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Maximum Principle and Existence of Weak Solutions for Nonlinear System Involving Singular *(p,q)*-Laplacian Operators

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Abstract.

We investigate in this work necessary and sufficient conditions for having the maximum principle for nonlinear system involving singular (p,q)-Laplacian operators on bounded domain Ω of R^n. Moreover, we prove the existence of positive weak solutions by the Browder theorem method for the considered system.

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1 Introduction:

Let us consider the following nonlinear system

$$\left.\begin{array}{lr} -\operatorname{div}[|x|^{-rp}|\nabla u|^{p-2}\nabla u] = a|x|^{-(r+1)p+\gamma}|u|^{p-2}u + b|u|^{\alpha}|v|^{\beta}v + f & \text{in } \Omega, \\ -\operatorname{div}[|x|^{-sq}|\nabla v|^{q-2}\nabla v] = c|u|^{\alpha}|v|^{\beta}u + d|x|^{-(s+1)q+\delta}|v|^{q-2}v + g & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1.1}$$

where Ω is a bounded domain of R^n with boundary $\partial\Omega$, $0 \in \Omega$, $1 < p, q < n$, $0 \leq r \leq \frac{n-p}{p}$, $0 \leq s \leq \frac{n-q}{q}$, $a, b, c, d, \alpha, \beta, \gamma, \delta$ are positive constants, f, g are given functions. The feature that needs to be highlighted in system (1.1) is the singularity in the weights. Due to this singularity in the weights, the extensions are challenging and nontrivial. A crucial milestone in the understanding of the elliptic problems involving the singular quasilinear elliptic operator $-\operatorname{div}[|x|^{-rp}|\nabla u|^{p-2}\nabla u]$ is the paper by Caffarelli, Kohn and Nirenberg [4] (see also [5, 16, 18]).

Many works have been devoted to the study of maximum principle for nonlinear systems either on a bounded domain (cf. [3, 9, 10]) or an unbounded domain (cf. [6, 8, 14]).

On the other hand, there have been many papers concerned with the existence of weak solutions for singular elliptic systems in recent years (see [1, 2, 12, 13, 15, 18] and related papers in their references.

This paper is organized as follows. In section 2, we introduce some technical results and definitions which will be used in the sequel. Section 3 is devoted to the maximum principle and the existence of positive weak solutions for the scalar case. Finally, in section 4, we consider the system case.

2 Technical results

We start this section by recalling some useful results in [17]. For $\sigma \in R^1$ and $p \geq 1$, we define $L_p(\Omega, |x|^{-\sigma})$ as being the subspace of $L_p(\Omega)$ of the Lebesgue measurable function $u : \Omega \to R^1$, satisfying

$$\|u\|_{L_p(\Omega,|x|^{-\sigma})} = \left[\int\limits_{\Omega} |x|^{-\sigma} |\nabla u|^p\right]^{\frac{1}{p}} < \infty. \tag{2.1}$$

If $1 < p < N$ and $-\infty < a < \frac{n-p}{p}$, we define $W_0^{1,p}(\Omega, |x|^{-ap}|)$ as being the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, |x|^{-ap}|)$ with respect to the norm defined by

$$\|u\|_{W_0^{1,p}(\Omega,|x|^{-ap}|)} = \left[\int\limits_{\Omega} |x|^{-ap} |\nabla u|^p\right]^{\frac{1}{p}} < \infty. \tag{2.2}$$

Then, the space $W_0^{1,p}(\Omega, |x|^{-ap}|)$ is reflexive and separable Banach space.

Let us recall some results on singular eigenvalue problems (see [17]) useful in the sequel for this work. It was known that the singular eigenvalue problems

$$\left.\begin{array}{ll} -div[|x|^{-ap}||\nabla u|^{p-2}\nabla u] = \lambda |x|^{-a(p+1)+c}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2.3}$$

where $\Omega \subset R^n$ is an open bounded domain with C^1 boundary $\partial\Omega$, $0 \in \Omega$, $1 < p < n$, $0 \leq a < \frac{(n-p)}{p}$ and $c > 0$, admits, an unique positive first eigenvalue $\lambda_1(p)$ with a nonnegative eigenfunction.

Moreover, this eigenvalue is isolated, simple and as a consequence of its variational characterization one has

$$\lambda_1(p) \int\limits_{\Omega} |x|^{-a(p+1)+c}|u|^p \leq \int\limits_{\Omega} |x|^{-ap}|\nabla u|^p, \tag{2.4}$$

where $u \geq 0$ *a.e. in* Ω (u not identical to 0) is the corresponding eigenfunction of $\lambda_1(p)$.

