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Suzuki Type Common Fixed Point Theorem For Four Maps Using α - admissible Functions In Partial Ordered Complex Valued Metric Spaces

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Abstract.

In this paper, we obtain a Suzuki type unique common fixed point theorem for four self maps using α -admissible function in partial ordered complex valued metric spaces. Also we give an example to illustrate our main theorem.

Keywords: Complex valued metric; compatible maps; Suzuki type contraction; admissible function.

Mathematics Subject Classification: 47H10,54E50,54H25.

1 Introduction and Preliminaries

It is a well-known fact that the mathematical results regarding fixed points of contraction type mappings are very useful for determining the existence and uniqeness of solutions to various mathematical models.

Azam et al.[1] introduced the notion of a complex valued metric space which is a generalization of the classical metric space and obtained sufficent conditions for the existence of common fixed points of a pair of mappings satisfying a rational contractive condition.Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. However, in complex valued metric spaces, one can study improvements of a host of results of analysis involving divisions.Later several authors proved fixed and common fixed point theorems in complex valued metric spaces ,for example, refer [3, 6, 9, 11, 13, 14, 16, 17, 19, 20, 23, 26].

In this paper,we prove a unique common fixed point theorem for two pairs of mappings satisfying a contractive condition of rational type in the frame work of complex valued metric spaces using α -admissible function. The proved result generalizes and extends some of the results in the literature.

To begin with, we recall some basic definitions, notations and results.

Throughout this paper $\mathcal{R}$, $\mathcal{R}^+$, $\mathcal{N}$ and $\mathbb{C}$ denote the set of all real numbers, non-negative real numbers , positive integers and complex numbers respectively.First we refer the following preliminaries.

Let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$.
Thus $z_1 \precsim z_2$ if one of the following holds:
(1)$Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(2)$Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(3)$Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(4)$Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
Clearly $z_1 \precsim z_2 \Rightarrow |z_1| \leq |z_2|$.
We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied. Also we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1.1 *One can easily check that the following statements :*
(i) if $0 \precsim z_1 \precnsim z_2$ then $|z_1| < |z_2|$;
(ii) if $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.2 *Let X be a non empty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z, \in X$ the following conditions are satisfied:*
(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.
The pair $(X, d)$is called a complex valued metric space.

Remark 1.3 *Let (X, d) be a complex valued metric space. Then*
(i) $|d(x, y)| < |1 + d(x, y)|$, for all $x, y \in X$.
(ii) $|d(x, y)| > 0$ if $x \neq y$.

Definition 1.4 *Let (X, d) be a complex valued metric space.*

(i) *A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.*

(ii) *A point $x \in X$ is called a limit point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) \cap (X - A) \neq \phi$.*

(iii) *A subset $B \subseteq X$ is called open whenever each point of B is an interior point of B.*

(iv) *A subset $B \subseteq X$ is called closed whenever each limit point of B is in B.*

(v) *The family $F = \{B(x,r) : x \in X$ and $0 \prec r\}$ is a sub basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.*

Let $\{x_n\}$ be a sequence in X and $x \in X$.If for every $c \in \mathbb{C}$ with $0 \preceq c$ there is $n_0 \in \mathcal{N}$ such that for all $n > n_0, d(x_n, x) \prec c$,then $\{x_n\}$ is said to be convergent to x and x is the limit point of $\{x_n\}$.We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathcal{N}$ such that for all $n > n_0, d(x_n, x_{n+m}) \prec c$, where $m \in \mathcal{N}$,then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d) then (X, d) is called a complete complex valued metric space. We require the follwing lemmas.

Lemma 1.5 *([1]) Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.6 *([1]) Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n, m \to \infty$.*

One can easily prove the following lemma

Lemma 1.7 *Let (X, d) be a complex valued metric space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to x and y respectively. Then $|d(x_n, y_n)| \to |d(x, y)|$ as $n \to \infty$.*

Now we extend the definition of compatible maps introduced by Jungck [7] in metric spaces to complex valued metric spaces as follows.

