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On Centralizing and Generalized Derivations Of prime Rings with involution

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Abstract

Let $(R, *)$ be a 2-torsion free $*$-prime ring with involution $*$, $L \neq 0$ be a nonzero square closed $*$-Lie ideal of R and Z the center of R. An additive mapping F: $R \longrightarrow R$ is called a generalized derivation on R if there exists a derivation d: $R \longrightarrow R$ commutes with $*$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper, we shall show that L is contained in the center of R such that R admits a generalized derivations F and G with associated derivations d and g commute with $*$ satisfying several conditions.

1 Introduction

Let R be an associative ring with center $Z(R)$ and involution $*$. For each $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$ and the symbol $x \circ y$ stands for the skew-commutator $xy + yx$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$. A left (resp. right, two sided) ideal L of R is called a left (resp. right, two sided) $*$-ideal if $L^* = L$. An ideal P of R is called $*$-prime ideal if $P(\neq R)$ is a $*$-ideal and for $*$-ideals L, J of R, $LJ \subseteq P$ implies that $L \subseteq P$ or

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$J \subseteq P$. An example: Let $\mathbb{Z}$ be the ring of integers. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$. We define a map $* : R \longrightarrow R$ as follows: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$. It is easy to check that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$ is a $*-$ideal of R. Now we give an example of $*-$ prime ideal: Let F be any field and $R = F[x]$ be the polynomial ring over F. Let $* : R \longrightarrow R$ be a map defined by $(f(x))^* = f(-x)$ for all $f(x) \in R$. Then it is easy to check that xR is a $*-$prime ideal of R. Note that an ideal I of R may be not a $*-$ideal: Let $\mathbb{Z}$ be the ring of integers and $R = \mathbb{Z} \times \mathbb{Z}$. Consider a map $* : R \longrightarrow R$ defined by $((a, b))^* = (b, a)$ for all $a, b \in R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of R, I is not a $*-$ideal of R since $I^* = \{0\} \times \mathbb{Z} \neq I$. A ring R equipped with an involution $*$ is said to be a $*$-prime ring if for any $a, b \in R$, $aRb = aRb^* = \{0\}$ implies $a = 0$ or $b = 0$. Obviously, every prime ring equipped with involution $*$ is $*$-prime. The converse need not be true in general. An example due to L. Oukhtite justifies the above statement is as following: Let R be a prime ring, $S = R \times R^\circ$ where R° is the opposite ring of R. Define involution $*$ on S as $(x, y)^* = (y, x)$. Since $(0, x)S(x, 0) = 0$, it follows that S is not prime. Further, it can be easily seen that if $(a, b)S(c, d) = (a, b)S(c, d)^* = 0$, then either $(a, b) = 0$ or $(c, d) = 0$. Hence S is $*-$prime but not prime. The set of symmetric and skew-symmetric elements of a $*-$ ring will be denoted by $S_*(R)$ i.e., $S_*(R) = \{x \in R \mid x^* = \pm x\}$.

An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal is said to be a $*-$Lie ideal if $L^* = L$. If L is a Lie (resp. $*$-Lie) ideal of R, then L is called a square closed Lie (resp. $*$- Lie) ideal of R if $x^2 \in L$ for all $x \in L$.

An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation.

An additive function $F : R \longrightarrow R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that

$$F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y) \quad \text{for all} \quad x, y \in R.$$

This observation leads to the following definition, an additive mapping $F : R \rightarrow R$ is called a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.
Familiar examples of generalized derivations are derivations and generalized inner derivations, and the latter includes left multipliers. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where c is a fixed element of R and d a derivation of R, is a generalized derivation; and if R has multiplicative identity 1, then all generalized derivations have this form. Over the last four decade, several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms, derivations or generalized derivations which are centralizing or commuting on appropriate subset of R (see [1], [2], [3] [7] and [17], for partial bibliography). In this paper, we shall discuss when $L \subseteq Z(R)$

