

**QUASI-IQC-INJECTIVITY**

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ABSTRACT. In this work, the notion of injectivity relative to a class of IQC submodules (namely, IQC-injectivity) has been introduced and studied, which is a generalization quasi-injective module. This notion is closed under direct summands. Several properties and characterizations have been given. We provide a characterization of semi simple Artinian ring, SI-ring and Dedekind domain in terms of IQC-injective \mathcal{R} -module.

Indexing terms/Keywords: Quasi -injective modules; IQC -injective R-module; Quasi-closed submodules; fully continuous modules; divisible modules.

Academic Discipline And Sub-Disciplines:

Mathematic: Algebra.

MATHEMATICS SUBJECT CLASSIFICATION:

16D50, 16D70.

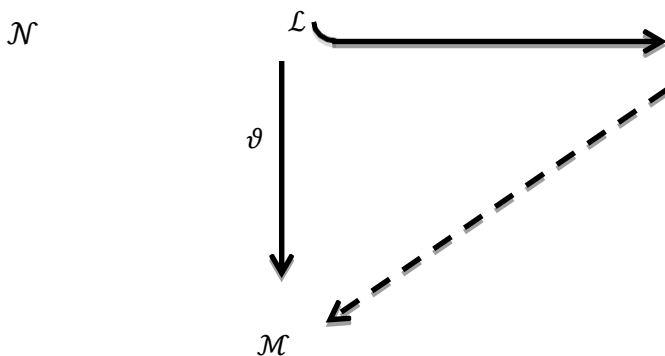
INTRODUCTION

Throughout, R represents an associative ring with identity and \mathcal{R} -modules are unitary left \mathcal{R} -modules. For an \mathcal{R} -modules \mathcal{M} and \mathcal{N} , $Hom_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ will denote the set of \mathcal{R} -module homomorphisms from \mathcal{M} to \mathcal{N} . The kernel of any $\beta \in Hom_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ is denoted by $ker(\beta)$ and its image by $\beta(\mathcal{M})$. $S = End_{\mathcal{R}}(\mathcal{M})$ will denote the ring of \mathcal{R} -endomorphisms of \mathcal{M} [1]. A submodule \mathcal{N} of \mathcal{R} -module \mathcal{M} is said to be an essential submodule of an \mathcal{R} -module \mathcal{M} , if \mathcal{N} has nonzero intersection with every nonzero submodule of \mathcal{M} [2]. A submodule \mathcal{N} of \mathcal{R} -module \mathcal{M} is said to be a closed in \mathcal{M} , if \mathcal{N} has no proper essential extensions in \mathcal{M} ([3], P.5). We shall use $\vartheta(\mathcal{R})$ to stand for the set of all essential right ideals of the ring \mathcal{R} . Given any \mathcal{R} -module \mathcal{M} , we set $Z(\mathcal{M}) = \{x \in \mathcal{M} \mid xI = 0, \text{ for some } I \in \vartheta(\mathcal{R})\}$ ([2], P.30). An \mathcal{R} -module \mathcal{M} , is singular provided $Z(\mathcal{M}) = \mathcal{M}$. At the other extreme, we say \mathcal{M} is a nonsingular provided $Z(\mathcal{M})=0$ ([2], P.31). A submodule \mathcal{N} of \mathcal{R} -module \mathcal{M} is said to be a direct summand of \mathcal{R} -module \mathcal{M} , if $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$, for some submodule \mathcal{L} of \mathcal{M} [2]. An \mathcal{R} -module \mathcal{M} is said to be semi

simple, if every sub module of \mathcal{M} is direct summand ([2], P.27). An \mathcal{R} -module \mathcal{M} is called CS-module (or extending ((C_1)-condition)), if \mathcal{M} satisfies any one of the following equivalent conditions (1) for every submodule \mathcal{N} of \mathcal{M} , there is a decomposition $\mathcal{M} = \mathcal{L} \oplus \mathcal{B}$ such that \mathcal{N} is essential in \mathcal{L} , (2) every closed submodule of \mathcal{M} is a direct summand [4]. A CS-module \mathcal{M} which satisfies (C_2)-condition: every sub module of \mathcal{M} which is isomorphic to a direct summand of \mathcal{M} is itself direct summand, is called continuous[4]. Let \mathcal{M} and \mathcal{N} be two \mathcal{R} -modules, \mathcal{N} is called \mathcal{M} -injective, if for every submodule \mathcal{L} of \mathcal{M} , any \mathcal{R} -homomorphism from \mathcal{L} to \mathcal{N} can be extended to an \mathcal{R} -homomorphism from \mathcal{M} to \mathcal{N} ([5], P.28). An \mathcal{R} -module \mathcal{N} is called injective, if it is \mathcal{M} -injective for all \mathcal{R} -module \mathcal{M} . A right \mathcal{R} -module \mathcal{M} is (minimal) quasi-injective, if every homomorphism from a (simple) submodule of \mathcal{M} to \mathcal{M} can be extended to an endomorphism of \mathcal{M} [6]([7]). A submodule \mathcal{N} of \mathcal{M} is called Quasi-closed submodule, if $\forall x \in \mathcal{M}$ with $x \notin \mathcal{N}$, there exists a closed submodule \mathcal{L} of \mathcal{M} containing \mathcal{N} and $x \notin \mathcal{L}$. it is clear that every closed submodule is a Quasi-closed –submodule[8]. Let \mathcal{M} be an \mathcal{R} -module. A submodule \mathcal{N} of \mathcal{M} is called IQC-submodule (simply $\mathcal{N} \leq^{IQC} \mathcal{M}$), if \mathcal{N} is \mathcal{R} -isomorphic to a Quasi-closed submodule of \mathcal{M} . It is clear that, every Quasi-closed submodule (and hence direct summand) is IQC-submodule, but the converse generally is not true, $n\mathbb{Z}$ is IQC-submodule of the \mathbb{Z} -module \mathbb{Z} which is not Quasi-closed for each positive integer $n > 2$. It is easy to prove that every submodule which is \mathcal{R} -isomorphic to IQC-submodule in \mathcal{M} is itself IQC-submodule in \mathcal{M} . Every IQC-submodule in a Quasi-closed submodule (direct summand) of \mathcal{M} is IQC-submodule in \mathcal{M} . Let \mathcal{M} and \mathcal{N} be two \mathcal{R} -modules. If $\mathcal{L} \leq^{IQC} \mathcal{M}$, then $f(\mathcal{L}) \leq^{IQC} \mathcal{N}$ where $f : \mathcal{M} \rightarrow \mathcal{N}$ is an \mathcal{R} -isomorphism [9]. An \mathcal{R} -module \mathcal{M} is fully (extending) continuous, if every I(QC)-submodule of \mathcal{M} is a direct summand [9], ([8]).

