



Generalized* \oplus Z^* Supplemented Modules and Generalized* \oplus Co- finitely Supplemented Modules

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Abstract: Let R be a commutative ring with identity, an R - module M is called $G^* \oplus Z^*$ supplemented modules, if every sub module containing $Z^*(M)$ has generalized* supplemented in M that is a direct summand of M . and an R -module M is called generalized* Co-finitely supplemented, if every Co-finitely has sub module of M has a generalized* supplemented in M and M is called \oplus Co-finitely generalized* supplemented, if every Co-finite sub module of M has G^*S that is direct summand of M .

Keywords: Generalized* $\oplus Z^*$ supplemented; \oplus Co-finitely generalized* supplemented modules; Generalized*Co-finitely supplemented modules.

In this paper we will prove some properties of these types of modules.

§1 Introduction

Let R be an associative ring with identity and let M be a unital left R -module. A submodule N of an R -module M is called small in denoted by $N \ll M$, if whenever $N + L = M$ for some submodule L of M , then $L = M$. equivalently for any proper submodule L of M , $N + L \neq M$ [1], let N and L be a submodule of M . N is called supplement of L in M if $M = N + L$ and N is minimal with respect to this property equivalently, $M = N + L$ and $N \cap L \ll N$ [1]. M is called supplemented, if every submodule of M has a supplemented in M [2]

For any R -module M , $Z^*(M) = \{m \in M, Rm \ll E(M)\}$ where $E(M)$ is the injective hull of M . equivalently $Z^*(M) = M \cap \text{Rad}(E(M)) = M \cap \text{Rad}(E_1)$ for any injective $E_1 \geq M$, where $\text{Rad}(E(M))$ and $\text{Rad}(E_1)$ denoted the Jacobson radical of $E(M)$. E_1 respectively. $Z^*(M)$ is called the co-singular submodule of M see [3], Notice that $\text{Rad}(M) \leq Z^*(M)$. But the converse does not hold in general for example Z as Z -module, $Z^*(Z) = Z \neq \text{Rad}(Z)$.

Let M be module, if $N, L \leq M$, and $M = N + L$, then L is called generalized supplement of N in case $N \cap L \leq \text{Rad}(L)$, M is called generalized supplemented or (briefly GS) in case each submodule N has generalized supplement in M [4]. A module M is called generalized \oplus supplemented, if every submodule has a generalized supplement that is a direct summand of M [5] As a generalization of \oplus supplemented modules. In [6] another were notation introduced called generalized \oplus radical supplemented module. A module M is called generalized \oplus radical supplemented, if every submodule containing radical has a generalized supplement that is a direct summand of M .

The concept generalized* supplement modules were introduced in [7]. Let M be a module, if $N, L \leq M$ and $M = N + L$ then L is called a generalized* supplement of N in case $N \cap L \leq Z^*(L)$. A submodule K of M is called co-finite if M/K is finitely generated. In [8] M is called \oplus co-finitely supplemented (briefly) \oplus cof-supplemented, if every co-finitely submodule of M has a supplement that is a direct summand of M .

In this paper we define generalized* $\oplus Z^*$ supplemented as a generalization of \oplus supplemented module. A module M is called $G^* \oplus Z^*$ supplemented if every submodule containing $Z^*(M)$ has generalized* supplement in M that is a direct summand of M (i.e. $\forall N \leq M, Z^*(M) \leq N, N$ has generalized* supplement L in M and L is direct summand). Clearly every semi-simple module is a $G^* \oplus Z^*$ S and every \oplus

As a generalization of \oplus cof-supplemented we define \oplus co-finitely generalized* supplement (briefly) \oplus cof-generalized* supplement, if every co-finitely submodule of M has generalized* supplement that is direct summand of M for short $G^* \oplus CS$. Clearly \oplus supplement are $G^* \oplus CS$ module and the converse is true if M is finitely generated.