Also, we introduce some basic definitions and theorems concerning the nonlinear operators which we use extensively in proving of existence of positive weak solution for our systems [7].

Definition 1 *Let $A : V \to V'$ be an operator on a real Banach space V. We say that the operator A is:*

- *bounded iff it maps bounded sets into bounded i.e. for each $r > 0$ there exists $M > 0$ (M depending on r) such that $\|u\| \leq r \implies \|A(u)\| \leq M, \quad \forall u \in V$;*
- *coercive: iff $\lim\limits_{\|u\|\to\infty} \frac{\langle A(u),u\rangle}{\|u\|} = \infty$;*
- *monotone iff $\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq 0 \quad$ for all $\quad u_1, u_2 \in V$;*

- *strictly monotone iff* $\langle A(u_1) - A(u_2), u_1 - u_2\rangle > 0$ *for all* $u_1, u_2 \in V$, $u_1 \neq u_2$;
- *strongly monotone iff* $\langle A(u_1) - A(u_2), u_1 - u_2\rangle > k\|u_1 - u_2\|$ *for all* $u_1, u_2 \in V$, $u_1 \neq u_2$;
- *continuous iff* $u_n \longrightarrow u$ *implies* $A(u_n) \longrightarrow A(u)$, *for all* $u_n, u \in V$;
- *strongly continuous iff* $u_n \xrightarrow{w} u$ *implies* $A(u_n) \longrightarrow A(u)$, *for all* $u_n, u \in V$;
- *demicontinuous iff* $u_n \longrightarrow u$ *implies* $A(u_n) \xrightarrow{w} A(u)$, *for all* $u_n, u \in V$;
- M_0 *−condition iff* $u_n \xrightarrow{w} u$, $A(u_n) \xrightarrow{w} b$, $\langle A(u_n), u_n\rangle \to \langle b, u\rangle \Rightarrow A(u) = b$ *or all* $u_n, u \in V$.

Remark 1 *From this Definition we have the following:*

1. *Every strongly continuous operator is continuous and bounded.*
2. *Every strongly continuous operator is continuous, which is demicontinuous and hence hemicontinuous.*
3. *Every strongly monotone operator is strictly monotone operator.*
4. *Every Monotone and continuous operator satisfies* M_0 *−condition.*

Theorem 2 (***Browder [11]***) *Let* V *be a reflexive real Banach space. Moreover let* $A : V \to V'$ *be an operator which is: bounded, demicontinuous, coercive, and monotone on the space* V*. Then, the equation* $A(u) = f$ *has at least one solution* $u \in V$ *for each* $f \in V'$*. If moreover,* A *is strictly monotone operator, then the equation (1.1) has precisely one solution* $u \in V$ *for every* $f \in V'$.

Definition 2 *By a solution* (u, v) *of (1.1), we mean a weak solution; i.e.,* $(u, v) \in W_0^{1,p}(\Omega, |x|^{-rp}) \times W_0^{1,q}(\Omega, |x|^{-sq})$ *such that*

$$\int_\Omega |x|^{-rp}|\nabla u|^{p-2}\nabla u\nabla\varphi + a\int_\Omega |x|^{-(a+1)p+c}|u|^{p-2}u\varphi + b\int_\Omega |u|^{\alpha}|v|^{\beta}v\varphi = \int_\Omega f\varphi,$$

$$\int_\Omega |x|^{-sq}|\nabla v|^{q-2}\nabla v\nabla\psi + c\int_\Omega |u|^{\alpha}|v|^{\beta}u\psi + d\int_\Omega |x|^{-(s+1)q+\delta}|v|^{q-2}v\psi = \int_\Omega g\psi,$$

for all $(\varphi, \psi) \in W_0^{1,p}(\Omega, |x|^{-rp}) \times W_0^{1,q}(\Omega, |x|^{-sq})$.

3 The case of a single equation for the singular p-Laplacian

In this section, we study the maximum principle and existence of positive weak solution for the following scalar case,

$$\left.\begin{array}{lr} -div[|x|^{-ap}||\nabla u|^{p-2}\nabla u] = \lambda|x|^{-a(p+1)+c}|u|^{p-2}u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{3.1}$$

where $\Omega \subset R^n$ is an open bounded domain with C^1 boundary $\partial\Omega$, $0 \in \Omega$, $1 < p < n$, $0 \leq a < \frac{(n-p)}{p}$, $c > 0$ and $0 \leq f \in L^{p^*}$ such that $\frac{1}{p} + \frac{1}{p^*} = 1$.