such that R is a $*$-prime ring admitting a generalized derivations F and G satisfying any one of the following properties: (i) $d(x)F(y) - xy \in Z(R)$, (ii) $[F(x), x] \in Z(R)$, (iii) $(F(x) \circ x) \in Z(R)$, (iv) $F(x \circ y) + [x, y] \in Z(R)$, (v) $F[x, y] - (x \circ y) \in Z(R)$, (vi) $F[x, y] - (F(x) \circ y) - [d(y), x] \in Z(R)$, (vii) $[F(x), F(y)] - [x, y] \in Z(R)$, $(viii)$ $F(x) \circ F(y) - (x \circ y) \in Z(R)$, (ix) $[F(x), F(y)] - (x \circ y) \in Z(R)$, (x) $(F(x) \circ F(y)) - [x, y] \in Z(R)$, (xi) $[F(x), x] - [x, G(x)] \in Z(R)$, (xii) $(F(x) \circ x) - (x \circ G(x)) \in Z(R)$, $(xiii)$ $[F(x), G(y)] - [x, y] \in Z(R)$, $(xiiii)$ $[F(x), F(y)] - F[x, y] \in Z(R)$, for all $x, y \in L$

2 Preliminary Result

We shall be frequently using the following identities without any specific mention:

- $[xy, z] = x[y, z] + [x, z]y$
- $[x, yz] = y[x, z] + [x, y]z$
- $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$
- $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$

We begin with the following known results which shall be used throughout to prove our theorems:

Lemma 2.1. [[13], **Lemma 4**] *Let R be a 2-torsion free $*$-prime ring and L a nonzero $*$-Lie ideal of R. If $a, b \in R$ such that $aLb = aLb^* = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2. [[14], **Theorem 1**] *Let R be a $2-$torsion free $*$-prime ring and L a square closed $*$-Lie ideal of R. If d is a derivation of R satisfying $[d(x), x] \in Z(R)$ for all $x \in L$, then $L \subset Z(R)$ or $d = 0$.*

Lemma 2.3. [[4], **Lemma3**] *Let R be a $*-$prime ring with characteristic not two and L be a nonzero $*-$Lie ideal of R. Suppose that $[L, L] \subseteq Z$, then $L \subseteq Z(R)$.*

Lemma 2.4. [[11], **Lemma2.4**] *Let R be a $2-$torsion free $*-$prime ring and L a nonzero $*-$Lie ideal of R. If d is a derivation of R which commutes with $*$ and satisfying $d(L) \subseteq Z(R)$, then $L \subseteq Z(R)$.*

Lemma 2.5. [[11], **Lemma 2.5**] *Let $d(\neq 0)$ be derivation of a 2-torsion free $*-$prime ring R which commutes with $*$. Let $L \nsubseteq Z(R)$ be a $*-$Lie ideal of R. If $t \in R$ satisfies $td(L) = 0$ or $d(L)t = 0$ then $t = 0$.*

Remark 2.1. Let b and ab be in the center of a $*-$prime ring R. If b is not zero, then $a \in Z(R)$.

Proof. $0 = [ab, r] = a[b, r] + [a, r]b = [a, r]b$ for all $r \in R$. Since $b \neq 0$, then by Lemma 2.5 $[a, r] = 0$ for all $r \in R$. Hence a must be in $Z(R)$.

3 Centralizing Derivation On $*-$Prime Rings

Theorem 3.1. *Let R be a $*-$prime ring with $char(R) \neq 2$ and L be a nonzero square closed $*$- Lie ideal of R. If R admits a generalized derivation (F, d) such that d commutes with $*$. Further, suppose that R satisfies the condition:*

$$d(x)F(y) - xy \in Z(R), \text{ for all } x \in L.$$

If $F = 0$ or $d \neq 0$ then $L \subseteq Z(R)$.

Proof. If $F = 0$, then $xy \in Z(R)$ for all $x, y \in L$. In particular $[xy, w] = 0$ i.e., $x[y, w] + [x, w]y = 0$ for all $x, y, w \in L$. Replacing x by $2mx$ and using $char(R) \neq 2$, we get $[m, w]Ly = [m, w]Ly^* = 0$. Applying Lemma 2.1, we get $[m, w] = 0$ for all $m, w \in L$ and hence by Lemma 2.3 we get the required result.