§2 Generalized* \oplus Z^* supplemented

In this section we define $G^* \oplus Z^*$ supplemented module as a generalization of \oplus supplemented in [6] and study some of properties of $G^* \oplus Z^* S$ supplemented module. Clearly every \oplus supplemented module is a $G^* \oplus Z^* S$ module, but the converse does not hold in general, for example Z -module, Q is a $G^* \oplus Z^* S$ module which is not \oplus supplemented module.

Definition (2.1) :- A module M is called $G^* \oplus Z^*$ supplemented if every sub module containing $Z^*(M)$ has generalized* supplement in M that is a direct summand of M (i.e. $\forall N \leq M, Z^*(M) \leq N, N$ has generalized* supplement L in M and L is direct summand (i.e. there exist $L, K \leq M$ such that $M = L + N = L \oplus K$ and $N \cap L \leq Z^*(L)$).

Recall that a sub module N of M is called fully invariant if for every $h \in \text{End}(M)$, $h(N) \leq N$ and M is called a duo module, if every sub module of M is fully invariant.[1]

Lemma (2.2):-Let M be a duo module, if $M = M_1 \oplus M_2$ then $N = (N \cap M_1) \oplus (N \cap M_2)$ for N is submodule of M .

Proof:- see[9]

Lemma (2.3):- Let M be any R -module, and let N be a submodule of M , then $Z^*(N) = Z^*(M) \cap N$. [10]

Proposition (2.4):- Let M be a $G^* \oplus Z^* S$ module, if N is a fully invariant submodule of M , then N is a $G^* \oplus Z^* S$ module.

Proof:- Let $K \leq N \leq M$ with $Z^*(N) \leq K$, then $Z^*(M) \leq K + Z(M)$ since M is a $G^* \oplus Z^* S$ module, then there exist $L, L' \leq M$ such that $M = (K + Z^*(M)) + L = L \oplus L'$ and $(K + Z^*(M)) \cap L \leq Z^*(L)$. Now $N = (K + Z^*(M)) + L \cap N = K + (Z^*(M) \cap N) + (L \cap N)$, hence $N = (K + Z^*(N)) + ((L \cap N)) = K + (L \cap N)$ since $Z^*(N) \leq K$ and $K \cap (L \cap N) = (K \cap L) \cap N \leq ((K + Z^*(M)) \cap L) \cap N \leq Z^*(L) \cap N = (Z^*(M) \cap (L \cap N)) = Z^*(L \cap N)$ since $M = L \oplus L'$, hence $N = (L \cap N) \oplus (L' \cap N)$ therefore N is a $G^* \oplus Z^* S$ submodule of M .

Proposition (2.5):-If M is a $G^* \oplus Z^* S$ module, then M / N is a $G^* \oplus Z^* S$ for every fully invariant submodule of M .

Proof:- Let N be a fully invariant submodule of M and let $K / N \leq M / N$ with $Z^*(M / N) \leq K / N$, since $Z^*(M) + K / N \leq Z^*(M / N)$ by [11], then $Z^*(M) \leq K$, then by assumption there exist $L \leq M$ such that $M = L + K$ with $L \cap K \leq Z^*(L)$ and $M = L \oplus L'$ for some $L' \leq M$, thus $M / N = K / N + (L + N) / N$ and $K / N \cap (L + N) / N = (K \cap L) + N / N \leq Z^*(L) + N / N \leq$

$Z^*(L + N) / N$. Since N is a fully invariant submodule of M then $N = (N \cap L) + (N \cap L')$ and $(N + L) / N \cap (N + L') / N = 0$ then $M / N = N + L / N \oplus N + L' / N$, hence M / N is a $G^* \oplus Z^*S$ module.

Corollary (2.6) :- The homomorphic image of a duo $G^* \oplus Z^*S$ is a $G^* \oplus Z^*S$ module

Proof:- Clear since every homomorphic image is isomorphic to a quotient module.

The following theorem shows that when M is a duo module, the direct sum of $G^* \oplus Z^*S$ is again a $G^* \oplus Z^*S$.

Theorem (2.7):- If $M = M_1 \oplus M_2$, if M is a duo module and M_1, M_2 are $G^* \oplus Z^*S$, then M is a $G^* \oplus Z^*S$.