3.1 Maximum principle

We are concerned with the following form of the maximum principle: The hypotheses $f \geq 0$ on Ω implies $u \geq 0$ for any solution u of system (3.1).

Theorem 3 *Maximum principle holds for system (3.1) iff* $\lambda_1(p) > \lambda$.

Proof: The condition is necessary

If $\lambda_1(p) \leq \lambda$, then the functions $f := (\lambda - \lambda_1(p))|x|^{-(a+1)p+c}\phi^{p-1} \geq 0$, is nonnegative, nevertheless $-\phi$ satisfies (3.1), which contradicts the maximum principle.

The condition is sufficient

Assume that $\lambda_1(p) > \lambda$ holds; if u is a solution of (3.1) for $f \geq 0$, we obtain by multiplying system (3.1) by $u^- := \max(0, -u)$ and integrating over Ω

$$\int_\Omega |x|^{-ap}||\nabla u^-|^p = \lambda \int_\Omega |x|^{-(a+1)p+c}|u^-|^p - \int_\Omega fu^- \leq \lambda \int_\Omega |x|^{-(a+1)p+c}|u^-|^p.$$

By using (2.4), we have

$$(\lambda_1(p) - \lambda) \int_\Omega |x|^{-(a+1)p+c}|u^-|^p \leq 0.$$

Then $u^- = 0$, which implies that $u \geq 0$, i.e. the maximum principle holds.

3.2 Existence of positive weak solution

In this subsection, using the Browder theorem, we prove the existence of positive weak solution for the scalar case (3.1).

Theorem 4 *For* $f \in L^{p^*}(\Omega)$, *there exists a weak solution* $u \in W_0^{1,p}(\Omega, |x|^{-ap}|)$ *for the scalar case (3.1) if* $\lambda_1(p) > \lambda$.

Proof. We transform the weak formulation of the scalar case (3.1) to the following operator form

$$J(u) = A(u) - \lambda B(u) = F, \tag{3.2}$$

where, A, B and F are given by

$$(A(u), \varphi) = \int_\Omega |x|^{-ap}||\nabla u|^{p-2}\nabla u \nabla\varphi, \ (B(u), \varphi) = \int_\Omega |x|^{-(a+1)p+c}|u|^{p-2}u\varphi, \ (F, \Phi) = (f, \varphi) = \int_\Omega f\varphi. \tag{3.3}$$

Now, we have the following properties of the operators A and B:

a) A and B are bounded operators. From (3.3), by using the Holder inequality, we have

$$\begin{aligned}
(A(u), \varphi) &= \int_\Omega |x|^{-ap}||\nabla u|^{p-2}\nabla u \nabla\varphi \\
&\leq \left(\int_\Omega |x|^{-ap}||\nabla u|^p\right)^{1/p^*} \left(\int_\Omega |x|^{-ap}||\nabla \phi|^p\right)^{1/p} \\
&= \|u\|^{p/p^*}_{W_0^{1,p}(\Omega,|x|^{-ap})}\|\phi\|_{W_0^{1,p}(\Omega,|x|^{-ap})}.
\end{aligned}$$

Also, one can prove that

$$(B(u),\varphi)=\int_\Omega |x|^{-(a+1)p+c}|u|^{p-2}u\varphi \le \|u\|^{p/p^*}_{L^p(\Omega,|x|^{-(a+1)p+c})}\|\phi\|_{W_0^{1,p}(\Omega,|x|^{-(a+1)p+c})}.$$

b) A and B are continuous operators. The operator $A(u)$ is the Frechet derivative of the functional $\frac{1}{p}\int_\Omega |x|^{-rp}|\nabla u|^p$. Hence $A(u)$ is a continuous operator. Also, the operator B is the Frechet derivative of the functional $\frac{1}{p}\int_\Omega |x|^{-(r+1)p+\gamma}|u|^p$. So it is a continuous operator. So A and B are continuous operators.

c) $J=A-B$ is Coercive operator.