Definition 1.8 *Let f and g be self mappings on a complex valued metric space (X, d). Then the pair(f, g) is said to be compatible if $\lim_{n\to\infty} |d(fgx_n, gfx_n)| = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.*

Definition 1.9 *([8])Let X be a non-empty set and $f, S : X \to X$. The pair (f, S) is said to be weakly compatible if $fSu = Sfu$ whenever $fu = Su$ for $u \in X$.*

The Banach contraction principle plays an important role in nonlinear analysis and has numerous generalizations and several applications. Suzuki proved generalized versions of Banach's and Edelstein's basic results. The importance of Suzuki contraction theorem is that the contractive condition required to be satisfied not for all points of the domain of mapping involved in it.
First we give the following theorem of Suzuki [24].

Theorem 1.10 *([24]) Let (X, d) be a complete metric space and let T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to \left(\frac{1}{2}, 1\right]$ by*

$$\theta(r) = \begin{cases} 1\ , & \text{if } \ 0 \le r \le \frac{(\sqrt{5}-1)}{2}, \\ (1-r)r^{-2}\ , & \text{if } \ \frac{(\sqrt{5}-1)}{2} \le r \le \ 2^{-\frac{1}{2}}, \\ (1+r)^{-1}\ , & \text{if } \ 2^{-\frac{1}{2}} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Later in this direction several authors, for example,[4, 5, 10, 21, 25] proved fixed and common fixed point theorems.

Samet et al. [2] introduced the notion of α- admissible mappings as follows

Definition 1.11 *([2]) Let X be a non empty set, $T : X \to X$ and $\alpha : X \times X \to \mathcal{R}^+$ be mappings. Then T is called α- admissible if for all $x, y \in X$, we have $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$.*

Later Shahi et al. [18] and Abdeljawad [22] defined the following

Definition 1.12 *([18]) Let X be a non empty set, $\alpha : X \times X \to \mathcal{R}^+$ and $f, g : X \to X$. Then f is said to be α- admissible with respect to g if $\alpha(gx, gy) \ge 1$ implies $\alpha(fx, fy) \ge 1$ for all $x, y \in X$.*

Definition 1.13 *([22]) Let X be a non empty set, $\alpha : X \times X \to \mathcal{R}^+$ and $f, g : X \to X$. Then the pair(f, g) is said to be α-admissible if $\alpha(x, y) \ge 1$ implies $\alpha(fx, gy) \ge 1$ and $\alpha(gx, fy) \ge 1$ for all $x, y \in X$.*

Using these definitions, we introduce the following α-admissible condition involving four maps.

Definition 1.14 *Let X be a non empty set, $\alpha : X \times X \to \mathcal{R}^+$ and $f, g, S, T : X \to X$. The pair (f, g) is said to be α- admissible w.r.to the pair (S, T) if $\alpha(Sx, Ty) \geq 1$ implies $\alpha(fx, gy) \geq 1$ and $\alpha(Tx, Sy) \geq 1$ implies $\alpha(gx, fy) \geq 1$ for all $x, y \in X$.*

Recently Abbas et al. [12] introduced the new concepts in a partially ordered set as follows

Definition 1.15 *([12]) Let $(X, \preceq)$ be a partially ordered set and $f, g : X \to X$.*
(i)f is said to be a dominating map if $x \preceq fx$.
(ii) f is said to be a weak annihilator of g if $fgx \preceq x$.

Definition 1.16 *$(X, d, \preceq)$ is called a partially ordered complex valued metric space if (i) $(X, \preceq)$ is a partially ordered set and (ii) (X, d) is a complex valued metric space.*

Now we prove our main result.

2 Main Result

Theorem 2.1 . *Let $(X, d, \preceq)$ be a partially ordered complete complex valued metric space and $\alpha : X \times X \to \mathcal{R}^+$ be a function. Let f, g, S and T be self mappings on X satisfying the following*

(2.1.1) *f and g are dominating maps and f and g are weak annihilators of T and S respectively,*

(2.1.2) *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$,*

(2.1.3) $\frac{1}{2}\min\{|d(fx, Sx)|, |d(gy, Ty)|\} \leq \max\{|d(Sx, Ty)|, |d(fx, gy)|\}$ *implies*

$$\begin{aligned}\alpha(Sx, Ty)\ d(fx, gy) \precsim\ & a_1 d(Sx, Ty) + a_2 d(Sx, fx) + a_3 d(Ty, gy) \\ & + a_4 d(Sx, gy) + a_5 d(Ty, fx) \\ & + a_6 \tfrac{d(fx,Sx)\ d(gy,Ty)}{1+d(Sx,Ty)} + a_7 \tfrac{d(Sx,gy)\ d(Ty,fx)}{1+d(Sx,Ty)}\end{aligned}$$

for all comparable elements $x, y \in X$, where $a_i, i = 1, 2, ..., 7$ are non-negative real numbers such that $\sum_{i=1}^{7} a_i < 1$,

