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A Note on Riemannian Submersions with Umbilical Fibres

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ABSTRACT.

In this paper, we discuss some geometric properties of Riemannian submersions whose fibres are totally contact umbilical. Some interrelations between totally contact umbilic, totally geodesic and minimality are established.

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1. INTRODUCTION

Riemannian submersions have been initiated by B. O'Neill in [6], but various classes are treated in [3] and the references therein. The case of totally umbilic fibres was introduced by G. Baditoiu and S. Ianus in [1].

In this note, we will be interested by almost contact metric submersions which are Riemannian submersions whose total space is an almost contact metric manifold studied by B. Watson [10].

The paper focuses on the geometry of the fibres, emphasing on the concept of totally contact umbilicity. It is organized in the following way.

In Section 2, we recall some basic notions related to almost contact metric manifolds.

Section 3 is devoted to almost contact metric submersions. Relationships between this concept with the minimality or the geodesibility are established. Furtheremore, we apply, to the study of almost contact metric submersions of type II, a result of Hong and Tripathi [4] concerning the asymptotic direction generated by the characteristic vector field of the total space.

2. BACKGROUND ON MANIFOLDS

Let M be a differentiable manifold of odd dimension $2m+1$. An almost contact structure on M is a triple (φ, ξ, η) where:

(1) ξ is a characteristic vector field,

(2) η is a $1-$form such that $\eta(\xi) = 1$,and

(3) φ is a tensor field of type (1, 1) satisfying

$$\varphi^2 = -\mathbb{I} + \eta \otimes \xi,$$

where $\mathbb{I}$ is the identity transformation.

If M is equipped with a Riemannian metric g

$$g(\varphi D, \varphi E) = g(D, E) - \eta(D)\eta(E),$$

then (g, φ, ξ, η) is called an almost contact metric structure. So, the quintuple $(M^{2m+1}, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. Any almost contact metric manifold admits a fundamental $2-$form, ϕ, defined by

$$\phi(D, E) = g(D, \varphi E).$$

An almost contact metric manifold is said to be:

(1) *closely cosymplectic* if $(\nabla_D \varphi)D = 0$ and $d\eta = 0$;

(2) *nearly cosymplectic* if $(\nabla_D \varphi)D = 0$

(3) *nearly-K- cosymplectic* if $(\nabla_D \varphi)E + (\nabla_E \varphi)D = 0$ and $\nabla_D \xi = 0$;

(4) *nearly Kenmotsu* if $(\nabla_D \varphi)D = -\eta(D)\varphi D$ and $d\eta = 0$;

(5) *K-contact* $\phi = d\eta$ if $(\nabla_D \eta)E + (\nabla_E \eta)D = 0.$

We begin by recalling some basic concepts before applying them to the case of Riemannian submersions.

Let $\bar{M}$ be a submanifold of a Riemannian manifold (M^m, g) and σ the second fundamental form of $\bar{M}$. It is well known that the mean curvature vector field, *H*, of $\bar{M}$ is given by $H = \frac{1}{m} tr(\sigma)$ where $tr(\sigma)$ denotes the trace of σ and m the dimension of $\bar{M}$

If $H = 0$, then $\bar{M}$ is said to be minimal;

If $\sigma = 0$,then $\bar{M}$ is said to be totally geodesic, and

If $\sigma(D, E) = g(D, E)H$,then $\bar{M}$ is totally umbilical.

In the case of contact geometry, M.M. Tripathi and S.S. Shukla, [7], have defined the concepts of totally contact umbilic and totally contact geodesic in the following way.

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost contact metric manifold. For a distribution $\mathcal{D}$ on $\bar{M}$, $\bar{M}$ is said to be $\mathcal{D}-$totally geodesic if, for all $D, E \in \mathcal{D}$,we have $\sigma(D, E) = 0.$

If for all $D, E \in \mathcal{D}$,we have $\sigma(D, E) = g(D, E)N$ for some normal vector field N, then $\bar{M}$ is $\mathcal{D}-$totally umbilical.

Suppose that $\bar{M}$ is tangent to the structure vector field ξ. Consider the distribution $\{\xi\}$ generated by this vector field and $\{\xi\}^{\perp}$ its complement. In this case, Tripathi and Shukla have defined the following two concepts.

The submanifold $\bar{M}$ is said to be:

(1) *totally contact umbilic* if it is $\{\xi\}^{\perp}$–totally umbilic;

(2) *totally contact geodesic* if it is $\{\xi\}^{\perp}$–totally geodesic.

Definition 2.1. *Let* $\bar{M}$ *be a submanifold of an almost contact metric manifold* $(M^{2m+1}, g, \varphi, \xi, \eta)$.

Then $\bar{M}$ is:

(a) totally contact umbilic if, $\sigma(\varphi^2 D, \varphi^2 E) = g(\varphi^2 D, \varphi^2 E)N,$

for all $D, E \in \chi(\bar{M})$ and N a unit normal vector field.