From (3.3), we have

$$(J(u),u)=\int_\Omega |x|^{-ap}||\nabla u|^p-\lambda\int_\Omega |x|^{-(a+1)p+c}|u|^p. \tag{3.4}$$

Using (2.4), (3.4) becomes

$$\begin{aligned}(J(u),u) &\ge \int_\Omega |x|^{-ap}||\nabla u|^p-\frac{\lambda}{\lambda_1(p)}\int_\Omega |x|^{-ap}||\nabla u|^p\\ &= [1-\frac{\lambda}{\lambda_1(p)}]\int_\Omega |x|^{-ap}||\nabla u|^p=[1-\frac{\lambda}{\lambda_1(p)}]\|u\|^p_{W_0^{1,p}(\Omega,|x|^{-ap})},\end{aligned}$$

and hence $(J(u),u)\to\infty$ as $\|u\|_{W_0^{1,p}(\Omega,|x|^{-ap})}\to\infty$.

d) Now, to apply the Browder theorem, it remains to prove that J is a strictly monotone operator.

From (2.4), we have

$$\lambda_1^{\frac{1}{p}}(p)\|u\|_{L^p(\Omega,|x|^{-(a+1)p+c})}\le \|u\|^p_{W_0^{1,p}(\Omega,|x|^{-ap})} \tag{3.5}$$

Now, form (3.3), we get

$$\begin{aligned}(J(u_1)-J(u_2),u_1-u_2) &= \int_\Omega |x|^{-ap}|\nabla u_1|^p-\lambda\int_\Omega |x|^{-(a+1)p+c}|u_1|^p\\ &+\int_\Omega |x|^{-ap}|\nabla u_2|^p-\lambda\int_\Omega |x|^{-(a+1)p+c}|u_2|^p\\ &-\int_\Omega |x|^{-ap}|\nabla u_1|^{p-2}\nabla u_1\nabla u_2+\lambda\int_\Omega |x|^{-(a+1)p+c}|u_1|^{p-2}u_1u_2\\ &-\int_\Omega |x|^{-ap}|\nabla u_2|^{p-2}\nabla u_2\nabla u_1+\lambda\int_\Omega |x|^{-(a+1)p+c}|u_2|^{p-2}u_2u_1.\end{aligned}$$

Using Holder inequality, we have

$$\begin{aligned}&(J(u_1)-J(u_2),u_1-u_2)\\ \ge\ & \|u_1\|^p_{W_0^{1,p}(\Omega,|x|^{-ap})}+\|u_2\|^p_{W_0^{1,p}(\Omega,|x|^{-ap})}\\ &-\|u_1\|^{p-1}_{W_0^{1,p}(\Omega,|x|^{-ap})}\|u_2\|^p_{W_0^{1,p}(\Omega,|x|^{-ap})}-\|u_2\|^{p-1}_{W_0^{1,p}(\Omega,|x|^{-ap})}\|u_1\|^p_{W_0^{1,p}(\Omega,|x|^{-ap})}\\ &-\lambda[\|u_1\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})}+\|u_2\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})}\\ &-\|u_1\|^{p-1}_{L^p(\Omega,|x|^{-(a+1)p+c})}\|u_2\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})}-\|u_2\|^{p-1}_{L^p(\Omega,|x|^{-(a+1)p+c})}\|u_1\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})}]\\ >\ & (\|u_1\|^{p-1}_{W_0^{1,p}(\Omega,|x|^{-ap})}-\|u_2\|^{p-1}_{W_0^{1,p}(\Omega,|x|^{-ap})})(\|u_1\|_{W_0^{1,p}(\Omega,|x|^{-ap})}-\|u_2\|_{W_0^{1,p}(\Omega,|x|^{-ap})})\\ &-\lambda(\|u_1\|^{p-1}_{L^p(\Omega,|x|^{-(a+1)p+c})}-\|u_2\|^{p-1}_{L^p(\Omega,|x|^{-(a+1)p+c})})(\|u_1\|_{L^p(\Omega,|x|^{-(a+1)p+c})}-\|u_2\|_{L^p(\Omega,|x|^{-(a+1)p+c})}).\end{aligned}$$

By using (3.5), we have

$$(J(u_1)-J(u_2),u_1-u_2) > (1-\frac{\lambda}{\lambda_1})(\|u_1\|^{p-1}_{W_0^{1,p}(\Omega,|x|^{-ap})}-\|u_2\|^{p-1}_{W_0^{1,p}(\Omega,|x|^{-ap})})(\|u_1\|_{W_0^{1,p}(\Omega,|x|^{-ap})}-\|u_2\|_{W_0^{1,p}(\Omega,|x|^{-ap})}) > 0,$$

This proves the strictly monotone condition and hence the existence of positive weak solution of the scalar case (3.1).