Quasi - IQC-injective module

Definition(2.1): Let \mathcal{M} and \mathcal{N} be two \mathcal{R} -modules. \mathcal{M} is said to be an IQC- \mathcal{N} -injective, if for each IQC-submodule \mathcal{L} of \mathcal{N} , every \mathcal{R} -homomorphism ϑ from \mathcal{L} to \mathcal{M} can be extended to an \mathcal{R} -homomorphism from \mathcal{N} into \mathcal{M} , see (1). The \mathcal{R} -module \mathcal{M} is called Quasi- IQC -injective, if it is IQC - \mathcal{M} -injective.



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Examples and remarks (2.2):

(1) Every fully continuous \mathcal{R} -module is Quasi- IQC -injective. But the converse may not be true, in general.

(2) Every quasi-injective \mathcal{R} -module is Quasi- IQC -injective. But the converse may not be true, in general. For example see [10, Remark (2.9)]. It may be fully continuous (Quasi- IQC - injective) , but not quasi-injective.

(3) Let \mathcal{M} and \mathcal{N} be two \mathcal{R} -modules. If \mathcal{M} is an IQC - \mathcal{N} -injective, then \mathcal{M} is an IQC - \mathcal{L} -injective for each Quasi-closed \mathcal{R} -submodule \mathcal{L} of \mathcal{N} .

Proof: Let \mathcal{L} be any Quasi-closed \mathcal{R} -submodule of \mathcal{N} , \mathcal{B} be any IQC -submodule of \mathcal{L} and $\vartheta: \mathcal{B} \rightarrow \mathcal{M}$ be any \mathcal{R} -homomorphism. Let $\iota_{\mathcal{B}}$ be the inclusion \mathcal{R} -homomorphism from \mathcal{B} into \mathcal{L} and $\iota_{\mathcal{L}}$ be the inclusion \mathcal{R} -homomorphism from Quasi-closed \mathcal{R} -submodule \mathcal{L} into \mathcal{N} . \mathcal{M} is an IQC - \mathcal{N} -injective, thus there exists an \mathcal{R} -homomorphism $\zeta: \mathcal{N} \rightarrow \mathcal{M}$ such that $(\zeta \iota_{\mathcal{L}} \iota_{\mathcal{B}})(b) = \vartheta(b)$, for all $b \in \mathcal{B}$. Put $\psi = \zeta \iota_{\mathcal{L}}: \mathcal{L} \rightarrow \mathcal{M}$. For each $b \in \mathcal{B}$, then $\psi(b) = (\zeta \iota_{\mathcal{L}})(b) = (\zeta \iota_{\mathcal{L}})(\iota_{\mathcal{B}}(b)) = (\zeta \iota_{\mathcal{L}} \iota_{\mathcal{B}})(b) = \vartheta(b)$. Therefore \mathcal{M} is an IQC - \mathcal{L} -injective \mathcal{R} -module.

(4) Let \mathcal{M} be an \mathcal{R} -module and $\{\mathcal{N}_i\}_{i \in I}$ a family of \mathcal{R} -modules. If $\prod_{i \in I} \mathcal{N}_i$ is an IQC - \mathcal{M} -injective, then for each $i \in I$, \mathcal{N}_i is an IQC - \mathcal{M} -injective.