Henceforth, we shall assume $d \neq 0$, then we have

$$d(x)F(y) - xy \in Z(R), \text{ for all } x \in L. \tag{3.1}$$

Replacing y by $2ym$ in (3.1) and using (3.1) & $char(R) \neq 2$, we get

$$(d(x)F(y) - xy)m + d(x)yd(m) \in Z(R).$$

This implies

$$[d(x)yd(m), m] = 0 \text{ for all } x, y, m \in L. \tag{3.2}$$

Thus, we have $d(x)[yd(m), m] + [d(x), m]yd(m) = 0$ for all $x, y, m \in L$, replacing y by $2d(x)y$ in the above expression, using (3.2) and $char(R) \neq 2$, we get $[d(x), m]d(x)yd(m) = 0$ i.e.,

$$[d(x), m]d(x)Ld(m) = 0 \text{ for all } x, y, m \in L. \tag{3.3}$$

Let $m \in L \cap S_*(R)$, then (3.3) yields that $[d(x), m]d(x)Ld(m) = [d(x), m]d(x)L(d(m))^* = 0$, and hence by Lemma 2.1 either $[d(x), m]d(x) = 0$ or $d(m) = 0$, since $m - m^* \in L \cap S_*(R)$, then either $[d(x), m - m^*]d(x) = 0$ or $d(m - m^*) = 0$. Suppose that $d(m - m^*) = 0$, then we have $d(m) = (d(m))^*$ and in view of (3.3) we get that $[d(x), m]d(x) = 0$ or $d(m) = 0$. Suppose that $[d(x), m - m^*]d(x) = 0$, then we have $[d(x), m - m^*]d(x) = 0$ or $d(m - m^*) = 0$. Since $m + m^* \in L \cap S_*(R)$. If $[d(x), m + m^*]d(x) = 0$, then $2[d(x), m]d(x) = 0$ and hence $[d(x), m]d(x) = 0$. On the other hand, if $d(m + m^*) = 0$, then $d(m) = -(d(m))^*$ and (3.3) yields that $[d(x), m]d(x) = 0$ or $d(m) = 0$. Consequently, for all $m \in L$, we get either $[d(x), m]d(x) = 0$ or $d(m) = 0$. Now let $A = \{m \in L \mid [d(x), m]d(x) = 0\}$ and $B = \{m \in L \mid d(m) = 0\}$. Then A, B are additive subgroups of L and $A \cup B = L$. But a group can not be a union of its two proper subgroups, and hence either $A = L$ or $B = L$. Now if $A = L$ then we have $[d(x), m]d(x) = 0$ for all $x, m \in L$. Replacing m by $2mn$ and using the fact that $chr(R) \neq 2$, we get $[d(x), m]nd(x) = 0$, that is

$$[d(x), m]Ld(x) = 0, \text{ for all } x, m \in L. \tag{3.4}$$

Suppose that $x \in L \cap S_*(R)$, since L is a $*-$Lie ideal and $^*d = d^*$, then (3.4) yields that $[d(x), m]Ld(x) = [d(x), m]L(d(x))^* = 0$ so by Lemma 2.1 either $[d(x), m] = 0$ or $d(x) = 0$ using the fact that $x - x^* \in L \cap S_*(R)$ we get $[d(x - x^*), m] = 0$ or $d(x - x^*) = 0$. If $[d(x - x^*), m] = 0$, then $[d(x), m] = ([d(x), m])^*$ and hence (3.4) yields that $[d(x), m] = 0$ or $d(x) = 0$. On the other hand, if $d(x - x^*) = 0$, then $d(x) = (d(x))^*$, in view of (3.4) we have $[d(x), m] = 0$ or $d(x) = 0$. In conclusion, for any $x \in L$, we find that either $[d(x), m] = 0$ or $d(x) = 0$ for all $m \in L$. Now let $A_1 = \{x \in L \mid [d(x), m] = 0\}$ and $B_1 = \{x \in L \mid d(x) = 0\}$. Then A_1, B_1 are both additive subgroups of L and $A_1 \cup B_1 = L$. But a group can not be a union of its two proper subgroups, and hence either $A_1 = L$ or $B_1 = L$. If $A_1 = L$ then $[d(x), m] = 0$ for all $x, m \in L$. In particular, $[d(x), x] = 0$ for all $x \in L$ and hence by Lemma 2.2 we get $L \subseteq Z(R)$. On the other hand, if $B_1 = L$, then we have $d(x) = 0$ and hence by Lemma 2.4 we get $L \subseteq Z(R)$.