(2.1.4) *the pair (f, g) is α-admissible w.r.to the pair (S, T),* (2.1.5) *$\alpha(Sx_1, fx_1) \geq 1$ and $\alpha(fx_1, Sx_1) \geq 1$ for some $x_1 \in X$,*

(2.1.6)(a) *S is continuous,the pair (f, S) is compatible and the pair (g, T) is weakly compatible and if there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq 1$, $\alpha(y_{n+1}, y_n) \geq 1$ for all $n \in \mathcal{N}$ and $y_n \to z$ for some $z \in X$, then we have $\alpha(Sy_{2n}, y_{2n-1}) \geq 1, \alpha(z, y_{2n-1}) \geq 1, \alpha(z, z) \geq 1$ and $\alpha(z, Tz) \geq 1$,*

(or)

(2.1.6)(b) *T is continuous,the pair (g, T) is compatible and the pair (f, S) is weakly compatible and if there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq 1$, $\alpha(y_{n+1}, y_n) \geq 1$ for all $n \in \mathcal{N}$ and $y_n \to z$ for some $z \in X$, then we have $\alpha(y_{2n}, Ty_{2n-1}) \geq 1, \alpha(y_{2n}, z) \geq 1, \alpha(z, z) \geq 1$ and $\alpha(Sz, z) \geq 1$,*

(2.1.7) *if for a non-decreasing sequence $\{x_n\}$ in X with $x_n \preceq y_n, \forall n \in \mathcal{N}$ and $y_n \to u$ implies $x_n \preceq u, \forall n \in \mathcal{N}$.*

Then f, g, S and T have a common fixed point in X .

(2.1.8) *Further if we assume that $\alpha(u, v) \geq 1$ whenever u and v are common fixed points of f, g, S and T and the set of common fixed points of f, g, S and T is well ordered then f, g, S and T have unique common fixed point in X.*

Proof. From (2.1.5), we have $\alpha(Sx_1, fx_1) \geq 1$ and $\alpha(fx_1, Sx_1) \geq 1$ for some $x_1 \in X$.
From (2.1.2), there exist sequences $\{x_n\}$ and $\{y_n\}$ as follows:
$y_{2n+1} = fx_{2n+1} = Tx_{2n+2}$, $y_{2n+2} = gx_{2n+2} = Sx_{2n+3}$, $n = 0, 1, 2, \cdots$.
Now

$$\begin{aligned} \alpha(Sx_1, fx_1) \geq 1 \ &\Rightarrow \alpha(Sx_1, Tx_2) \geq 1, \quad \text{from definition of } \{y_n\} \\ &\Rightarrow \alpha(fx_1, gx_2) \geq 1, \quad \text{from (2.1.4), } i.e \ \alpha(y_1, y_2) \geq 1 \\ &\Rightarrow \alpha(Tx_2, Sx_3) \geq 1, \quad \text{from definition of } \{y_n\} \\ &\Rightarrow \alpha(gx_2, fx_3) \geq 1, \quad \text{from (2.1.4), } i.e \ \alpha(y_2, y_3) \geq 1 \\ &\Rightarrow \alpha(Sx_3, Tx_4) \geq 1, \quad \text{from definition of } \{y_n\} \\ &\Rightarrow \alpha(fx_3, gx_4) \geq 1, \quad \text{from (2.1.4), } i.e \ \alpha(y_3, y_4) \geq 1 \end{aligned}$$