4 The case of a system for the (p,q)-Laplacian

In this section, we assume the following hypotheses

(H_1) $1 < p, q < n$, $0 < r < \frac{n-p}{p}$, $0 < s < \frac{n-q}{q}$

(H_2) $\alpha, \beta \geq 0$, $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$ and $\gamma, \delta > 0$,

(H_3) $a, b, c, d > 0$ and $b, c \leq \inf\left[\left(|x|^{-(r+1)p+\gamma}\right)^{\frac{\alpha+1}{p}}\left(|x|^{-(s+1)q+\delta}\right)^{\frac{\beta+1}{q}}\right]$,

(H_4) $\left(\frac{b}{|x|^{-(r+1)p+\gamma}}\right)^{\frac{\alpha+1}{p}}\left(\frac{c}{|x|^{-(s+1)q+\delta}}\right)^{\frac{\beta+1}{q}} \leq 1$,

(H_5) $f \in L^{p^*}(\Omega)$, $g \in L^{q^*}(\Omega)$, $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\frac{1}{q} + \frac{1}{q^*} = 1$.

4.1 Maximum principle

Our aim in this subsection is to construct a maximum principle for system (1.1) which means that if f, g are nonnegative functions then any solution (u, v) of system (1.1) imply $u \geq 0$, $v \geq 0$.

Theorem 5 *Assume that hypotheses (H_1)-(H_5) are satisfied. Then, the maximum principle holds for system (1.1) if*

$$\lambda_1(p) > a, \quad \lambda_1(q) > d, \tag{4.1}$$

$$(\lambda_1(p)-1)^{\frac{\alpha+1}{p}}(\lambda_1(q)-1)^{\frac{\beta+1}{q}} > 1. \tag{4.2}$$

Conversely, if the maximum principle holds, then (4.1) and (4.3) are satisfied, where

$$(\lambda_1(p)-1)^{\frac{\alpha+1}{p}}(\lambda_1(q)-1)^{\frac{\beta+1}{q}} > \Theta \tag{4.3}$$

with,

$$0 < \Theta = \frac{\inf_\Omega\left(\frac{\phi^p}{\psi^q}\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\sup_\Omega\left(\frac{\phi^p}{\psi^q}\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}} \leq 1,$$

and ϕ (respectively ψ) is the positive eigenfunction associated to $\lambda_1(p)$ (respectively $\lambda_1(q)$) normalized by $\|\phi\|_\infty = \|\psi\|_\infty = 1$.

Proof: The conditions are necessary

If $\lambda_1(p) \leq a$, then the functions $f := (a-\lambda_1(p))|x|^{-(r+1)p+\gamma}\phi^{p-1}$, $g := 0$ are nonnegative, nevertheless $(-\phi, 0)$ satisfies (1.1), which contradicts the maximum principle.

Similarly, if $\lambda_1(q) \leq d$, then the functions $f := 0$, $g := (d-\lambda_d(q))|x|^{-(s+1)q+\delta}\psi^{q-1}$ are nonnegative, nevertheless $(0, -\psi)$ satisfies (1.1), which means that the maximum principle does not hold.

Now suppose that $\lambda_1(p) > a$, $\lambda_1(q) > d$ and (4.3), (and hence (4.2)), does not hold, i.e.

$$(\lambda_1(p) - a)^{\frac{\alpha+1}{p}} (\lambda_1(q) - d)^{\frac{\beta+1}{q}} \le \Theta.$$

Now, we want to fined a positive real number ξ such that

$$\left.\begin{array}{ll} A\left(\frac{\phi^p}{\psi^q}\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le \xi, & A > 0, \\ B\left(\frac{\psi^q}{\phi^p}\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le \frac{1}{\xi}, & B > 0. \end{array}\right\} \tag{4.4}$$

Equation (4.4) is satisfied if

$$A \sup_{\Omega} \left(\frac{\phi^p}{\psi^q}\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le \xi \le \frac{1}{B} \inf_{\Omega} \left(\frac{\phi^p}{\psi^q}\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}. \tag{4.5}$$

Let $A = \left[\frac{\lambda_1(p)-a}{b} \sup(|x|^{-(r+1)p+\gamma})\right]^{\frac{\alpha+1}{p}}$ and $B = \left[\frac{\lambda_1(q)-d}{c} \inf(|x|^{-(s+1)q+\delta})\right]^{\frac{\beta+1}{q}}$, then, after some calculations, ξ exists if (4.3), (and hence (4.2)), does not hold.

