty$. by (3.2). Thus
$(\varphi_{rs}a_{mn}) \in \left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V =$
$\left[\chi_g^{2qB_\eta^\mu}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V$ and hence
$(a_{mn}) \in \left[\chi_g^{2qB_\eta^\mu}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V$. This gives that

(3.3)
$$\left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^{V\alpha} \subset \left[\chi_g^{2qB_\eta^\mu}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V$$

we are granted with (3.1) and (3.3)
$$\left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^{V\alpha} =$$
$$\left[\chi_g^{2qB_\eta^\mu}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^V$$

(ii) Similarly, one can prove that $\left[\chi_g^{2qB_\eta^\mu}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^\varphi\right]^{V\alpha} \subset \left[\chi_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0\right.\right.$
if the sequence (g_{mn}) satisfies uniform Δ_2- condition.

3.3. **Proposition.** If $f = (f_{mn})$ be any Musielak Orlicz function. Then
$\left[\Lambda_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^{\varphi^*}\right]^V \subset$
$\left[\Lambda_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^{\varphi^{**}}\right]^V$ if and only if $sup_{r,s\geq 1}\frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty$.

Proof: Let $x \in \left[\Lambda_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^{\varphi^*}\right]^V$ and $N = sup_{r,s\geq 1}\frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} <$
∞. Then we get
$$\left[\Lambda_{fB_\eta^\mu}^{2q}, \left\|B_\eta^\mu(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p^{\varphi_{rs}^{**}}\right]^V =$$

$N\left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi^*_{rs}}\right]^V = 0.$

Thus $x \in \left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi^{**}}\right]^V$. Conversely, suppose that $\left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi^*}\right]^V \subset$

$\left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi^{**}}\right]^V$ and $x \in \left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi^*}\right]^V$. Then $\left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi^*}\right]^V < \epsilon$, for every $\epsilon > 0$. Suppose that $sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} = \infty$, then there exists a sequence of members (rs_{jk}) such that $lim_{j,k\to\infty}\frac{\varphi^*_{jk}}{\varphi^{**}_{jk}} = \infty$. Hence, we have $\left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi^*_{rs}}\right]^V = \infty$. Therefore $x \notin \left[\Lambda^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi^{**}}\right]^V$, which is a contradiction. This completes the proof.

3.4. **Proposition.** The sequence space $\left[\chi^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi}\right]^V$ is not solid

Proof: The result follows from the following example.

Example: Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1 & 1 & ... & 1 \end{pmatrix} \in \left[\chi^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi}\right]^V. \text{ Let}$$

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & ... & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & ... & -1^{m+n} \\ . & & & \\ . & & & \\ . & & & \\ -1^{m+n} & -1^{m+n} & ... & -1^{m+n} \end{pmatrix}, \text{ for all } m,n \in \mathbb{N}.$$

Then $\alpha_{mn}x_{mn} \notin \left[\chi^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi}\right]^V$. Hence $\left[\chi^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi}\right]^V$ is not solid.

3.5. **Proposition.** The sequence space $\left[\chi^{2q}_{fB^{\mu}_{\eta}}, \left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p^{\varphi}\right]^V$ is not monotone

Proof: The proof follows from Proposition 3.4.

A sequence $x = (x_{mn})$ is said to be $\varphi-$ statistically convergent or $s_{\varphi}-$ statistically convergent to 0 if for every $\epsilon > 0$,

$$lim_{rs}\left|\left\{\left[f_{mn}\left(\left\|B^{\mu}_{\eta}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\right| \geq \epsilon\right\} = 0$$

where the vertical bars indicates the number of elements in the enclosed set. In this case we write $s_\varphi - limx = 0$ or $x_{mn} \to 0\,(s_\varphi)$ and $s_\varphi = \{x : \exists 0 \in \mathbb{R} : s_\varphi - limx = 0\}$.

3.6. **Proposition.** For any sequence of Musielak Orlicz functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. Then
$\left[\chi^{2q}_{fB^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V \subset$
$\left[s^{2q}_{\varphi f B^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V$.

Proof: Let $x \in \left[\chi^{2q}_{fB^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V$ and $\epsilon > 0$. Then
$\left[f_{mn}\left(\left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \geq$
$\left|\left\{\left[f_{mn}\left(\left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\right| \geq \epsilon\right\}$
from which it follows that $x \in \left[s^{2q}_{\varphi f B^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V$.

To show that $\left[s^{2q}_{\varphi f B^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V$ strictly contain $\left[\chi^{2q}_{fB^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V$. We define $x = (x_{mn})$ by $(x_{mn}) = mn$ if $rs - \left[\sqrt{\varphi_{rs}}\right] + \leq mn \leq rs$ and $(x_{mn}) = 0$ otherwise. Then
$x \notin \left[\Lambda^{2q}_{fB^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V$ and for every $\epsilon\,(0 < \epsilon \leq 1)$,

$$\left|\left\{\left[f_{mn}\left(\left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\right| \geq \epsilon\right\} = \frac{\left[\sqrt{\varphi_{rs}}\right]}{\varphi_{rs}} \to 0 \text{ as } r,s \to \infty$$

i.e $x \to 0\left(\left[s^{2q}_{\varphi f B^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V\right)$, where [] denotes the greatest integer function. On the other hand,

$$\left[f_{mn}\left(\left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \to \infty \text{ as } r,s \to \infty$$

i.e $x_{mn} \not\to 0 \left[\chi^{2q}_{fB^\mu_\eta}, \left\|B^\mu_\eta(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\right\|^\varphi_p\right]^V$. This completes the proof.