Continuing in this way, we have

$$\alpha(y_n, y_{n+1}) \geq 1, \quad \forall \ n \in \mathcal{N} \tag{1}$$

Similarly by using $\alpha(fx_1, Sx_1) \geq 1$, we can show that

$$\alpha(y_{n+1}, y_n) \geq 1, \quad \forall \ n \in \mathcal{N} \tag{2}$$

From(2.1.1), we have
$x_{2n+1} \preceq fx_{2n+1} = Tx_{2n+2} \preceq fTx_{2n+2} \preceq x_{2n+2}$,
$x_{2n+2} \preceq gx_{2n+2} = Sx_{2n+3} \preceq gSx_{2n+3} \preceq x_{2n+3}$. Thus

$$x_n \preceq x_{n+1}, \forall n \in \mathcal{N} \tag{3}$$

Case (i): Suppose $y_{2m} = y_{2m+1}$ for some m.
Assume that $y_{2m+1} \neq y_{2m+2}$.
Now $\alpha(Sx_{2m+1}, Tx_{2m+2}) = \alpha(y_{2m}, y_{2m+1}) \geq 1$, from (1).
Also we have
$\frac{1}{2}\min\{|d(fx_{2m+1}, Sx_{2m+1})|, |d(gx_{2m+2}, Tx_{2m+2})|\}$
$\quad \leq \max\{|d(Sx_{2m+1}, Tx_{2m+2})|, |d(fx_{2m+1}, gx_{2m+2})|\}$,from def.of $\{y_n\}$.
From (2.1.3) and (3),we have
$d(y_{2m+1}, y_{2m+2}) = d(fx_{2m+1}, gx_{2m+2})$
$\precsim \alpha(Sx_{2m+1}, Tx_{2m+2})d(fx_{2m+1}, gx_{2m+2})$
$\precsim a_1 d(y_{2m}, y_{2m+1}) + a_2 d(y_{2m}, y_{2m+1}) + a_3 d(y_{2m+1}, y_{2m+2})$
$\quad +a_4 d(y_{2m}, y_{2m+2}) + a_5 d(y_{2m+1}, y_{2m+1})$
$\quad +a_6 \frac{d(y_{2m}, y_{2m+1})d(y_{2m+1}, y_{2m+2})}{1+d(y_{2m}, y_{2m+1})} + a_7 \frac{d(y_{2m}, y_{2m+2})d(y_{2m+1}, y_{2m+1})}{1+d(y_{2m}, y_{2m+1})}$
$\precsim a_3 d(y_{2m+1}, y_{2m+2}) + a_4 d(y_{2m+1}, y_{2m+2})$
$= (a_3 + a_4)d(y_{2m+1}, y_{2m+2})$
Thus $|d(y_{2m+1}, y_{2m+2})| \leq (a_3 + a_4)\,|d(y_{2m+1}, y_{2m+2})|$.
It is a contradiction.Hence $y_{2m+1} = y_{2m+2}$.
Continuing in this way we can conclude that $y_n = y_{n+l}$ for all positive integers l. Thus $\{y_n\}$ is a Cauchy sequence in X.
Case (ii): Suppose that $y_n \neq y_{n+1}$ for all $n \in \mathcal{N}$.
Now $\alpha(Sx_{2n+1}, Tx_{2n+2}) = \alpha(y_{2n}, y_{2n+1}) \geq 1$, from (1).
As in Case (i), we have
$|d(y_{2n+1}, y_{2n+2})| \leq k_1\ |d(y_{2n}, y_{2n+1})|$, where $k_1 = \frac{a_1+a_2+a_4}{1-a_3-a_4-a_6} < 1$.
Similarly using (2),we can show that $|d(y_{2n}, y_{2n+1})| \leq k_2\ |d(y_{2n-1}, y_{2n})|$, where $k_2 = \frac{a_1+a_3+a_5}{1-a_2-a_5-a_6} < 1$.
Let $k = max\,\{k_1, k_2\}$. Then $k < 1$.
Thus, we have $|d(y_n, y_{n+1})| \leq k\ |d(y_{n-1}, y_n)|$,for $n = 2, 3, 4, ...$

$$|d(y_n, y_{n+1})| \leq k^{n-1}\ |d(y_1, y_2)|, n = 2, 3, 4, ... \tag{4}$$

$$|d(y_n, y_{n+1})| \to 0 \quad as \quad n \to \infty \tag{5}$$

For $m > n$,using (4) we have

$$\begin{aligned} |d(y_n, y_m)| &\leq |d(y_n, y_{n+1})| + |d(y_{n+1}, y_{n+2})| + ... + |d(y_{m-1}, y_m)| \\ &\leq [k^{n-1} + k^{n-2} + ... + k^{m-2}]\ |d(y_1, y_2)| \\ &\leq \frac{k^{n-1}}{1-k}\ |d(y_1, y_2)| \to 0\ as\ n, m \to \infty. \end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is complete , there exists $z \in X$ such that $\lim_{n\to\infty} |d(y_n, z)| = 0$, from Lemma 1.5.