If we take $\xi = [(\frac{D^q}{C^p})]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$ with $C, D > 0$, then (4.4) implies

$$\begin{aligned} (\lambda_1(p) - a)|x|^{-(r+1)p+\gamma}(C\phi)^{p-1} &\le b(C\phi)^{\alpha}(D\psi)^{\beta+1}, \\ (\lambda_1(q) - d)|x|^{-(s+1)q+\delta}(D\psi)^{q-1} &\le c(C\phi)^{\alpha+1}(D\psi)^{\beta}, \end{aligned}$$

Then,

$$f = -\,(\lambda_1(p) - a)|x|^{-(r+1)p+\gamma}(C\phi)^{p-1} + b(C\phi)^{\alpha}(D\psi)^{\beta+1},$$

and

$$g = -\,(\lambda_1(q) - d)|x|^{-(s+1)q+\delta}(D\psi)^{q-1} + c(C\phi)^{\alpha+1}(D\psi)^{\beta},$$

are nonnegative functions, nevertheless $(-\,C\phi, -\,D\psi)$ is a solution of (1.1). This is a contradiction with the maximum principle.

The conditions are sufficient

Assume that (4.1) and (4.2) hold; if (u, v) is a solution of (1.1) for $f, g \ge 0$, we obtain by multiplying the first equation of (1.1) by $u^- := \max(0, -u)$ and integrating over Ω

$$\int_{\Omega} |x|^{-rp}|\nabla u^-|^p \le a \int_{\Omega} |x|^{-(r+1)p+\gamma}|u^-|^p + b \int_{\Omega} |u^-|^{\alpha+1}|v^-|^{\beta+1}.$$

From the characterization of the first eigenvalue $\lambda_1(p)$, we have

$$(\lambda_1(p) - a) \int_{\Omega} |x|^{-(r+1)p+\gamma}|u^-|^p \le b \int_{\Omega} |u^-|^{\alpha+1}|v^-|^{\beta+1}.$$

Using (H_3), we have

$$(\lambda_1(p) - a) \int_{\Omega} |x|^{-(r+1)p+\gamma}|u^-|^p \le \int_{\Omega} \left[|x|^{-(r+1)p+\gamma}|u^-|^p\right]^{\frac{\alpha+1}{p}} \left[|x|^{-(s+1)q+\delta}|v^-|^q\right]^{\frac{\beta+1}{q}}. \tag{4.6}$$

Applying Hölder inequality, (4.6) becomes

$$\left[(\lambda_1(p) - a)\left(\int_{\Omega} \left[|x|^{-(r+1)p+\gamma}|u^-|^p\right]\right)^{\frac{\beta+1}{q}} - \left(\int_{\Omega} \left[|x|^{-(s+1)q+\delta}|v^-|^q\right]\right)^{\frac{\beta+1}{q}}\right] \times \left[\left(\int_{\Omega} \left[|x|^{-(r+1)p+\gamma}|u^-|^p\right]\right)^{\frac{\alpha+1}{p}}\right] \le 0.$$

Now, if $\int\limits_{\Omega}\left[|x|^{-(r+1)p+\gamma}|u^-|^p\right]=0$, then $u^-=0$ and hence $u\geq 0$. If not, then we have

$$(\lambda_1(p)-a)^{\frac{\alpha+1}{p}}\left(\int\limits_{\Omega}\left[|x|^{-(r+1)p+\gamma}|u^-|^p\right]\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\leq\left(\int\limits_{\Omega}\left[|x|^{-(s+1)q+\delta}|v^-|^q\right]\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}. \tag{4.7}$$

Similarly, for the second equation of (1.1), we have

$$\left[(\lambda_1(q)-d)\left(\int\limits_{\Omega}\left[|x|^{-(s+1)q+\delta}|v^-|^q\right]\right)^{\frac{\alpha+1}{p}}-\left(\int\limits_{\Omega}\left[|x|^{-(r+1)p+\gamma}|u^-|^p\right]\right)^{\frac{\alpha+1}{p}}\right]\times\left[\left(\int\limits_{\Omega}\left[|x|^{-(s+1)q+\delta}|v^-|^q\right]\right)^{\frac{\beta+1}{q}}\right]\leq 0$$