Proof:- Let $N \leq M$ with $Z^*(M) \leq N$, then $Z^*(M_i) \leq N \cap M_i \forall i=1,2$ hence there exist V_i, V'_i of M_i ($\forall i=1,2$) such that $M_i = (N \cap M_i) + V_i, (N \cap M_i) \cap V_i \leq Z^*(V_i)$ and $M_i = V_i \oplus V'_i$ since N is fully invariant submodule of M hence $N = N \cap M_1 \oplus N \cap M_2$, let $V = V_1 \oplus V_2, V' = V'_1 \oplus V'_2$, hence there exist $V, V' \leq M$ such that $M = M_1 \oplus M_2 = (N \cap M_1) \oplus (N \cap M_2) + (V_1 \oplus V_2) = N + V$ and $N \cap V = (N \cap M_1) + (N \cap M_2) \cap (V_1 \oplus V_2) \leq Z^*(V_1) \oplus Z^*(V_2) = Z^*(V)$ by [11] and $V \oplus V' = (V_1 \oplus V_2) \oplus (V'_1 \oplus V'_2) = M_1 \oplus M_2 = M$, hence M is $G^* \oplus Z^*S$

Corollary (2.8) :- Let $\{M_i\}_{i=1}^\infty$ be any infinite collection of R -modules and $M = \bigoplus_{i \in I} M_i$ is duo module, then M is $G^* \oplus Z^*S$ if M_i are $G^* \oplus Z^*S$ for each $i \in I$.

Lemma (2.9):- For any R -module $M \neq 0, Z^*(M) = 0$ if and only if $\text{Rad}(E(M)) = 0$.

Proof:- see [11]

Proposition (2.10) :- Let M be a non-zero R -module with $\text{Rad}(E(M)) = 0$ then M is $G^* \oplus Z^*S$ if and only if M is semi-simple.

Proof:- \Rightarrow Clear since $\text{Rad}(E(M)) = 0$ then by Lemma (2.9) $Z^*(M) = 0$, hence $\forall 0 \leq N \leq M, N$ has generalized* supplement in M i.e. there exist $K \leq M$ such that $M = N + K$ and $N \cap K = Z^*(K) = 0 \Rightarrow N$ is a direct summand of M

\Leftarrow Clearly since every semi-simple is a $G^* \oplus Z^*S$.

$\S 3 \oplus$ Co-finitely generalized* supplemented modules

In this section we introduce a \oplus co-finitely generalized* supplemented module as a generalization \oplus co-finitely generalized module.[8]

Definition (3.1):- An R-module M is called generalized*co-finitely supplemented if every co-finite has submodule of M has a generalized* supplement in M for short we will refer to these module by G*CS. M is called \oplus co- finitely generalized* supplemented or (briefly) \oplus -cof G*S, if every co- finite submodule of M has G*S that is direct summand of M for short (G*\oplus CS)(i.e. $\forall N \leq M$ with M/N is finitely generated, there exist $L \leq M$ such that L is a G*S of N in M ($M = N + L$), $N \cap L \leq Z^*(L)$ and there exist $K \leq M$ such that $M = L \oplus K$.

It easy to see that \oplus supplement modules are G*\oplus CS module and the converse is true if M is finitely generated, Notice that hollow modules are G*\oplus CS modules.

The following proposition shows that under certain condition the quotient of G*\oplus CS is a G*\oplus CS

Proposition (3.2):- Assume that M is a G*\oplus CS due module then M/N is a G*\oplus CS module.

Proof:- Let $N \leq K \leq M$ with K/N a co- finite submodule of M/N , then $M/K \cong (M/N)/(K/N)$ is finitely generated since M is a G*\oplus CS module, there exist a submodule L and L' of M such that $M = K + L = L \oplus L'$ and $K \cap L \leq Z^*(L)$.