Proof: Put $\mathcal{N} = \prod_{i \in I} \mathcal{N}_i$, suppose that \mathcal{N} is an IQC - \mathcal{M} -injective and \mathcal{A} is an IQC-submodule of \mathcal{M} , and $f: \mathcal{A} \rightarrow \mathcal{N}_i, \forall i \in I$. There exists $h: \mathcal{M} \rightarrow \mathcal{N}$ such that $h i_{\mathcal{A}} = \varphi_i f$ where $i_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{M}$ is inclusion mapping and $\varphi_i: \mathcal{N}_i \rightarrow \mathcal{N}$ is injection mapping. We now define $h': \mathcal{M} \rightarrow \mathcal{N}_i$, by $h'(m) = \pi_i h(m), \forall m \in \mathcal{M}$ where $\pi_i: \mathcal{N} \rightarrow \mathcal{N}_i$ is projection mapping, $\forall i = 1, 2$. Then h' is an \mathcal{R} -homomorphism and if $\forall a \in \mathcal{A}$, $h' i_{\mathcal{A}}(a) = \pi_i h i_{\mathcal{A}}(a) = \pi_i \varphi_i f(a) = f(a)$, this shows that \mathcal{N}_i is an IQC - \mathcal{M} -injective.

(5) Let \mathcal{M} and \mathcal{N}_i be \mathcal{R} -modules where $i \in I$ and I is finite index set, if $\bigoplus_{i \in I} \mathcal{N}_i$ is an IQC - \mathcal{M} -injective $\forall i \in I$, then \mathcal{N}_i is an IQC - \mathcal{M} -injective. In particular every direct summand of IQC- \mathcal{N} -injective \mathcal{R} -module is IQC- \mathcal{N} -injective.

Proof: Let \mathcal{M} be any IQC- \mathcal{N} -injective \mathcal{R} -module and \mathcal{L} be any direct summand \mathcal{R} -submodule of \mathcal{M} . Thus there exists an \mathcal{R} -submodule \mathcal{A} of \mathcal{M} such that $\mathcal{M} = \mathcal{L} \oplus \mathcal{A}$. Let \mathcal{B} be any IQC-submodule of \mathcal{N} and $f: \mathcal{B} \rightarrow \mathcal{L}$ be any \mathcal{R} -homomorphism. Define $g: \mathcal{B} \rightarrow \mathcal{M} = \mathcal{L} \oplus \mathcal{A}$ by $g(b) = (f(b), 0)$, for all $b \in \mathcal{B}$. It is clear that g is an \mathcal{R} -homomorphism, since \mathcal{M} is an IQC- \mathcal{N} -injective \mathcal{R} -module, thus there exists an \mathcal{R} -homomorphism $h: \mathcal{N} \rightarrow \mathcal{M}$ such that $h(b) = g(b)$ for all $b \in \mathcal{B}$. Let $\pi_{\mathcal{L}}$ be the natural projection \mathcal{R} -homomorphism of $\mathcal{M} = \mathcal{L} \oplus \mathcal{A}$ into \mathcal{L} . Put $h_1 = \pi_{\mathcal{L}} h: \mathcal{N} \rightarrow \mathcal{L}$. Thus h_1 is an \mathcal{R} -homomorphism and for each $b \in \mathcal{B}$, then $h_1(b) = (\pi_{\mathcal{L}} h)(b) = \pi_{\mathcal{L}}(g(b)) = \pi_{\mathcal{L}}((f(b), 0)) = f(b)$. Therefore \mathcal{L} is an IQC- \mathcal{N} -injective \mathcal{R} -module.

(6) Let \mathcal{M} be an \mathcal{R} -module and $\{\mathcal{N}_i\}_{i \in I}$ a family of \mathcal{R} -modules. if \mathcal{M} is IQC - $\bigoplus_{i \in I} \mathcal{N}_i$ -injective $\forall i \in I$, then \mathcal{M} is IQC - \mathcal{N}_i -injective.

Proof: Suppose that \mathcal{M} is an IQC - $\bigoplus_{i=1}^n \mathcal{N}_i$ -injective \mathcal{R} -module. Let \mathcal{A} is an IQC-submodule of \mathcal{N}_i (inclusion homomorphism $\iota_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{N}_i$) and $\mu: \mathcal{A} \rightarrow \mathcal{M}$ be an \mathcal{R} -homomorphism. By $\iota_{\mathcal{N}_i}: \mathcal{N}_i \rightarrow \bigoplus_{i=1}^n \mathcal{N}_i$ is inclusion homomorphism and hypothesis, there exists \mathcal{R} -homomorphism $\gamma: \bigoplus_{i=1}^n \mathcal{N}_i \rightarrow \mathcal{M}$ such that $\gamma \iota_{\mathcal{N}_i} \iota_{\mathcal{A}} = \mu$. Put $g = \gamma \iota_{\mathcal{N}_i}: \mathcal{N}_i \rightarrow \mathcal{M}$ such that $g \iota_{\mathcal{A}} = \mu$.

(7) Isomorphic to Quasi- IQC -injectivity is Quasi- IQC -injectivity.

(8) Let \mathcal{N} be any IQC -submodule of \mathcal{L} such that \mathcal{N} is IQC - \mathcal{M} -injective. Then every \mathcal{R} -monomorphism from \mathcal{N} into \mathcal{M} splits. In particular, if \mathcal{M} is an \mathcal{R} -module whose Quasi-closed submodules are IQC - \mathcal{M} -injective, then \mathcal{M} is fully extending module.