Hence

$$\lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} gx_{2n+2} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = z. \qquad (6)$$

Suppose (2.1.6)(a) holds.
Suppose $Sz \neq z$.
Since the pair (f, S) is compatible, we have $\lim_{n\to\infty} |d(fSx_{2n+1}, Sfx_{2n+1})| = 0$.
Since S is continuous at z, we have $SSx_{2n+1} \to Sz$ and $Sfx_{2n+1} \to Sz$. Also $|d(fSx_{2n+1}, Sz)| \leq |d(fSx_{2n+1}, Sfx_{2n+1})| + |d(Sfx_{2n+1}, Sz)|$.
Letting $n \to \infty$, we get $\lim_{n\to\infty} fSx_{2n+1} = Sz$ by Lemma 1.5.

$$\text{If} \quad \tfrac{1}{2}\min\{|d(fSx_{2n+1}, SSx_{2n+1})|, |d(gx_{2n}, Tx_{2n})|\} > \max\{|d(SSx_{2n+1}, Tx_{2n})|, |d(fSx_{2n+1}, gx_{2n})|\}$$

then letting $n \to \infty$ and using Lemma 1.7, we get $0 \geq |d(Sz, z)|$. It is a contradiction. Hence

$$\tfrac{1}{2}\min\{|d(fSx_{2n+1}, SSx_{2n+1})|, |d(gx_{2n}, Tx_{2n})|\} \leq \max\{|d(SSx_{2n+1}, Tx_{2n})|, |d(fSx_{2n+1}, gx_{2n})|\}.$$

Clearly $\alpha(SSx_{2n+1}, Tx_{2n}) = \alpha(Sy_{2n}, y_{2n-1}) \geq 1$, from (2.1.6)$(a)$.
From(2.1.1), we have $x_{2n} \preceq gx_{2n} = Sx_{2n+1}$.
From (2.1.3) and Lemma 1.7, we have

$$\begin{aligned}
&|d(Sz, z)| = \lim_{n\to\infty} |d(fSx_{2n+1}, gx_{2n})| \\
&\leq \lim_{n\to\infty} \alpha(SSx_{2n+1}, Tx_{2n})\, |d(fSx_{2n+1}, gx_{2n})| \\
&\leq \lim_{n\to\infty} \left[\begin{array}{l} a_1 |d(SSx_{2n+1}, Tx_{2n})| + a_2 |d(SSx_{2n+1}, fSx_{2n+1})| + a_3 |d(Tx_{2n}, gx_{2n})| \\ +a_4 |d(SSx_{2n+1}, gx_{2n})| + a_5 |d(Tx_{2n}, fSx_{2n+1})| \\ +a_6 \frac{|d(SSx_{2n+1}, fSx_{2n+1})|\ |d(Tx_{2n}, gx_{2n})|}{|1+d(SSx_{2n+1}, Tx_{2n})|} + a_7 \frac{|d(SSx_{2n+1}, gx_{2n})|\ |d(Tx_{2n}, fSx_{2n+1})|}{|1+d(SSx_{2n+1}, Tx_{2n})|} \end{array} \right] \\
&\leq a_1 |d(Sz, z)| + a_4 |d(Sz, z)| + a_5 |d(Sz, z)| + a_7 \frac{|d(Sz,z)|\ |d(Sz,z)|}{|1+d(Sz,z)|} \\
&< (a_1 + a_4 + a_5 + a_7) |d(Sz, z)| < |d(Sz, z)|, \mathit{from\ Remark}\ 1.3(i)
\end{aligned}$$

It is a contradiction. Thus $Sz = z$.
Suppose $fz \neq z$.
If $\frac{1}{2}\min\{|d(Sz, fz)|, |d(gx_{2n}, Tx_{2n})|\} > \max\{|d(Sz, Tx_{2n})|, |d(fz, gx_{2n})|\}$
then letting $n \to \infty$, we get $0 \geq |d(fz, z)|$, from (7) and Lemma 1.7 .
It is a contradiction. Hence

$$\frac{1}{2}\min\{|d(Sz, fz)|, |d(gx_{2n}, Tx_{2n})|\} \leq \max\{|d(Sz, Tx_{2n})|, |d(fz, gx_{2n})|\}.$$

Also $\alpha(Sz, Tx_{2n}) = \alpha(z, y_{2n-1}) \geq 1$, from (2.1.6)$(a)$.
Since $x_{2n} \preceq gx_{2n}$ and $gx_{2n} \to z$,by (2.1.7), we have $x_{2n} \preceq z$.