Theorem 3.2. *Let R be a $*-$prime ring with $char(R) \neq 2$ and L be a nonzero square closed $*-$ Lie ideal of R. If R admits a generalized derivation (F, d) such that d commutes with $*$ and $d(Z(L)) \neq 0$. Further, Suppose that R satisfies the conditions:*

(i) $[F(x), x] \in Z(R)$, *for all* $x \in L$, *or*

(ii) $(F(x) \circ x) \in Z(R)$ *for all* $x \in L$,

then $L \subseteq Z(R)$.

Proof. (i) Linearizing the above expression, we get

$$[F(x), y] + [F(y), x] \in Z(R) \text{ for all } x, y \in L. \tag{3.5}$$

Replacing y by $2yz$ for any $z \in Z(L)$ in (3.5), and using the fact that $char(R) \neq 2$, we get

$$y[F(x), z] + F(y)[z, x] + [y, x]d(z) + y[d(z), x] \in Z(R). \tag{3.6}$$

Since $z \in Z(L)$ and d is a derivation $d(z) \in Z(L)$, then (3.6) yields that $[y, x]d(z) \in Z(R)$, Again since $0 \neq d(z) \in Z(L)$ so according to Remark 2.1 and $*-$primeness of R, we get $[y, x] \in Z(R)$ for all $x, y \in L$. This implies $[r, [y, x]] = 0$ for all $r \in R$ and $x, y \in L$. Replacing y by $2yx$ and using $char(R) \neq 2$, we get $[r, [x, yx]] = [r, [x, y]x] = [x, y][r, x]$. Again replacing r by ry, we get $[y, x]R[y, x] = 0$ for all $x, y \in L$. Therefore, $[y, x]R[y, x] = [y, x]R([y, x])^* = 0$, and hence $*-$primeness of R forces $[y, x] = 0$ for all $x, y \in L$. Applying Lemma 2.3, we get the required result.

(ii) For all $x \in L$, we have

$$(F(x) \circ x) \in Z(R).$$

Linearizing the last expression, we get

$$(F(x) \circ y) + (F(y) \circ x) \in Z(R) \text{ for all } x, y \in L. \tag{3.7}$$

Replacing y by $2yz$ for any $z \in Z(L)$ in (3.7) and using the fact that $char(R) \neq 2$, we get

$$-y[F(x), z] + F(y)[z, x] + (y \circ x)d(z) + y[d(z), x] \in Z(R).$$

Since $z \in Z(L)$ and d is a derivation so $d(z) \in Z(L)$, and hence we get $(y \circ x)d(z) \in Z(R)$, again since $0 \neq d(z) \in Z(L)$ and R is a $*-$prime so by Remark 2.1, we get

$$(y \circ x) \in Z(R) \text{ for all } x, y \in L. \tag{3.8}$$

This implies $[r, (y \circ x)] = 0$ for all $r \in R$, $x, y \in L$. Replacing y by $2yx$ in the last expression and using the fact that $char(R) \neq 2$, we get $(y \circ x)[r, x] = 0$ for all $r \in R$, $x, y \in L$. Again replacing r by sr, we get