Proof: Let $\gamma: \mathcal{N} \rightarrow \mathcal{M}$ be an \mathcal{R} -monomorphism, and $\gamma^{-1}: \gamma(\mathcal{N}) \rightarrow \mathcal{N}$. As \mathcal{N} is an IQC - \mathcal{M} -injective module, there exists an \mathcal{R} -homomorphism $\beta: \mathcal{M} \rightarrow \mathcal{N}$, such that $\beta \gamma = I_{\mathcal{N}}$. For $m \in \mathcal{M}$ then $\beta(m) \in \mathcal{N}$, there exists $\gamma(n) \in \gamma(\mathcal{N})$ such that $\gamma^{-1}(\gamma(n)) = \beta(m) = \beta(\gamma(n))$ and hence $m - \gamma(n) \in \ker(\beta)$. It follows that $m = \gamma(n) + (m - \gamma(n)) \in \gamma(\mathcal{N}) + \ker(\beta)$. Moreover, $\gamma(\mathcal{N}) \cap \ker(\beta) = \ker(\gamma^{-1}) = 0$. Thus $\mathcal{M} = \gamma(\mathcal{N}) \oplus \ker(\beta)$.

(9) If \mathcal{M} is Quasi- IQC -injective \mathcal{R} -module then any \mathcal{R} -monomorphism $\gamma: \mathcal{M} \rightarrow \mathcal{M}$ splits.

Proposition(2.3): Every Quasi-IQC-injective \mathcal{R} -module \mathcal{M} has C_2 -condition.

Proof: Let \mathcal{M} be a Quasi- IQC -injective \mathcal{R} -module, \mathcal{A} and \mathcal{B} two sub modules of \mathcal{M} with \mathcal{A} is a direct summand in \mathcal{M} and \mathcal{B} is \mathcal{R} -isomorphic to \mathcal{A} . Let $f: \mathcal{B} \rightarrow \mathcal{A}$ be an \mathcal{R} -isomorphism. Then \mathcal{A} is an IQC - \mathcal{M} -injective, Examples and remarks (2.2), \mathcal{B} is an IQC - \mathcal{M} -injective. The inclusion mapping $\iota_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{M}$, there exists an \mathcal{R} -homomorphism $g: \mathcal{M} \rightarrow \mathcal{B}$ such that $g \iota_{\mathcal{B}} = I_{\mathcal{B}}$. Then $\mathcal{M} = \mathcal{B} \oplus \ker(g)$. That is; \mathcal{B} is a direct summand in \mathcal{M} , then \mathcal{M} has C_2 -condition.

The submodule $n\mathbb{Z}$ (where $n \geq 2$) of \mathbb{Z} as \mathbb{Z} -module which is isomorphic to \mathbb{Z} is not a direct summand in \mathbb{Z} as \mathbb{Z} -module.

Corollary(2.4): Let \mathcal{M} be a Quasi-IQC-injective \mathcal{R} -module. Then every submodule of \mathcal{M} which is \mathcal{R} -isomorphic to \mathcal{M} is a direct summand in \mathcal{M} .

Proposition(2.5): Let \mathcal{M} be a Quasi- IQC -injective \mathcal{R} -module. Then every submodule of \mathcal{M} which is isomorphic to closed submodule in \mathcal{M} is closed in \mathcal{M} .

Proof: Let \mathcal{M} be a Quasi- IQC -injective \mathcal{R} -module, \mathcal{K} a closed in \mathcal{M} and \mathcal{A} a submodule of \mathcal{M} with an \mathcal{R} -isomorphism $f: \mathcal{A} \rightarrow \mathcal{K}$. Consider the following diagram where $\iota_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{M}$, $\iota_{\mathcal{K}}: \mathcal{K} \rightarrow \mathcal{M}$ are two inclusion homomorphism. Then f extends to some g in $\text{End}(\mathcal{M})$ such that $\iota_{\mathcal{K}} f = g \iota_{\mathcal{A}}$, by a Quasi- IQC - \mathcal{M} -injectivity of \mathcal{M} . Now let Ω be collection of the set of all essential extension of \mathcal{A} in

\mathcal{M} . $\Omega \neq \phi$, since $\mathcal{A} \in \Omega$. By Zorn's lemma, there exists maximal essential member \mathcal{A}' . That is, \mathcal{A}' is maximal essential extension sub module in \mathcal{M} , which is evidently, it is closed submodule of \mathcal{M} . Thus $g|_{\mathcal{A}'}$ is an \mathcal{R} -homomorphism. Since $g(\mathcal{A}) = f(\mathcal{A})$, hence $\mathcal{K} = g(\mathcal{A})$ is essential in $g(\mathcal{A}')$, by \mathcal{A} is essential sub module in \mathcal{A}' . Since \mathcal{K} is a closed in \mathcal{M} . This implies $\mathcal{K} = g(\mathcal{A})$, whence $\mathcal{A} = \mathcal{A}'$. The conclusion follows.

An \mathcal{R} -module \mathcal{M} is multiplication, if each submodule is of the form $\mathcal{M}\mathcal{A}$ for some right ideal \mathcal{A} of \mathcal{R} [13].

Proposition(2.6): Every Quasi - closed submodule of a multiplication a Quasi – IQC -injective is a Quasi - IQC - injective.