(b) totally contact geodesic if, $\sigma(\varphi^2 D, \varphi^2 E) = 0,$ for all $D, E \in \chi(\bar{M})$.

In his study of lightlike hypersurfaces of indefinite Sasakian manifolds, F. Massamba [5] used also these two concepts.

Proposition 2.2. *Let* $\bar{M}$ *be a submanifold of an almost contact metric manifold* $(M^{2m+1}, g, \varphi, \xi, \eta)$*.if* $\bar{M}$ *is:*

(1) *totally contact umbilic, then,*

$$\sigma(D, E) = g(\varphi D, \varphi E)N + \eta(D)\sigma(E, \xi) + \eta(E)\sigma(D, \xi) - \eta(D)\eta(E)\sigma(\xi, \xi),$$

(2) *totally contact geodesic, then,*

$$\sigma(D, E) = \eta(D)\sigma(E, \xi) + \eta(E)\sigma(D, \xi) - \eta(D)\eta(E)\sigma(\xi, \xi).$$

Proof. (1) Let us consider the relation

$$\sigma(\varphi^2 D, \varphi^2 E) = g(\varphi^2 D, \varphi^2 E)N.$$

It is known that

$$g(\varphi^2 D, \varphi^2 E) = g(D, E) - \eta(D)\eta(E),$$

from which $\sigma(\varphi^2 D, \varphi^2 E) = (g(D, E) - \eta(D)\eta(E))N.$ On the other hand,

$$\begin{aligned} \sigma(\varphi^2 D, \varphi^2 E) &= \sigma(-D + \eta(D)\xi, -E + \eta(E)\xi) \\ &= \sigma(D, E) - \eta(E)\sigma(D, \xi) - \eta(D)\sigma(E, \xi) + \eta(D)\eta(E)\sigma(\xi, \xi). \end{aligned}$$

Remembering that

$$g(D, E) - \eta(D)\eta(E) = g(\varphi D, \varphi E),$$

we get the proof of assertion (1).

Concerning assertion (2), since $\bar{M}$ is totally contact geodesic, we have

$\sigma(\varphi^2 D, \varphi^2 E) = 0,$ which implies that $\sigma(-D + \eta(D)\xi, -E + \eta(E)\xi) = 0.$ But

$$\begin{aligned} \sigma(-D + \eta(D)\xi, -E + \eta(E)\xi) &= \sigma(D, E) - \sigma(D, \eta(E)\xi) \\ &- \sigma(\eta(D)\xi, E) + \sigma(\eta(D)\xi, \eta(E)\xi) \\ &= \sigma(D, E) - \eta(E)\sigma(D, \xi) \\ &- \eta(D)\sigma(E, \xi) + \eta(D)\eta(E)\sigma(\xi, \xi). \end{aligned}$$

which is $\sigma(D,E)=\eta(D)\sigma(E,\xi)+\eta(E)\sigma(D,\xi)-\eta(D)\eta(E)\sigma(\xi,\xi)$. □

Proposition 2.3. ([4]) *Let $\bar{M}$ be a submanifold of an almost contact metric manifold tangent to ξ. If $\bar{M}$ otally contact umbilic or totally contact geodesic, then ξ is an asymptotic direction.*

Proof. Let us recall that a vector field D defines an asymptotic direction if $\sigma(D,D)=0$. With this in mind, putting $D=\xi=E$ in the defining equations in Proposition 2.2, it is clear that the structure vector field ξ defines an asymptotic direction. □

3. ALMOST CONTACT METRIC SUBMERSIONS

In [6], O'Neill has defined a Riemannian submersion as a surjective mapping $\pi: M \longrightarrow B$ between two Riemannian manifolds such that

(i) π is of maximal rank;

(ii) $\pi_{*|(Ker\pi_*)^\perp}$ is a linear isometry.

The tangent bundle $T(M)$, of the total space M, admits an orthogonal decomposition

$$T(M)=V(M)\oplus H(M),$$

where $V(M)$ and $H(M)$ denote respectively the vertical and horizontal distributions.

We denote by $\mathcal{V}$ and $\mathcal{H}$ the vertical and horizontal projections respectively. A vector field X of the horizontal distribution is called a basic vector field if it is $\pi-$ related to a vector field X_* of the base space B. Such a vector field means that $\pi_* X = X_*$.

On the base space, tensors and other objects will be denoted by a prime' while those tangent to the fibres will be specified by a carretˆ. Herein, vector fields tangent to the fibres will be denoted by U, V W.

When the base space is an almost Hermitian manifold, $(B^{2m'},g',J')$, the Riemannian submersion

$$\pi: M^{2m+1}\rightarrow B^{2m'}$$

is called an almost contact metric submersion of type II, [8], if II, $\pi_*\varphi = J'\pi_*$.

This type of submersions are called $(\varphi,J)-$holomorphic in [2].

Now, let us turn to some implications of the total umbilicity.