$$(y \circ x)R[r, x] = 0, \text{ for all } r \in R, \ x, y \in L. \tag{3.9}$$

For all $x \in L \cap S_*(R)$, relation (3.9) yields that $(y \circ x)R[r, x] = 0 = (y \circ x)R([r, x])^*$. Since R is $*$-prime ring and hence we obtain either $(y \circ x) = 0$ or $[r, x] = 0$. Now for any $x \in L$, using the fact $x - x^* \in L \cap S_*(R)$, then $(y \circ (x - x^*)) = 0$ or $[r, x - x^*] = 0$. If $y \circ (x - x^*) = 0$, then $y \circ (x - x^*) = (y \circ x^*) = 0$, then we have either $(y \circ x) = 0$ or $[r, x)] = 0$. On other hand, if $[r, x - x^*] = 0$ then $[r, x - x^*] = [r, x^*] = 0$. In conclusion, for all $x, y \in L$ and $r \in R$ we have either $(y \circ x) = 0$ or $[r, x] = 0$. Let $A = \{x \in L \mid (y \circ x) = 0\}$, $B = \{x \in L \mid [r, x] = 0 \ for \ all \ r \in R\}$. Then A and B are both additive and $A \cup B = L$, but $(L, +)$ is not union of two its proper subgroups shows that either $A = L$ or $B = L$. If $A = L$, then $(y \circ x) = 0$ for all $x, y \in L$, replacing x by $[x, rx]$ in the last expression, we get $[x, r][y, x] = 0$, for all $x, y \in L$, $r \in R$. Again replacing r by sr, we get

$$[x, R]R[y, x] = 0 \text{ for all } x, y \in L. \tag{3.10}$$

If $x \in L \cap S_*(R)$, then $[x, R]R[y, x] = ([x, R])^*R[y, x] = 0$. Thus, $*-$primeness of R yields that either $[x, R] = 0$ or $[y, x] = 0$, but for any $x \in L$, $x - x^*$, $x + x^* \in L \cap S_*(R)$. Thus, for any $x \in L$ either $[x - x^*, R] = 0$ or $[y, x - x^*] = 0$. If $[x - x^*, R] = 0$ then equation (3.10) yields that $[x, R]R[y, x] = ([x, R])^*R[y, x] = 0$. for all $x, y \in L$, hence either $[x, R] = 0$ or $[y, x] = 0$. Let $A_1 = \{x \in L \mid [x, R] = 0\}$ and $B_1 = \{x \in L \mid [y, x] = 0\}$. Again A_1 and B_1 are both additive and $A_1 \cup B_1 = L$, but $(L, +)$ is not union of two its proper subgroups shows that either $A_1 = L$ or $B_1 = L$. If $A_1 = L$, then $[x, R] = 0$ for all $x \in L$ that is $L \subseteq Z(R)$ and if $B_1 = L$ then we have $[y, x] = 0$ for all $x, y \in L$ and hence $L \subseteq Z(R)$ by Lemma 2.3. Thus, in both cases we find that $L \subseteq Z(R)$. On the other hand if $B = L$ then we have $[x, R] = 0$ for all $x \in L$ and again $L \subseteq Z(R)$, hence in both cases we find that $L \subseteq Z(R)$.

Theorem 3.3. *Let R be a $*-$prime ring with $char(R) \neq 2$ and L be a nonzero square closed $*-$ Lie ideal of R. If R admits a generalized derivation (F, d) such that d commutes with $*$ and $d(Z(L)) \neq 0$. Further, Suppose that R satisfies the conditions:*

(*i*) *$F(x \circ y) + [x, y] \in Z(R)$ for all $x, y \in L$.*

(*ii*) *$F[x, y] - (x \circ y) \in Z(R)$, for all $x, y \in L$.*

(iii) $F[x,y]-(F(x)\circ y)-[d(y),x]\in Z(R)$ *for all* $x,y\in L$

(iv) $[F(x),F(y)]-[x,y]\in Z(R)$, *for all* $x,y\in L$

(v) $F(x)\circ F(y)-(x\circ y)\in Z(R)$, *for all* $x,y\in L$.

(vi) $[F(x),F(y)]-(x\circ y)\in Z(R)$, *for all* $x,y\in L$

(vii) $(F(x)\circ F(y))-[x,y]\in Z(R)$, *for all* $x,y\in L$

If $F=0$ *or* $d\neq 0$, *then* $L\subseteq Z(R)$.

Proof. (i) If $F=0$, we have $[x,y]\in Z(R)$. Using the same manner as the last paragraph of proof of Theorem 3.2 (i), we find that $L\subseteq Z(R)$.