Proof: Let \mathcal{L} be an IQC- submodule of a Quasi- closed submodule \mathcal{N} of \mathcal{M} and let $\theta : \mathcal{L} \rightarrow \mathcal{N}$ be an \mathcal{R} -homomorphism. Since \mathcal{N} is an Quasi - closed submodule of \mathcal{R} -module \mathcal{M} . By hypothesis, there exists $\xi : \mathcal{M} \rightarrow \mathcal{M}$, by multiplication property of \mathcal{M} , then $\mathcal{N} = \mathcal{M}\mathcal{A}$ for some right ideal \mathcal{A} of \mathcal{R} , $\xi|_{\mathcal{N}} = \xi(\mathcal{N}) = \xi(\mathcal{M}\mathcal{A}) = \xi(\mathcal{M})\mathcal{A} \subseteq \mathcal{M}\mathcal{A} = \mathcal{N}$.

In the following, we characterize fully continuous modules in terms of IQC - \mathcal{M} -injectivity.

Proposition(2.7): The following statements are equivalent for an \mathcal{R} -module \mathcal{M} :

- (1) \mathcal{M} is fully continuous.
- (2) Every \mathcal{R} -module is IQC - \mathcal{M} - injective.
- (3) Every IQC-submodule of \mathcal{M} is IQC - \mathcal{M} - injective.
- (4) Every Quasi-closed submodule of \mathcal{M} is IQC - \mathcal{M} - injective.

Proof: (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) It is clear. (4) \Rightarrow (1). Let \mathcal{K} be any submodule of \mathcal{M} which is isomorphic to Quasi-closed submodule \mathcal{L} of \mathcal{M} . By (4) \mathcal{L} is IQC – \mathcal{M} –injective. Then \mathcal{K} is IQC - \mathcal{M} -injective The identity mapping $i_{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{K}$, there exists an \mathcal{R} -homomorphism $\mathbb{G} : \mathcal{M} \rightarrow \mathcal{K}$ such that $\mathbb{G}i_{\mathcal{K}} = I_{\mathcal{K}}$. Then $\mathcal{M} = \mathcal{K} \oplus \ker(\mathbb{G})$. That is, $\mathcal{K} \leq^{\oplus} \mathcal{M}$.

An \mathcal{R} -module \mathcal{M} is said to be fully IQC- stable, if every IQC-submodule of \mathcal{M} is stable [9].

Proposition(2.8): Every multiplication Quasi-IQC-injective is a fully IQC- stable.

Proof: Let \mathcal{N} be an IQC-submodule of \mathcal{M} and an \mathcal{R} -monomorphism $g : \mathcal{N} \rightarrow \mathcal{M}$. Since \mathcal{M} is multiplication, then $\mathcal{N} = \mathcal{M}\mathcal{A}$ for some ideal \mathcal{A} of \mathcal{R} . Then g can be extended to an \mathcal{R} -homomorphism $h : \mathcal{M} \rightarrow \mathcal{M}$, since \mathcal{M} is Quasi-IQC -injective. Now $g(\mathcal{N}) = h(\mathcal{N}) = h(\mathcal{M}\mathcal{A}) = h(\mathcal{M})\mathcal{A} \subseteq \mathcal{M}\mathcal{A} = \mathcal{N}$.

Proposition(2.9): If \mathcal{M} is a fully extending and fully IQC-stable, then \mathcal{M} is Quasi- IQC – injective module.

Proof: It follows by [9 ,Proposition(2. 10)] and Proposition(2.3).

Theorem(2.10): The following statements are equivalent for an \mathcal{R} -module \mathcal{M} :

- (1) \mathcal{M} is fully continuous.
- (2) \mathcal{M} is Quasi- IQC - injective module and fully extending.

Proof: (1) \Rightarrow (2). By Examples and remarks (2.2).(2) \Rightarrow (1). By Proposition(2.3).

According to the definition of an IQC-injectivity, every \mathcal{R} -homomorphism of IQC-submodule of \mathcal{M} to \mathcal{M} is extendable to all \mathcal{M} . In the following, we consider a direct sum of IQC-submodules instead of individual IQC-submodule.

We consider the following condition for an \mathcal{R} -module \mathcal{M} and a positive integer n .

(ω_n) : For any submodule K of \mathcal{M} such that $K = K_1 \oplus K_2 \oplus \dots \oplus K_n$ where K_i is IQC-submodule of \mathcal{M} , $\forall i=1,2, \dots, n$, every \mathcal{R} -homomorphism $\vartheta: K \rightarrow \mathcal{M}$ can be extended to an \mathcal{R} -endomorphism of \mathcal{M} . It is clear that, if \mathcal{M} satisfies (ω_n) , then \mathcal{M} satisfies (ω_{n-1}) , $\forall n \geq 2$.

Theorem(2.11):The following statements are equivalent for a fully extending module \mathcal{M} :

- (1) \mathcal{M} is fully continuous.
- (2) \mathcal{M} satisfies $(\omega_n) \forall n \in \mathbb{Z}^+$.
- (3) \mathcal{M} satisfies $(\omega_n) \forall (n \geq 2) \in \mathbb{Z}^+$.
- (4) \mathcal{M} satisfies (ω_2) .
- (5) \mathcal{M} is Quasi- IQC-injective.

Proof: (1) \Rightarrow (2). [9 , Definition (2.2)] implies that K_i is direct summand of \mathcal{M} for each $i=1,2, \dots, n$. So K_i is direct summand of \mathcal{M} , Theorem(2.10) and hence each \mathcal{R} -homomorphism from K into \mathcal{M} can be extended to an \mathcal{R} -endomorphism.