Notice that $M/N = K/N + (L+N)/N$ by modularity $K \cap (L+N) = (K \cap L) + N$ since $K \cap L \leq Z^*(L)$, we have $K/N \cap (L+N)/N = [(K \cap L) + N]/N \leq Z^*(L+N)/N$ this implies that $(L+N)/N$ is G*S of K/N in M/N . Now $N = (N \cap L) \oplus (N \cap L')$ by lemma(2.2) implies that $(L+N) \cap (L'+N) = N + (L+N+L+N \cap L') \cap L'$ it following that $(L+N) \cap (L'+N) \leq N$ and $M/N = (L+N)/N \oplus (L'+N)/N$ then $(L+N)/N$ is direct summand of M/N . Consequently M/N is a G*\oplus CS module.

Proposition (3.3):- For any ring R, if $M = M_1 \oplus M_2$ with M_1 and M_2 are G*\oplus CS module if M is a duo module, then M is a G*\oplus CS.

Proof: - Let L be a co-finite submodule of M i.e. M/L is finitely generated. Now $M = M_1 + M_2 + L$ then $M_1 + M_2 + L$ has a G*S 0 in M i.e. $M = M_1 + M_2 + L + 0$ and $M_1 + M_2 + L \cap \{0\} \leq Z^*(0) = 0$ and so $M/L + M_1$ is finitely generated. Notice that $M/L + M_1 = (M_2 + (M_1 + L))/M_1 + L \cong M_2 / M_2 \cap (M_1 + L)$ hence $M_2 / M_2 \cap (M_1 + L)$ is finitely generated , but M_2 is a G*\oplus CS module, then there exist $H \leq M_2$ such that $M_2 = M_2 \cap (L + M_1) + H$ and $M_2 = H \oplus H'$ for some $H' \leq M_2$ also $M = M_1 + M_2 = M_1 + M_2 \cap (L + M_1) + H$, hence $M = (M_1 + L) + H$ and $(M_1 + L) \cap H \leq Z^*(H)$, thus H is a G*S of $M_1 + L$ in M .Note that $M/L + H = M_1 + (L + H)/(L + H) \cong M_1 / M_1 \cap (L + H)$, then $M_1 \cap (L + H)$ is co-finite submodule of M, since M_1 is a G*\oplus CS module then there exist $K \leq M_1$ such that $M_1 = M_1 \cap (L + H) + K$ with $M_1 \cap (L + H) \cap K = (L + H) \cap K \leq Z^*(K)$ and there exist $K' \leq M_1$ such that $M_1 = K \oplus K'$, hence L is a G*S of $H + K$ in M i.e. $M = M_1 + M_2 = (L + H + K) + (M_2 \cap (L + M_1))$

$+ H = L + H + K$ and $L \cap (H + K) \leq Z^*(K) \leq Z^*(K + H)$ and $H + K = H \oplus K$ since H is a direct summand of M_1 , hence $H \oplus K$ is a direct summand of M .

Corollary (3.4):- Any finite direct sum of $G^* \oplus CS$ module is a $G^* \oplus CS$ module.

Proof:- follows from proposition (3.3)

Before we give next result we need the following definitions:-

Definition (3.5): A module M is said to be have the summand intersection property (SIP) if the intersection of any pair of direct summands of M is a direct summand of M (i.e. if N and K are direct summand of M then $N \cap K$ is also a direct summand of M).

A module M is said to have the summand sum property (SSP) if the sum of any pair of direct summand of M is a summand of M (i.e. if N and K are direct summand of M then $N + K$ is also a direct summand of M).

Recall that a module M distributive if for submodule K, L, N of M $N+(K \cap L) = (N + K) \cap (N + L)$ or $N \cap (K + L) = (N \cap K) + (N \cap L)$.

Hence we have the following:-

Theorem (3.6):- 1- Let M be a $G^* \oplus CS$ -module and N a submodule of M , if for every direct summand K of M , $(N+K)/N$ is direct summand of N/M then M/N is a $G^* \oplus CS$ module.

2- Let M be a $G^* \oplus CS$ -distributive module then M/N is a $G^* \oplus CS$ module for every submodule N of M .

3- Let M be a $G^* \oplus CS$ module with SSP then every direct summand of M is a $G^* \oplus CS$ module.