Therefore, we shall assume that $d\neq 0$. Then for any $x,y\in L$, we have

$$F(x\circ y)+[x,y]\in Z(R). \tag{3.11}$$

For any $z\in Z(L)$, replacing y by $2yz$ in (3.11) and using (3.11) & $char(R)\neq 2$, we get

$$(F(x\circ y)z+(x\circ y)d(z)-F(y)[x,z]-yd[x,z]+y[x,z]-[x,y]z\in Z(R). \tag{3.12}$$

Using (3.11) and the fact that $z\in Z(L)$, we get $(x\circ y)d(z)\in Z(R)$. Now, using the same argument in the last paragraph of the proof of Theorem 3.2 (ii) after equation (3.8), we get $L\subseteq Z(R)$.

(ii) If $F=0$, then $(x\circ y)\in Z(R)$ for all $x,y\in L$, and hence by using the same technique as used in proof of Theorem 3.2 (ii) after equation (3.8) , we get the required result.

Therefore, we shall assume that $d\neq 0$, then we have for any $x,y\in L$

$$F[x,y]-(x\circ y)\in Z(R). \tag{3.13}$$

For any $z\in Z(L)$, replacing y by $2yz$ in (3.13) and using (3.13) & the fact that $char(R)\neq 2$, we get

$$F(y)[x,z]+yd[x,z]+[x,y]d(z)-y[x,z]\in Z(R), \text{ for all } x,y\in L \text{ and } z\in Z(L).$$

Since $z\in Z(L)$ then, we get $[x,y]d(z)\in Z(R)$. Using the same manner as used in the proof of the last paragraph of Theorem 3.2 (i), we get the required result.

(iii) If $F=0$ then $-[d(y),x]\in Z(R)$ for all $x,y\in I$, in particular $[d(x),x]\in Z(R)$ for all $x\in L$, and hence $L\subseteq Z(R)$ by Lemma 2.2.

Therefore, we shall assume $d\neq 0$ then for any $x,y\in L$, we have

$$F[x,y]-(F(x)\circ y)-[d(y),x]\in Z(R). \tag{3.14}$$

For any $z \in Z(L)$, replacing y by $2yz$ in (3.14) using (3.14) & the fact that $char(R) \neq 2$, we get

$$F(y)[x,z] + yd[x,z] + [x,y]d(z) - y[F(x),z] - d(y)[z,x]$$

$$-y[d(z),x] - [y,x]d(z) \in Z(R). \tag{3.15}$$

Since $z \in Z(L)$ and d is a derivation so $0 \neq d(z) \in Z(L)$, and hence we get $2[x,y]d(z) \in Z(R)$. Using the fact that $char(R) \neq 2$ and $0 \neq d(z) \in Z(L)$, we get $[x,y] \in Z(R)$, using the same arguments as used above to get the required result.

(iv) If $F = 0$, then $[x,y] \in Z(R)$ for all $x, y \in L$, hence we get the required result by using the same argument which used in the last paragraph of the proof of Theorem 3.2 (i).

Therefore, we shall assume that $d \neq 0$, then for any $x, y \in L$, we have

$$[F(x), F(y)] - [x,y] \in Z(R). \tag{3.16}$$

Replacing y by $2yz$ for any $z \in Z(L)$ in (3.16) and use (3.16) & the fact that $char(R) \neq 2$, we get

$$F(y)[F(x),z] + y[F(x),d(z)] + [F(x),y]d(z) - y[x,z] \in Z(R). \tag{3.17}$$

Now, since $z \in Z(L)$ and d is a derivation, $0 \neq d(z) \in Z(L)$ so we have $[F(x),y]d(z) \in Z(R)$, again since $0 \neq d(z) \in Z(L)$ and R is $*-$prime, then according to Remark 2.1, we get $[F(x),y] \in Z(R)$ for all $x, y \in L$. In particular we have $[F(x),x] \in Z(R)$ for all $x \in L$. Now using Theorem 3.2 (i), we get the required result.

If the commutator is replaced by the anti-commutator in the the last theorem, then we see that the conclusion of these theorem hold good.

(v) If $F = 0$, then $-(x \circ y) \in Z(R)$ for all $x, y \in L$, and hence we get the required result by using same argument which used in the proof of Theorem 3.2 (i) after equation (3.8) .