From (2.1.3) and Lemma 1.7, we have

$$\begin{aligned}
|d(fz,z)| &= \lim_{n\to\infty} |d(fz,gx_{2n})| \\
&\leq \lim_{n\to\infty} \alpha(Sz,Tx_{2n}) \ |d(fz,gx_{2n})| \\
&\leq \lim_{n\to\infty} \begin{bmatrix} a_1 |d(z,Tx_{2n})| + a_2 |d(z,fz)| + a_3 |d(Tx_{2n},gx_{2n})| \\ +a_4 |d(z,gx_{2n})| + a_5 |d(Tx_{2n},fz)| \\ +a_6 \frac{|d(z,fz)| \ |d(Tx_{2n},gx_{2n})|}{|1+d(z,Tx_{2n})|} + a_7 \frac{|d(z,gx_{2n})| \ |d(Tx_{2n},fz)|}{|1+d(z,Tx_{2n})|} \end{bmatrix} \\
&\leq a_2 |d(z,fz)| + a_5 |d(z,fz)| \\
&= (a_2+a_5) |d(z,fz)| < |d(z,fz)|
\end{aligned}$$

It is a contradiction. Thus $fz = z$.
Since $f(X) \subseteq T(X)$, there exists $w \in X$ such that $z = fz = Tw$. Also we have $z = fz = Tw \preceq fTw \preceq w$, from (2.1.1).
From(2.1.6)(a), we have $\alpha(Sz,Tw) = \alpha(z,z) \geq 1$.
Suppose $z \neq gw$. Clearly we have

$$\begin{aligned}
\tfrac{1}{2}\min\{|d(Sz,fz)|,|d(gw,Tw)|\} &= \min\{|d(z,z)|,|d(gw,z)|\} \\
&= 0 < \max\{|d(Sz,Tw)|,|d(fz,gw)|\}.
\end{aligned}$$

From(2.1.3), we have

$$\begin{aligned}
& d(z,gw) = d(fz,gw) \\
& \precsim \alpha(Sz,Tw) d(fz,gw) \\
& \precsim \begin{bmatrix} a_1 d(z,z) + a_2 d(z,z) + a_3 d(z,gw) \\ +a_4 d(z,gw) + a_5 d(z,z) \\ +a_6 \frac{d(z,z) g(z,gw)}{1+d(z,z)} + a_7 \frac{d(z,gw) d(z,z)}{1+d(z,z)} \end{bmatrix} \\
& = a_3 d(z,gw) + a_4 d(z,gw)
\end{aligned}$$

Thus $|d(z,gw)| \leq (a_3+a_4)|d(z,gw)| < |d(z,gw)|$.
It is a contradiction. Thus $z = gw$.
Since the pair(g,T) is weakly compatible , we have $gz = gTw = Tgw = Tz$.
From (2.1.6)(a),we have $\alpha(Sz,Tz) = \alpha(z,Tz) \geq 1$.
Suppose $z \neq gz$. Then

$$\begin{aligned}
\tfrac{1}{2}\min\{|d(Sz,fz)|,|d(gz,Tz)|\} &= \min\{|d(z,z)|,|d(gz,Tz)|\} \\
&= 0 < \max\{|d(Sz,Tz)|,|d(fz,gz)|\}.
\end{aligned}$$

From (2.1.3), we have

$$\begin{aligned}
& d(z,gz) = d(fz,gz) \\
& \precsim \alpha(Sz,Tz) d(fz,gz) \\
& \precsim \begin{bmatrix} a_1 d(z,Tz) + a_2 d(z,z) + a_3 d(Tz,gz) \\ +a_4 d(z,gz) + a_5 d(gz,z) \\ +a_6 \frac{d(z,z) d(Tz,gz)}{1+d(z,Tz)} + a_7 \frac{d(z,gz) d(gz,z)}{1+d(z,Tz)} \end{bmatrix} \\
& = a_1 d(z,gz) + a_4 d(z,gz) + a_5 d(z,gz) + a_7 \frac{d(z,gz) d(gz,z)}{1+d(z,Tz)}.
\end{aligned}$$

Thus from Remark 1.3(i),$|d(z,gz)| < (a_1+a_4+a_5+a_7)|d(z,gz)| < |d(z,gz)|$.
It is a contradiction. Hence $z = gz = Tz$.
Thus z is a common fixed point of f, g, S and T.
Suppose z' is another common common fixed point of f, g, S and T.