Proof: 1- Any co -finite submodule of M/N has the form L/N where L there exist a direct summand K of M such that $M = L + K = K \oplus K'$ and $L \cap K \leq Z^*(K)$ for some submodule $K' \leq M$. Now $M/N = L/N + (K + N)/N$, by hypothesis $(K + N)/N$ is direct summand of M/N , Note that $(L/N) \cap (K + N)/N = [(L \cap (K + N))/N] = [N + (K \cap L)]/N$ since $L \cap K \leq Z^*(K)$.we have $[(K \cap L + N)/N] \leq Z^*(K + N)/N$. This implies that $(K+N)/N$ is G^*S submodule of L/N in M/N . hence M/N is a $G^* \oplus CS$ module.

Proof: 2- Since M is a $G^* \oplus CS$ then any co-finite submodule of M has a G^*S that is a direct summand of M . Let L be a direct summand of M i.e. $M = L \oplus L'$ for some submodule L' of M . Now $M/N = [(L + N)/N] + [(L' + N)/N]$ and $N = N + (L \cap L') = (N + L) \cap (N +$

L') since M is distributive, thus $M/N[(L+N)/N] \oplus [(L'+N)/N]$ by (1) hence M/N is $G^* \oplus CS$ module.

Proof:3- Let N be a direct summand of M i.e. $M = N \oplus N'$ for some $N' \leq M$, to prove that M/N' is a $G^* \oplus CS$ module. Let L be a direct summand of M , since M has the SSP, then $L + N'$ is a direct summand of M . i.e. $M = (L + N') \oplus K$ for some $K \leq M$, then $M/N' = (L + N')/N' \oplus K/N'$, hence by (1) M/N' is a $G^* \oplus CS$ module.

Weimin Xue in [12] introduce the notation of generalized projective covers to characterize semi perfect modules and rings.

An epimorphism $f: P \rightarrow M$ is called a generalized cover in case $\ker f \leq \text{Rad}(P)$, when P is a projective module then f is called a generalized projective cover.

As a generalization of this concept we introduce the following definition:

Definition (3.7):- If P and M are modules, we call an epimorphism $f: P \rightarrow M$ a (generalized*) cover in case $(\ker f \leq Z^*(P))$, If P is a projective module, then f is called (generalized*) projective cover. Clearly every projective cover is generalized* projective cover.

We have the following basis properties of generalized* cover.

Lemma (3.8):1- If $f: P \rightarrow M$ and $g: M \rightarrow N$ are generalized* cover for M and N , with $f(Z^*(P)) = Z^*(M)$, then $g \circ f: P \rightarrow N$ is a generalized* cover for N .

Proof: - If both f and g are covers, then $g \circ f$ is cover by [2], Now let both f and g be generalized* cover. It is enough to prove that $\ker(g \circ f) \leq Z^*(P)$. Let $x \in \ker(g \circ f)$, then $g \circ f(x) = 0$, hence $f(x) \in \ker g \leq Z^*(M)$, since $\ker f \leq Z^*(P)$, then there exist $x' \in Z^*(P)$ such that $f(x) = f(x')$, for some $x' \in Z^*(P)$, hence $x - x' \in \ker f \leq Z^*(P)$, therefore $x \in Z^*(P)$.

2- If each $f_i: P_i \rightarrow M_i$, $i = 1, \dots, n$, is a generalized* cover, then $\bigoplus_{i=1}^n f_i: \bigoplus_{i=1}^n P_i \rightarrow \bigoplus_{i=1}^n M_i$ is a generalized* cover.

Proof:- Since $\ker f_i \leq Z^*(P_i)$, $\forall i = 1, 2, \dots, n$ we have $\ker(\bigoplus_{i=1}^n f_i) = \bigoplus_{i=1}^n \ker f_i$, thus $\ker(\bigoplus_{i=1}^n f_i) \leq \bigoplus_{i=1}^n Z^*(P_i)$, i.e. $\bigoplus_{i=1}^n f_i$ is a generalized* cover.