Therefore, we shall assume that $d \neq 0$ i.e., we have

$$F(x) \circ F(y) - x \circ y \in Z(R) \text{ for all } x, y \in L. \tag{3.18}$$

Replacing y by $2yz$, for all $z \in Z(L)$ in (3.18) and using (3.18) & $char(R) \neq 2$, we get

$$-F(y)[F(x),z] + (F(x) \circ y)d(z) - y[F(x),d(z)] - y[x,z] \in Z(R). \tag{3.19}$$

Since $z \in Z(L)$ and d is a derivation so $0 \neq d(z) \in Z(L)$ and hence, we get $(F(x) \circ y)d(z) \in Z(R)$. Again since $0 \neq d(z) \in Z(L)$, and according to Remark 2.1 and $*-$ primeness of R, we get $(F(x) \circ y) \in Z(R)$. In particular we have $(F(x) \circ x) \in Z(R)$ for all $x \in L$. Now using Theorem 3.2 (ii), we get the required result.

(vi) If $F = 0$, then $-(x \circ y) \in Z(R)$ for all $x, y \in L$, and hence $L \subseteq Z(R)$ by using the similar arguments which used in the proof of the last paragraph of Theorem 3.2 (ii).

Therefore, we shall assume that $d \neq 0$ then for all $x, y \in L$, we have

$$[F(x), F(y)] - (x \circ y) \in Z(R). \tag{3.20}$$

Replacing y by $2yz$, for all $z \in Z(L)$ in (3.20) and using (3.20) & the fact that $char(R) \neq 2$, we get

$$F(y)[F(x), z] + y[F(x), d(z)] + [F(x), y]d(z) + y[x, z] \in Z(R). \tag{3.21}$$

Since $z \in Z(L)$ and d is a derivation so $0 \neq d(z) \in Z(L)$ and hence for all $x, y \in L$, we have $[F(x), y]d(z) \in Z(R)$, again since $0 \neq d(z) \in Z(L)$ and R is $*-$prime then according to Remark 2.1, we find that $[F(x), y] \in Z(R)$ for all $x, y \in L$. In particular we have $[F(x), x] \in Z(R)$ for all $x \in L$. Hence using Theorem 3.2 (i), we get the required result.

(vii) If $F = 0$, then $-[x, y] \in Z(R)$ for all $x, y \in L$. Using the same technique as used in the first paragraph of the last Theorem, we get the required result.

Therefore, we shall assume that $d \neq 0$ then for any $x, y \in L$, we have

$$(F(x) \circ F(y)) - [x, y] \in Z(R). \tag{3.22}$$

For all $z \in Z(L)$, replacing y by $2yz$ in (3.22) and using (3.22) & the fact that $char(R) \neq 2$, we get

$$-F(y)[F(x), z] + (F(x) \circ y)d(z) - y[F(x), d(z)] - y[x, z] \in Z(R). \tag{3.23}$$

Since $z \in Z(L)$, and $0 \neq d(z) \in Z(L)$, so we get $(F(x) \circ y)d(z) \in Z(R)$ for all $x, y \in L$. But R is $*-$prime then then according to Remark 2.1, we get $(F(x) \circ y) \in Z(R)$ for all $x, y \in L$. In particular we have $(F(x) \circ x) \in Z(R)$ for all $x \in L$. Now using Theorem 3.2 (ii), we get the required result.

Theorem 3.4. *Let R be a $*-$prime ring with $char(R) \neq 2$ and L be a nonzero square closed $*-$ Lie ideal of R. If R admits a generalized derivations F and G with associated derivation d and g commute with $*$ such that $d(Z(L))$, $g(Z(L)) \neq 0$. Further, Suppose that R satisfies the conditions:*

(i) $[F(x), x] - [x, G(x)] \in Z(R)$, for all $x \in L$, or

(ii) $(F(x) \circ x) - (x \circ G(x)) \in Z(R)$, for all $x \in L$, or

(iii) $[F(x), G(y)] - [x, y] \in Z(R)$, for all $x, y \in L$.

If $F = 0$ (or $G = 0$) or $d \neq 0$ (or $g \neq 0$) then $L \subseteq Z(R)$.