(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5). It is clear. (5) \Rightarrow (1): It follows from Proposition (2.3).

An \mathcal{R} -module \mathcal{M} is said to be co-Hopfian if every injective endomorphism $f: \mathcal{M} \rightarrow \mathcal{M}$ is an automorphism [14]. An \mathcal{R} -module \mathcal{M} is directly finite, if $fg = I_{\mathcal{M}}$ implies that $gf = I_{\mathcal{M}}$ for all $f; g \in \text{End}(\mathcal{M})$ ([2], Lemma (6.9)). An \mathcal{R} -module \mathcal{M} is called weakly co-Hopfian, if any injective \mathcal{R} -endomorphism $f: \mathcal{M} \rightarrow \mathcal{M}$ is essential, that is; $f(\mathcal{M})$ is an essential submodule of \mathcal{M} [15]. In the following proposition, a sufficient condition for Quasi- IQC -injective modules to be co-Hopfian is given.

Proposition (2.12): A Quasi- IQC-injective \mathcal{R} -module \mathcal{M} is directly finite if and only if it is co-Hopfian.

Proof: Let f be injective \mathcal{R} -endomorphism of \mathcal{M} and $I_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{M}$ the identity map. Since \mathcal{M} is a Quasi-IQC -injective, there exists a map $g: \mathcal{M} \rightarrow \mathcal{M}$ such that, $gf = I_{\mathcal{M}}$. By directly finite of \mathcal{M} , we have $fg = I_{\mathcal{M}}$ which shows that f is an automorphism. Hence \mathcal{M} is co-Hopfian. The converse is clear.

In the following proposition, we give a condition for weakly co-Hopfian modules to be co-Hopfian.

Proposition (2.13): The following conditions are equivalent for a Quasi-IQC-injective \mathcal{R} -module \mathcal{M} :

- (1) \mathcal{M} is weakly co-Hopfian .
- (2) \mathcal{M} is co-Hopfian.

Proof: (1) \Rightarrow (2) Let $f: \mathcal{M} \rightarrow \mathcal{M}$ be an \mathcal{R} -monomorphism. By (1) we have $f(\mathcal{M})$ is essential in \mathcal{M} . f splits and hence $f(\mathcal{M})$ is a direct summand of \mathcal{M} , since \mathcal{M} is a Quasi-IQC -injective. Therefore $f(\mathcal{M}) = \mathcal{M}$. This shows that \mathcal{M} is co-Hopfian. (2) \Rightarrow (1) is obvious.

It is well-known that an \mathcal{R} -module \mathcal{M} is injective if and only if \mathcal{M} is \mathcal{N} -injective for each \mathcal{R} -module \mathcal{N} .

Proposition(2.14): The following statements are equivalent for an \mathcal{R} -module \mathcal{M} :

- (1) \mathcal{M} is injective.
- (2) \mathcal{M} is IQC - \mathcal{N} -injective, for each \mathcal{R} -module \mathcal{N} .

Proof: (1) \Rightarrow (2): Obvious, (2) \Rightarrow (1): Let $E = E(\mathcal{M})$ be the injective hull of \mathcal{M} . Let $i: \mathcal{M} \rightarrow E$ be the inclusion mapping and $j: E \rightarrow \mathcal{M} \oplus E$ the natural injection. By IQC - $\mathcal{M} \oplus E$ - injectivity of \mathcal{M} , implies that the identity mapping $I_{\mathcal{M}}$ of \mathcal{M} , can be extended to an \mathcal{R} -homomorphism $f: \mathcal{M} \oplus E \rightarrow \mathcal{M}$ such that $gi = I_{\mathcal{M}}$ where $g = fj$. Then $E = \mathcal{M} \oplus \ker(g)$, then $\mathcal{M} = E$, hence \mathcal{M} is injective.

It is well-known that if \mathcal{R} is a semi simple Artinian ring, then every \mathcal{R} -module is injective ([2], Theorem(1.18)). Also, Osofsky in [16] proved that ring \mathcal{R} is semi simple Artinian if and only if every cyclic \mathcal{R} -module is injective. Recall that \mathcal{R} is a right V-ring, if every simple \mathcal{R} -module is injective [17]. We now provide a characterization of semi simple Artinian rings in terms of Quasi-IQC -injective modules.

Theorem (2.15) : The following conditions are equivalent for a ring \mathcal{R} .

- (1) \mathcal{R} is semi simple Artinian,
- (2) \mathcal{R} is a right V-ring and every minimal quasi-injective right \mathcal{R} -module is Quasi-IQC -injective,
- (3) Every \mathcal{R} -module is Quasi-IQC -injective,

(4) The direct sum of every two Quasi- IQC -injective modules is Quasi- IQC - injective. And every cyclic \mathcal{R} -module is Quasi- IQC -injective,

Proof:(1) \Rightarrow (2).It follows from([2],Theorem(1.18)). (2) \Rightarrow (3). Since \mathcal{R} is a right V-ring, every simple \mathcal{R} -module is injective and hence every simple right \mathcal{R} -module is a direct summand of each module containing it. So every \mathcal{R} -module is minimal quasi-injective, hence is Quasi- IQC -injective \mathcal{R} -module.(3) \Rightarrow (4).It is clear. (4) \Rightarrow (1). Let \mathcal{M} be Quasi- IQC -injective module and E the injective hull of \mathcal{M} . By(4) $\mathcal{M} \oplus E$ is Quasi- IQC -injective. Then Examples and remarks (2.2), \mathcal{M} is IQC - $\mathcal{M} \oplus E$ -injective and Proposition (2.14), hence \mathcal{M} is injective. By every cyclic \mathcal{R} -module is Quasi- IQC -injective, then every cyclic \mathcal{R} -module is injective , that is; \mathcal{R} is semi simple Artinian , by Osofsky's theorem in [16].