Lemma (3.9):- Let N be a submodule of the module M and $f: M \rightarrow M/N$ be canonical epimorphism also, let P any module, $g: P \rightarrow M/N$ and $h: P \rightarrow M$ with $h(Z^*(P)) = Z^*(M)$ such that g is h composed with f . Then the map g is a generalized* cover epimorphism if and only if $\text{Im}(h)$ is a generalized* supplemented of N and $\ker h \leq Z^*(P)$.

Proof:- \Rightarrow) Let $x \in N \cap \text{Im}h$, then $x \in N = \text{ker}f$ and $x \in \text{Im}h$ i.e. there exist $y \in P$ such that $x = h(y)$. Now $g(y) = f(h(y)) = f(x) = 0$ (since $x \in \text{ker}f = N$), thus $y \in \text{ker}g$ and $h(y) \in h(\text{ker}g)$. Now let $x \in h(\text{ker}g)$, then $x = h(y)$, $y \in \text{ker}g$ (i.e. $g(y) = 0$, hence $fh(y) = g(y) = 0$, $f(x) = g(y) = 0$, thus $N \cap \text{Im}h = h(\text{ker}g) \leq Z^*(\text{Im}h) = Z^*(h(P))$ then $\text{Im}h = h(P)$ is a generalized* supplement of N , since g is an epimorphism then $\text{ker}h \leq \text{ker}g$ thus $\text{ker}h \leq Z^*(P)$.

\Leftarrow) the converse is clearly by lemma (3.8 (1))

Recall that an R -module M is called semi perfect module if every factor module has a projective cover. As a generalization of semi perfect modules, we will introduce the following [1].

An R -module M is called a generalized* co-finitely semi perfect, if every finitely generated factor has a generalized* projective cover. Clearly every semi perfect module is a generalized* co-finitely semi perfect.

Theorem (3.10):- Let M be a module in which every generalized* projective cover f satisfies $f(Z^*(P)) = Z^*(M)$, the following are equivalent:

1. M is a generalized* co-finitely semi perfect module.
2. M is a generalized* co-finitely module by supplements which have generalized* projective cover.

Proof:- $1 \Rightarrow 2$) Let $M = N + L$ with M/N is finitely generated projective cover for M/N , P is a projective R -module. Now $M/N = N + L/N \simeq L/L + N/N$ since P is projective, then the map f lefts $g : P \rightarrow L$, and since f is a generalized* cover, then by Lemma(3.9), we get $\text{Im}g$ is a generalized* cover of $(L \cap N)$ i.e. $\text{Im}g + (L \cap N) = L$ and $\text{Im}g \cap (L \cap N) \leq Z^*(\text{Im}g)$, $\text{ker}g \leq \text{ker}(\pi \circ i \circ g) = \text{ker}f \leq Z^*(P)$.

$2 \Rightarrow 1$) Let N be a co-finite submodule of M , then M/N is finitely generated by (2) there exist $L \leq M$ such that $M = L + N$ and $L \cap N \leq Z^*(L)$. Let $f : P \rightarrow L$ be a generalized* projective cover of L the natural epimorphism $g : L \rightarrow L/L \cap N \simeq N + K/N = M/N$ is a generalized* cover (for $\text{ker}g = L \cap N \leq Z^*(L)$), hence $h = g \circ f : P \rightarrow M/N$ is generalized* projective cover for M/N by Lemma (3.8).

Corollary (3.11):- Let M be a projective $G^* \oplus CS$ module, then M is a generalized* co-finitely semi perfect module.

Proof:- Let N be a co-finite submodule of M , i.e. M/N is finitely generated since M is a $G^* \oplus CS$ module, then there exist $K, K' \leq M$ such that $M = N + K = K \oplus K'$ and $N \cap K \leq Z^*(K)$, K is projective, let $i : K \rightarrow M$ be the inclusion homomorphism and let $\pi : M \rightarrow M/N$ be the natural epimorphism, hence $\pi \circ i : K \rightarrow M/N$ is an epimorphism, $\text{ker}(\pi \circ i) = N \cap K \leq Z^*(K)$ thus M is a generalized* co-finitely semi perfect module.

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