Proof. (i) It is given that F and G are generalized derivations of R such that $[F(x), x] - [x, G(x)] \in Z(R)$. If $G = 0$ $(or\ F = 0)$ then $[F(x), x] \in Z(R)$ or $([x, G(x)] \in Z(R))$ for all $x \in L$, and hence in both cases we obtain $L \subseteq Z(R)$ by Theorem 3.2 (i).

Henceforth, we shall assume that $0 \neq d$ or $(g \neq 0)$. Linearizing the above expression, we get

$$[F(x), y] + [F(y), x] - [x, G(y)] - [y, G(x)] \in Z(R). \tag{3.24}$$

Replacing y by $2yz$ for any $z \in Z(L)$ in (3.24) and using (3.24) & the fact that $char(R) \neq 2$, we get

$$y[F(x), z] + F(y)[z, x] + y[d(z), x] + [y, x]d(z) - G(y)[x, z] - y[x, g(z)] - [x, y]g(z) - y[z, G(x)] \in Z(R)$$

Since $z \in Z(L)$ and d, g are derivations so $0 \neq d(z)$, $g(z) \in Z(L)$, and hence we get $[y, x](d(z) + g(z)) \in Z(R)$.

Since $0 \neq (d(z) + g(z)) \in Z(R)$ and R is $*-$prime, then by Remark 2.1, we get $[y, x] \in Z(R)$, By using similar argument which used in the proof of last paragraph of Theorem 3.2 (i), we get the required result.

(ii) It is given that F and G are generalized derivations of R such that $(F(x) \circ x) - (x \circ G(x)) \in Z(R)$. If $G = 0$ or $(F = 0)$ then $(F(x) \circ x) \in Z(R)$ or $(x \circ G(x)) \in Z(R))$ for all $x \in L$, and hence in both cases we obtain $L \subseteq Z(R)$ by Theorem 3.2 (ii).

Henceforth, we shall assume that $0 \neq d$ or $(g \neq 0)$. Linearizing the last expression, we get

$$(F(x) \circ y) + (F(y) \circ x) - (x \circ G(y)) - (y \circ G(x)) \in Z(R). \tag{3.25}$$

For any $z \in Z(L)$, replacing y by $2yz$ in (3.25) and using (3.25) & the fact that $char(R) \neq 2$, we get

$$-y[F(x), z] + F(y)[z, x] + (y \circ x)d(z) + y[d(z), x] - G(y)[x, z]$$
$$-(x \circ y)g(z) - y[x, g(z)] - y[z, G(x)] \in Z(R). \tag{3.26}$$

Since $z \in Z(L)$ and g is a derivation so $0 \neq g(z) \in Z(L)$, and hence we get $(y \circ x)(d(z) + g(z)) \in Z(R)$, but $0 \neq (d(z) + g(z)) \in Z(L)$ and R is $*-$prime, then according to Remark 2.1, we get $(y \circ x) \in Z(R)$. Applying the similar technique as used in the last paragraph of the proof of Theorem 3.2 (ii), we obtain that $L \subseteq Z(R)$.

(iii) If $F = 0$ or $(G = 0)$, then $-[x, y] \in Z(R)$ for all $x, y \in L$, and hence by using the same argument which used in the last paragraph of proof of Theorem 3.2 (i), we get the required result.

Therefore, we shall assume that $d \neq 0$ or $(g \neq 0)$. For any $x, y \in L$, we have

$$[F(x), G(y)] - [x, y] \in Z(R). \tag{3.27}$$

For any $z \in Z(L)$, replacing y by $2yz$ in (3.27) and using (3.27) & the fact that $char(R) \neq 2$, we get

$$G(y)[F(x), z] + y[F(x), g(z)] + [F(x), y]g(z) - y[x, z] \in Z(R). \quad (3.28)$$

Since $z \in Z(L)$ and g is a derivation so $0 \neq g(z) \in Z(L)$ and hence we get $[F(x), y]g(z) \in Z(R)$. Again since, $0 \neq g(z) \in Z(L)$ so using $*-$primeness of R and Remark 2.1, we get $[F(x), y] \in Z(R)$. In particular $[F(x), x] \in Z(R)$ for all $x \in L$ and hence by Theorem 3.2 (i), we get the required result.

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