Theorem (2.16): The following statements are equivalent for a ring :

(1) \mathcal{R} is a semi-simple Artinian ring .

(2) For each \mathcal{R} -module \mathcal{M} , if \mathcal{N}_1 and \mathcal{N}_2 are Quasi- IQC -injective \mathcal{R} -submodules of \mathcal{M} , then $\mathcal{N}_1 \cap \mathcal{N}_2$ is a Quasi- IQC -injective \mathcal{R} -module .

(3) For each \mathcal{R} -module \mathcal{M} , if \mathcal{N}_1 and \mathcal{N}_2 are quasi-injective \mathcal{R} -submodules of \mathcal{M} , then $\mathcal{N}_1 \cap \mathcal{N}_2$ is a Quasi- IQC-injective \mathcal{R} -module.

(4) For each \mathcal{R} -module \mathcal{M} , if \mathcal{N}_1 and \mathcal{N}_2 are injective \mathcal{R} -submodules of \mathcal{M} , then $\mathcal{N}_1 \cap \mathcal{N}_2$ is a Quasi- IQC -injective \mathcal{R} -module.

Proof: (1) \Rightarrow (2).It follows from Theorem (2.15). (2) \Rightarrow (3) and (3) \Rightarrow (4) are obvious. (4) \Rightarrow (1)Let \mathcal{M} be any \mathcal{R} -module and $\Xi = \Xi(\mathcal{M})$ is the injective envelope of \mathcal{M} ,let $\mathcal{Q} = \Xi \oplus \Xi$, $\mathcal{K} = \{(x, x) \in \mathcal{Q} \mid x \in \mathcal{M}\}$ and let $\mathcal{Q} = \mathcal{Q} / \mathcal{K}$.Also, put $\mathcal{M}_1 = \{ \mathcal{Y} + \mathcal{K} \in \mathcal{Q} \mid y \in \Xi \oplus (0) \}$ and $\mathcal{M}_2 = \{ \mathcal{Y} + \mathcal{K} \in \mathcal{Q} \mid y \in (0) \oplus \Xi \}$. It is clear that $\mathcal{Q} = \mathcal{M}_1 + \mathcal{M}_2$ Define $\tau_1: \Xi \rightarrow \mathcal{M}_1$ by $\tau_1(y) = (y, 0) + \mathcal{K}$, for all $y \in \Xi$ and $\tau_2: \Xi \rightarrow \mathcal{M}_2$ by $\tau_2(y) = (0, y) + \mathcal{K}$, for all $y \in \Xi$.Since $(\Xi \oplus (0)) \cap \mathcal{K} = (0)$ and $((0) \oplus \Xi) \cap \mathcal{K} = (0)$, thus we have τ_1 and τ_2 are \mathcal{R} -isomorphisms. Since Ξ is an injective \mathcal{R} -module , therefore \mathcal{M}_i is injective \mathcal{R} -submodule of \mathcal{Q} , for $i=1,2$. Thus by (4) , we have $\mathcal{M}_1 \cap \mathcal{M}_2$ is a Quasi- IQC -injective \mathcal{R} -module. Define $f: \mathcal{M} \rightarrow \mathcal{M}_1 \cap \mathcal{M}_2$ by $f(m) = (m, 0) + \mathcal{K}$, for all $m \in \mathcal{M}$. Since $\mathcal{M}_1 \cap \mathcal{M}_2 = \{ \mathcal{Y} + \mathcal{K} \in \mathcal{Q} \mid y \in \mathcal{M} \oplus (0) \}$, thus it is easy to prove that f is an \mathcal{R} -isomorphism. Thus \mathcal{M} is a Quasi- IQC -injective \mathcal{R} -module, by remark ((2.2),7) . Hence every \mathcal{R} -module is Quasi- IQC -injective and this implies that \mathcal{R} is a semi-simple Artinian ring , by Theorem (2.15).

Recall that an \mathcal{R} -module \mathcal{M} is direct injective, if given any direct summand A of \mathcal{M} , an injection $i_A: A \rightarrow \mathcal{M}$ and every \mathcal{R} -monomorphism $f: A \rightarrow \mathcal{M}$, there is an \mathcal{R} -endomorphism g of \mathcal{M} such that $gf = i_A$ [18].

Nicholson in ([19], Theorem(7.13)) proved that direct injective \mathcal{R} -module is equivalent to C_2 -condition. Proposition(2.3) shows that every Quasi- IQC injective \mathcal{R} -module is a direct injective and every direct injective \mathcal{R} -module is divisible [18]. Then we have the following:

Proposition(2.17): Every Quasi- IQC -injective \mathcal{R} -module is divisible.