Competing Interests: The authors declare that there is no conflict of interests regarding the publication of this research paper.

Acknowledgement

The authors are extremely grateful to the anonymous learned referee(s) for their keen reading, valuable suggestion and constructive comments for the improvement of the manuscript. The authors are thankful to the editor(s) and reviewers of Journal of Progressive Research in Mathematics. The research of the first author Deepmala is supported by the Science and Engineering Research Board (SERB), Department of Science and Technology (DST), Government of India under SERB National Post-Doctoral fellowship scheme File Number:PDF/2015/000799.

References

[1] Bromwich, T.J.I'A. 1965. An introduction to the theory of infinite series *Macmillan and Co.Ltd.*, New York.

[2] Hardy, G.H. 1917. On the convergence of certain multiple series, *Proceedings of the Cambridge Philosophical Society*, **19**, 86-95.

[3] Moricz, F. 1991. Extentions of the spaces c and c_0 from single to double sequences, *Acta Mathematica Hungarica* , **57(1-2)**, 129-136.

[4] Moricz, F. and Rhoades, B.E. 1988. Almost convergence of double sequences and strong regularity of summability matrices, *Mathematical Proceedings of the Cambridge Philosophical Society*, **104**, 283-294.

[5] Başar, F. and Sever, Y. 2009. The space $\mathcal{L}_p$ of double sequences, *Mathematical Journal of Okayama University*, **51**, 149-157.

[6] Tripathy, B.C. and Sarma, B. 2009. Vector valued double sequence spaces defined by Orlicz function, *Mathematica Slovaca*, **59(6)** , 767-776.

[7] Tripathy, B.C. and Dutta, A.J. 2010. Bounded variation double sequence space of fuzzy real numbers, *Computers and Mathematics with Applications*, **59(2)**, 1031-1037.

[8] Tripathy, B.C. and Sarma, B. 2011. Double sequence spaces of fuzzy numbers defined by Orlicz function, *Acta Mathematica Scientia*, **31 B(1)(2011)**, 134-140.

[9] Tripathy, B.C. and Chandra, P. 2011. On some generalized difference paranormed sequence spaces associated with multiplier sequences defined by modulus function, *Analysis in Theory and Applications*, **27(1)** , 21-27.

[10] Tripathy, B.C. and Dutta, A.J. 2013. Lacunary bounded variation sequence of fuzzy real numbers, *Journal in Intelligent and Fuzzy Systems*, **24(1)**, 185-189.

[11] Turkmenoglu, A.1999. Matrix transformation between some classes of double sequences, *Journal of Institute of Mathematics and Computer Science Mathematics Series.*, **12(1)**, 23-31.

[12] Raj, K.and Sharma, S.K. 2011. Some sequence spaces in 2-normed spaces defined by Musielak Orlicz function, *Acta Universitatis Sapientiae Mathematica*, **Vol. 3, no.1**, pp. 97-109.

[13] Raj, K.Sharma, A.K. and Sharma, S.K. 2012. Sequence spaces defined by Musielak-Orlicz function in 2-normed spaces, *Journal of Computational Analysis and Applications*, **Vol. 14**.

[14] Raj, K.and Sharma, S.K.2013. Lacunary sequence spaces defined by Musielak-Orlicz function, *Le Matematiche*, **68** , 33-51.

[15] Kamthan, P.K. and Gupta, M. 1981. Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c., New York.*

[16] Lindenstrauss, J. and Tzafriri, L. 1971. On Orlicz sequence spaces, *Israel Journal of Mathematics*,**10**, 379-390.

[17] Wilansky, A. 1984. Summability through Functional Analysis, *North-Holland Mathematical Studies, North-Holland Publishing, Amsterdam*, **Vol.85**.

[18] Mishra, V.N. 2007. Some problems on Approximations of functions in Banach spaces, Ph.D, thesis , *Indian Institute of Technology Roorkee, 247 667, Uttarakhand, India* .

[19] Mishra, V.N. Subramanian, N. and Khan, Idrees A. 2016. Riesz Triple Almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ Lacunary $\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$ sequence spaces defined by a Orlicz function-III, *International Journal of Mathematics and Mathematical Sciences* , **in press**.

[20] Deepmala, Subramanian, N. and Mishra, V.N. 2016. The Generalized difference of $\int \chi^{2I}$ of fuzzy real numbers over $p-$ metric spaces defined by Musielak-Orlicz function, *New Trends in Mathematical Sciences (2016)*, **in press**.

[21] Deepmala, Subramanian, N. and Mishra, V.N. 2016. Double Almost $(\lambda_m\mu_n)$ in χ^2-Riesz Space, *Southeast Asian Bulletin of Mathematics (2016)*, **in press**.