Proposition 3.1. *Let $\pi:(M,g)\longrightarrow(M',g')$ be a Riemannian submersion with totally umbilic fibres. If the mean curvature vector field, H, of the fibres is parallel,*

(a) $\mathcal{R}(U,V,U,V)=\hat{\mathcal{R}}(U,V,U,V)+[g(U,V)^2-g(U,U)g(V,V)]g(H,H)$;
(b) $\mathcal{R}(X,U,X,U)=g(A_X U, A_X U)$;
(c) $\mathcal{R}(X,Y,X,Y)=\mathcal{R}'(X_*,Y_*,X_*,Y_*)-3g(A_X Y, A_X Y)$.

Proof. Assertions (a) and (c) follow from O'Neill's equations [6].

Let us consider (b). Recall from Baditoiu and Ianus [1], that

$$\mathcal{R}(X,U,X,U) = g(U,U)[g(\nabla_X H, X) - g(X,H)^2] + g(A_X U, A_X U).$$

Since, H is vertical and X is horizontal, we have $g(X,H) = 0$;on the other hand, the parallelism of H implies that $\nabla_X H = 0$ so that $g(\nabla_X H, X) = 0$;
therefore

$$\mathcal{R}(X,U,X,U) = g(A_X U, A_X U).$$

□

Let us examine some applications in the case of almost contact metric submersions.

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ and $(M'^{2m'+1}, g', \varphi', \xi', \eta')$be two almost contact metric manifolds. By an almost contact metric submersion of type I, in the sense of Watson [10], one understands a Riemannian submersion

$$\pi : M^{2m+1} \to M'^{2m'+1}$$

satisfying

(i) $\pi_* \varphi = \varphi' \pi_*$,
(ii) $\pi_* \xi = \xi'$.

When the base space is an almost Hermitian manifold, $(M'^{2m'}, g', J')$, the Riemannian submersion

$$\pi : M^{2m+1} \to M'^{2m'}$$

is called an almost contact metric submersion of type II, if $\pi_* \varphi = J' \pi_*$, [10].

Now, we overview some of the fundamental properties of these submersions.

Proposition 3.2. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ *be an almost contact metric submersion of type I. Then*

(a) $\pi^* \phi' = \phi$;
(b) $\pi^* \eta' = \eta$;
(c) $\eta(U) = 0$ *for all* $U \in V(M)$;
(d) $\mathcal{H}(\nabla_X \varphi) Y$ *is the basic vector field associated to* $(\nabla'_{X_*} \varphi') Y_*$ *if* X *and* Y *are basic.*

Proof. See [8, 9, 10].

□

Proposition 3.3. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ *an almost contact metric submersion of type II. Then*

(a) $\pi^* \Omega' = \phi$;
(b) $\eta(X) = 0$ *for all* $X \in H(M)$;
(c) $\mathcal{H}(\nabla_X \varphi) Y$ *is the basic vector field associated to* $(\nabla'_{X_*} J') Y_*$ *if* X *and* Y *are basic.*

Proof. See again Watson [10]. □

Proposition 3.4. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ *be an almost contact metric submersion of type I. If the fibres are totally contact umbilic, then*

$$T_U V = g(U,V)H.$$

Proof. It is known that the O'Neill configuration tensor T is the second fundamental form of the fibres. Since η vanishes on vertical vector fields, then in Proposition 2.2, the defining equation (a) becomes $T_U V = g(\varphi U, \varphi V)H$, which is $T_U V = g(U,V)H$ because

$$\begin{aligned} g(\varphi U, \varphi V) &= g(U,V) - \eta(U)\eta(V) \\ &= g(U,V). \end{aligned}$$

□

Corollary 3.5. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ *be an almost contact metric submersion of type I with totally contact umbilic fibres. If the fibres are minimal, then they are totally geodesic.*

Proof. In such a case, we have

$$T_U V = g(U,V)H.$$

because of umbilicity of the fibres. If moreover the fibres are minimal, we have $T_U V = 0$ which shows that $T = 0$. □

Proposition 3.6. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ *be an almost contact metric submersion of type II. Suppose that the total space* M *is a* $K-$*contact manifold. If the fibres are totally contact umbilic, then they are totally geodesic.*

Proof. See [4]. □

It is easy to show that if $T_U \varphi V = \varphi T_U V$, then $T_U \xi = 0$. With this in mind, we can deduce that

The fibres of an almost contact metric submersion whose total space is closely cosymplectic, nearly cosymplectic, nearly- K*-- cosymplectic or nearly Kenmotsu are totally geodesic.*

Proposition 3.7. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ *be an almost contact metric submersion of type II. If the fibres are totally contact umbilic, then the vector fields* ξ *defines an asymptotic direction..*

Proof. Since, for a type II submersion, ξ is tangent to the fibres, then we can apply Proposition 2.3. □

Note. Let $\pi : B \times_h F \longrightarrow B$ be a Riemannian submersion whose total space is a warped product. In this case, we have a submersion with totally umbilic fibres as noted in [3]. Since a Kenmotsu manifold is a warped product, [3], it is clear that any almost contact metric submersion with total space a Kenmotsu manifold is of totally umbilic fibres.

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