The converse of Proposition(2.17) may not be true.

Quasi- IQC - injectivity is not closed under direct sums in general, as we see in the following

$\mathcal{R} = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ Where $F = \frac{Z}{2Z}$. It is easy to see that the \mathcal{R} -modules \mathcal{A} and \mathcal{B} are quasi-injective. And hence by Examples and remarks (2.2), they are Quasi- IQC -injective. However $\mathcal{R} = \mathcal{A} \oplus \mathcal{B}$ is not Quasi- IQC -injective, since otherwise \mathcal{R} satisfies (C2)-condition, by Proposition(2.3). But \mathcal{A} is isomorphic to \mathcal{C} and \mathcal{C} is not a direct summand in \mathcal{R} , contradiction.

Since \mathcal{A} and \mathcal{B} are two divisible \mathcal{R} -modules. And every direct sum of divisible \mathcal{R} -modules is divisible. That is; $\mathcal{A} \oplus \mathcal{B}$ is divisible. But it is not Quasi- IQC -injective.

In the following, we show that the distinction between Quasi- IQC -injectivity and divisibility vanishes over Dedekind domain. A domain \mathcal{R} is called Dedekind ring, if every divisible \mathcal{R} -module is injective ([20], Theorem(4.24)). We now provide a characterization of domain \mathcal{R} is Dedekind rings in terms of Quasi- IQC -injective \mathcal{R} -modules.

Theorem(2.18): The following conditions are equivalent for a ring \mathcal{R} .

- (1) \mathcal{R} is Dedekind domain,
- (2) Every divisible \mathcal{R} -module is Quasi- IQC -injective.

Proof: (1) \Rightarrow (2). By ([20], Theorem(4.24)). (2) \Rightarrow (1). Let \mathcal{M} be a divisible \mathcal{R} -module and $\Xi(\mathcal{M})$ an injective hull of \mathcal{M} . By ([5], proposition (2.6)), $\Xi(\mathcal{M})$ is divisible and by ([5], Lemma(2.5)), then $\mathcal{M} \oplus \Xi$ is divisible. By (2) $\mathcal{M} \oplus \Xi$ is Quasi- IQC -injective. Then Examples and remarks (2.2), \mathcal{M} is IQC- $\mathcal{M} \oplus \Xi$ -injective and Proposition(2.14). That is; \mathcal{M} is injective, implies \mathcal{R} is Dedekind domain [20].

Recall that a ring \mathcal{R} is SI-ring, if every singular \mathcal{R} -module is injective ([3], below Corollary (7.16)). Over non singular ring; we provide a characterization of SI-ring in terms of Quasi- IQC -injective \mathcal{R} -modules.

Proposition(2.19): The following statements are equivalent for non singular ring \mathcal{R} :

- (1) \mathcal{R} is SI-ring.

(2) Every singular \mathcal{R} -module is Quasi- IQC -injective,

Proof: (1) \Rightarrow (2) is clear.(2) \Rightarrow (1). Let \mathcal{M} be a singular \mathcal{R} -module and $\mathfrak{E}(\mathcal{M})$ the injective hull of \mathcal{M} . ([2], Proposition(1.23) and (1.22)), then $\mathcal{M} \oplus \mathfrak{E}(\mathcal{M})$ is singular. By(2) $\mathcal{M} \oplus \mathfrak{E}$ is Quasi- IQC -injective. Then Examples and remarks (2.2), \mathcal{M} is IQC - $\mathcal{M} \oplus \mathfrak{E}$ -injective and Proposition(2.14), hence \mathcal{M} is injective. That is; \mathcal{R} is SI-ring .

In the next part we characterize some rings by Quasi- IQC -injectivity. In the following, Noetherian rings are characterize as in terms of Quasi- IQC -injective. Recall that a \mathcal{R} -module \mathcal{M} is F-injective, if for any finitely generated ideal \mathcal{L} of \mathcal{R} , every \mathcal{R} -homomorphism of \mathcal{L} into \mathcal{M} , can be extended to an \mathcal{R} -homomorphism \mathcal{M} into \mathcal{M} [21].

Proposition (2.20) : The following conditions are equivalent:

- (1) \mathcal{R} is Noetherian ring ;
- (2) Every F-injective \mathcal{R} -modules are injective;
- (3) Every F-injective \mathcal{R} -module is Quasi- IQC -injective.

Proof: (1) implies (2) and (2) implies (3) are evidently.

Assume (3) . Let \mathcal{M} be a F-injective \mathcal{R} -module, \mathfrak{E} the injective hull of \mathcal{M} . Write $\mathcal{Q} = \mathcal{M} \oplus \mathfrak{E}$ is F-injective \mathcal{R} -module. By(3) $\mathcal{M} \oplus \mathfrak{E}$ is Quasi- IQC -injective. Then Examples and remarks (2.2) , \mathcal{M} is IQC - $\mathcal{M} \oplus \mathfrak{E}$ -injective and Proposition(2.14), hence \mathcal{M} is injective. We have shown that every F-injective \mathcal{R} -module is injective. Since any direct sum of F-injective \mathcal{R} -modules is F-injective, then every direct sum of injective modules is injective which implies that \mathcal{R} is Noetherian, by ([20], P.82). Thus (3) implies (2) and (2) implies (1).

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