Also, if $\int\limits_{\Omega}\left[|x|^{-(s+1)q+\delta}|v^-|^q\right]=0$, then $v^-=0$ and hence $v\geq 0$. If not, then we have

$$(\lambda_1(q)-d)^{\frac{\beta+1}{q}}\left(\int\limits_{\Omega}\left[|x|^{-(s+1)q+\delta}|v^-|^q\right]\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\leq\left(\int\limits_{\Omega}\left[|x|^{-(r+1)p+\gamma}|u^-|^p\right]\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \tag{4.8}$$

Multiplying (4.7) by (4.8), we obtain

$$\left((\lambda_a(p)-1)^{\frac{\alpha+1}{p}}(\lambda_d(q)-1)^{\frac{\beta+1}{q}}-1\right)\left[\int\limits_{\Omega}a(x)|u^-|^p\right]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\left[\int\limits_{\Omega}d(x)|v^-|^q\right]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\leq 0.$$

Using (4.1), (4.2), (4.7) and (4.8), we have $u^-=v^-=0$, which implies that $u\geq 0,\ v\geq 0$, i.e. the maximum principle holds for system (1.1).

Corollary 6 *If $p=q$, then the maximum principle holds for system (1.1) if and only if (4.1) and (4.2) are satisfied.*

Remark 7 *When $r=s=0,\gamma=p$ and $\delta=q$ our result is reduced to the one in [3].*

4.2 Existence of positive weak solutions for the system case

In this subsection, using the Browder theorem, we prove the existence of positive weak solution for system (1.1). We have the following existence theorem:

Theorem 8 *For $(f,g)\in L^{p^*}(\Omega)\times L^{q^*}(\Omega)$, there exists a weak solution $(u,v)\in W_0^{1,p}(\Omega,|x|^{-rp})\times W_0^{1,q}(\Omega,|x|^{-sq})$ for the system (1.1) if hypotheses (H_1-H_5) and (4.1) are satisfied.*

Proof: As in the scalar case, we transform the weak formulation of the system (1.1) to the following operator form

$$S(u,v)=A(u,v)-B(u,v)=F, \tag{4.9}$$

where, A, B and F are given by

$$(A(u,v),(\varphi,\psi))=\int\limits_{\Omega}|x|^{-rp}|\nabla u|^{p-2}\nabla u\nabla\varphi+\int\limits_{\Omega}|x|^{-sq}|\nabla v|^{q-2}\nabla v\nabla\psi \tag{4.10}$$

$$\begin{aligned}(B(u,v),(\varphi,\psi)) &= a\int\limits_{\Omega}|x|^{-(r+1)p+\gamma}|u|^{p-2}u\varphi+d\int\limits_{\Omega}|x|^{-(s+1)q+\delta}|v|^{q-2}v\psi \\ &\quad +b\int\limits_{\Omega}|u|^{\alpha}|v|^{\beta}v\varphi+c\int\limits_{\Omega}|u|^{\alpha}|v|^{\beta}u\psi, \end{aligned} \tag{4.11}$$

$$(F,\Phi)=((f,g),(\varphi,\psi))=\int\limits_{\Omega}f\varphi+\int\limits_{\Omega}g\psi.$$

Now, we have the following properties for the operator $S=A-B$:

a) The operator S is Bounded and continuous. The operator $A(u,v)$ can be written as the sum of the two operators $A_1(u), A_2(v)$, where

$$(A_1(u),(\varphi))=\int\limits_{\Omega}|x|^{-rp}|\nabla u|^{p-2}\nabla u\nabla\varphi \ \text{ and } \ (A_2(v),(\psi))=\int\limits_{\Omega}|x|^{-sq}|\nabla v|^{q-2}\nabla v\nabla\psi.$$

As in the scalar case, operators $A_1(u)$ and $A_2(v)$ are bounded and continuous; so their sum, the operator A, will be the same. Also, for the operator $B(u,v)$.

b) The operator $S(u,v)$ is strictly monotone, to do this, since maximum principle holds for system (1.1), i.e., $u\geq 0, v\geq 0$, then we have

$$\begin{aligned}&(S(u_1,v_1)-S(u_2,v_2),(u_1,v_1)-(u_2,v_2))\\ =\ &(A(u_1,v_1)-A(u_2,v_2),(u_1,v_1)-(u_2,v_2))-(B(u_1,v_1)-B(u_2,v_2),(u_1,v_1)-(u_2,v_2)),\end{aligned}$$