From (2.1.8), we have $\alpha(Sz, Tz') = \alpha(z, z') \geq 1$ and $z \preceq z'$.
Now

$$\begin{aligned} \tfrac{1}{2}\min\{|d(fz, Sz)|, |d(gz', Tz')|\} &= \tfrac{1}{2}\min\{0, 0\} \\ &= 0 < \max\{|d(Sz, Tz')|, |d(fz, gz')|\}. \end{aligned}$$

Hence from (2.1.3),we have

$$\begin{aligned} & d(z, z') = d(fz, gz') \\ & \precsim \alpha(Sz, Tz')\ d(fz, gz') \\ & \precsim \left[\begin{array}{l} a_1 d(z, z') + a_2 d(z, z) + a_3 d(z', z') \\ +a_4 d(z, z') + a_5 d(z, z') \\ +a_6 \frac{d(z,z)\ d(z',z')}{1+d(z,z')} + a_7 \frac{d(z,z')\ d(z,z')}{1+d(z,z')} \end{array} \right] \\ & = a_1 d(z, z') + a_4 d(z, z') + a_5 d(z, z') + a_7 \frac{d(z,z')\ d(z,z')}{1+d(z,z')}. \end{aligned}$$

Hence from Remark 1.3(i),we have
$|d(z, z')| < (a_1 + a_4 + a_5 + a_7)\ |d(z, z')| < |d(z, z')|$.
It is a contradiction. Thus $z = z'$.
Thus f, g, S and T have a unique common fixed point.
Similarly we can prove Theorem 2.1 when (2.1.6)(b) holds.
Now we give an example to support Theorem 2.1.

Example 2.2 *Let* $X = [0, \infty)$, $d(x, y) = i\ |x - y|, \forall\ x, y \in X$ *and define* $x \preceq y$ *if* $y \leq x$. *Define* $f, g, S, T : X \to X$ *by* $fx = \frac{x}{2}$, $gx = \frac{x}{4}$, $Sx = 8x$ *and* $Tx = 4x$.

Define $\alpha : X \times X \to \mathcal{R}^+$ *by* $\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$

We have $fx = \frac{x}{2} \leq x \Rightarrow x \preceq fx$ *and* $gx = \frac{x}{4} \leq x \Rightarrow x \preceq gx$.
Also $fTx = 2x \geq x \Rightarrow fTx \preceq x$ *and* $gSx = 2x \geq x \Rightarrow gSx \preceq x$.
If $x > \frac{1}{8}$ *and* $y \in X$ *then* $\alpha(Sx, Ty) = 0$.
If $x \leq \frac{1}{8}$ *and* $y > \frac{1}{4}$ *then* $\alpha(Sx, Ty) = 0$.
In these cases,the condition (2.1.3) is clearly satisfied.
Suppose $x \leq \frac{1}{8}$ *and* $y \in [0, \frac{1}{4}]$ *then* $\alpha(Sx, Ty) = 1$.
In this case, we have

$$\begin{aligned} \alpha(Sx, Ty) d(fx, gy) &= i\ \left|\tfrac{x}{2} - \tfrac{y}{4}\right| \\ &= \tfrac{i}{4}\ |2x - y| \\ &= \tfrac{i}{16}\ |8x - 4y| \\ &= \tfrac{1}{16}\ d(Sx, Ty) \end{aligned}$$

Thus (2.1.3) is satisfied for all $x, y \in X$ *with* $a_1 = \frac{1}{16}$ *and* $a_i = 0$ *for* $i = 2, 3, 4, 5, 6, 7$.
One can easily verify the remaining conditions of Theorem 2.1. Clearly 0 is the common fixed point of f, g, S *and* T.

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