where

$$\begin{aligned}&(A(u_1,v_1)-A(u_2,v_2),(u_1,v_1)-(u_2,v_2))\\ =\ &\int\limits_{\Omega}|x|^{-rp}|\nabla u_1|^{p}+\int\limits_{\Omega}|x|^{-sq}|\nabla v_1|^{q}+\int\limits_{\Omega}|x|^{-rp}|\nabla u_2|^{p}+\int\limits_{\Omega}|x|^{-sq}|\nabla v_2|^{q}\\ &-\int\limits_{\Omega}|x|^{-rp}|\nabla u_1|^{p-2}\nabla u_1\nabla u_2-\int\limits_{\Omega}|x|^{-rp}|\nabla u_2|^{p-2}\nabla u_2\nabla u_1\\ &-\int\limits_{\Omega}|x|^{-sq}|\nabla v_1|^{q-2}\nabla v_1\nabla v_2-\int\limits_{\Omega}|x|^{-sq}|\nabla v_2|^{q-2}\nabla v_2\nabla v_1.\end{aligned}$$

and

$$\begin{aligned}&(B(u_1,v_1)-B(u_2,v_2),(u_1,v_1)-(u_2,v_2))\\ =\ &a\int\limits_{\Omega}|x|^{-(r+1)p+\gamma}|u_1|^{p}+d\int\limits_{\Omega}|x|^{-(s+1)q+\delta}|v_1|^{q}+a\int\limits_{\Omega}|x|^{-(r+1)p+\gamma}|u_2|^{p}+d\int\limits_{\Omega}|x|^{-(s+1)q+\delta}|v_2|^{q}\\ &-a\int\limits_{\Omega}|x|^{-(r+1)p+\gamma}|u_1|^{p-1}u_1u_2-a\int\limits_{\Omega}|x|^{-(r+1)p+\gamma}|u_2|^{p-1}u_2u_1\\ &-d\int\limits_{\Omega}|x|^{-(s+1)q+\delta}|v_1|^{q-1}v_1v_2-d\int\limits_{\Omega}|x|^{-(s+1)q+\delta}|v_2|^{q-1}v_2v_1\\ &+b\int\limits_{\Omega}|u_1|^{\alpha+1}|v_1|^{\beta+1}+b\int\limits_{\Omega}|u_2|^{\alpha+1}|v_2|^{\beta+1}+c\int\limits_{\Omega}|u_1|^{\alpha+1}|v_2|^{\beta+1}+c\int\limits_{\Omega}|u_2|^{\alpha+1}|v_1|^{\beta+1}\\ &-b\int\limits_{\Omega}|u_1|^{\alpha}u_2|v_1|^{\beta+1}-b\int\limits_{\Omega}|u_2|^{\alpha}u_1|v_2|^{\beta+1}-c\int\limits_{\Omega}|u_1|^{\alpha+1}|v_2|^{\beta}v_1-c\int\limits_{\Omega}|u_2|^{\alpha+1}|v_1|^{\beta}v_1.\end{aligned}$$

Applying Holder inequality, after complicated calculations, one have

$$(S(u_1, v_1) - S(u_2, v_2), (u_1, v_1) - (u_2, v_2)) > 0,$$

c) Now, to apply the Browder theorem, it remains to prove that $S(u, v)$ is a coercive operator. From (4.9-4.11), we have

$$\begin{aligned}(S(u,v),(u,v)) &= \int_\Omega |x|^{-rp}|\nabla u|^p + \int_\Omega |x|^{-sq}|\nabla v|^q - a\int_\Omega |x|^{-(r+1)p+\gamma}|u|^p - d\int_\Omega |x|^{-(s+1)q+\delta}|v|^q \\ &\quad -b\int_\Omega |u|^\alpha|v|^\beta vu - c\int_\Omega |u|^\alpha |v|^\beta uv.\end{aligned}$$

Using the validity of the maximum principle for system (1.1) and (2.4) we have

$$\begin{aligned}(S(u,v),(u,v)) &\geq (1-\frac{a}{\lambda_1(p)})\int_\Omega |x|^{-rp}|\nabla u|^p + (1-\frac{d}{\lambda_1(q)})\int_\Omega |x|^{-sq}|\nabla v|^q \\ &\quad -(b+c)\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}.\end{aligned}$$

Applying Holder inequality and (H_2), after some calculations, we have

$$(S(u,v),(u,v)) \longrightarrow \infty \quad \text{as} \quad \|(u,v)\|_{W_0^{1,p}(\Omega,|x|^{-rp})\times W_0^{1,q}(\Omega,|x|^{-sq})} \longrightarrow \infty.$$

This proves the coercive condition and so, the existence of positive weak solution for systems (1.1).

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