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On the order and type of entire functions defined by multiple Dirichlet series

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Abstract

In this paper, we shallobtain some relations between the orders and types of entire functions represented by multiple Dirichlet series. By taking asymptotic behavior in their coefficients. The results has been given in the form of theorems.

Keywords: Multiple Dirichlet series; order and type of the Dirichlet series.

1-Introduction.

Consider themultipleDirichlet series:

$f(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n}\ \exp(s_1\lambda_m+s_2\mu_n)$, $(s_j = \sigma_j+it_j,\ j = 1,2)$ (1.1)

Where $a_{m,n}$ ∈ C the field of complex numbers, λ_m, μ_n are real $0<\lambda_1<\lambda_2<\ldots\lambda_m\to\infty, \mu_1<\mu_2<\ldots<\mu_n\to\infty$.A.I Jansauskas [1] has proved that if:

$\lim_{m\to\infty} \frac{\log m}{\lambda_m} = 0, \lim_{m\to\infty} \frac{\log n}{\mu_n} = 0$ (1.2)

Then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence, Also sarkar [2, pp.99] has shown that the necessary and sufficient condition that the series (1.1) satisfying (1.2) to be entire is that:

$\lim_{m+n\to\infty} \frac{\log|a_{m,n}|}{\lambda_m+\mu_n} = \infty$ (1.3)

Let $M(\sigma_1,\sigma_2)=\sup\{|f(\sigma_1+it_1,\ \sigma_2+it_2|\}$, be the maximum modulus of f (s_1,s_2) on the tube Re $s_j=\sigma_j$, j=1,2.

(1.4)

We define the order ρ $(0\le \rho \le \infty)$ and typeT $(0\le T \le \infty)$,of $f(s_1,s_2)$,[4] as:

$\lim_{\sigma_1,\sigma_2\to\infty} \frac{\log M(\sigma_1,\sigma_2)}{\log(e^{\sigma_1}+e^{\sigma_2})} = \rho$ .eρT= $\lim_{m,n\to\infty} sup\{(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n})\left|a_{m,n}\right|^{\rho}\}^{\frac{1}{\lambda_m+\mu_n}}$ (1.5)

Now we have the following theorem [3, pp.52], which will be used in the next section.

Theorem 1.1: If f (s_1,s_2) is an entire function of order ρ ($0\leq \rho \leq \infty$) , then

$$\rho=\lim_{m,n\to\infty} \frac{\lambda_m \log\lambda_m + \mu_n \log\mu_n}{\log\left|a_{m,n}\right|^{-1}} . \quad (1.6)$$

In this paper, we shall obtain some relations between two or more entireDirichlet series and study the relations between the coefficients in the Taylor expansion of entire Dirichlet series and their orders and type.

2-Main Results

Theorem 2.1:

If $f_1(s_1,s_2)=\sum_{m,n=0}^{\infty} a_{m,n}^{(1)} \exp(s_1\lambda_{1,m}+s_2\mu_{1,n})$, $f_2(s_1,s_2)=\sum_{m,n=0}^{\infty} a_{m,n}^{(2)} \exp(s_1\lambda_{2,m}+s_2\mu_{2,n})$ be entire functions of orders ρ_1 $(0\leq \rho_1 \leq \infty)$ and ρ_2 $(0\leq \rho_2 \leq \infty)$. Then the function $f(s_1,s_2)=\sum_{m,n=0}^{\infty} a_{m,n} \exp(s_1\lambda_m+s_2\mu_n)$, where

(i) $\lambda_{1,m}\sim \lambda_{2,m}\sim\lambda_m$, $\mu_{1,n}\sim \mu_{2,n}\sim\mu_n$ and

(ii) $\left|a_{m,n}\right|\sim\left|a_{m,n}^{(1)}\right|\left|a_{m,n}^{(2)}\right|$ is an entire function of order ρ, such that

$$\frac{1}{\rho}\geq\frac{1}{\rho_1}+\frac{1}{\rho_2} \quad (2.1)$$

Proof

Since $\lambda_{1,m}\sim \lambda_{2,m}\sim\lambda_m$, $\mu_{1,n}\sim \mu_{2,n}\sim\mu_n$ it is evident that :

$$\lim_{m\to\infty} sup\frac{\log m}{\lambda_m}=\lim_{m\to\infty} sup\frac{\log m}{\lambda_{1,m}}\lim_{m\to\infty} sup\frac{\log m}{\lambda_{2,m}}=0, \text{ Also} \quad (2.2)$$

$$\lim_{n\to\infty} sup\frac{\log n}{\mu_n}=\lim_{n\to\infty} sup\frac{\log n}{\mu_{1,n}}=\lim_{m\to\infty} sup\frac{\log n}{\mu_{2,n}}=0$$

Since $f_1(s_1,s_2)$ and $f_2(s_1,s_2)$ satisfy (1.2) by hypothesis ,further since $f_1(s_1,s_2)$ and $f_2(s_1,s_2)$ are entire functions, bounded on

$(\sigma_1,\sigma_2)<(x_1,x_2)<(\infty,\infty)$ for any$(x_1,x_2)<(\infty,\infty)$.

The series $\sum_{m,n=0}^{\infty}\left|a_{m,n}^{(1)}\right| \exp(\sigma_1\lambda_{1,m}+\sigma_2\mu_{1,n})$,$\sum_{m,n=0}^{\infty}\left|a_{m,n}^{(2)}\right| \exp(\sigma_1\lambda_{2,m}+\sigma_2\mu_{2.n})$ (2.3)

Are convergent for every $\sigma=(\sigma_1,\sigma_2)$and as $\lambda_{1,m}\sim \lambda_{2,m}\sim\lambda_m$, $\mu_{1,n}\sim \mu_{2,n}\sim\mu_n$, we shall have :

$\sum_{m,n=0}^{\infty}\left|a_{m,n}^{(1)}\right|\exp(\sigma_1\lambda_m+\sigma_2\mu_n)<\infty$ and $\sum_{m,n=0}^{\infty}\left|a_{m,n}^{(2)}\right| \exp(\sigma_1\lambda_m+\sigma_2\mu_n)<\infty$,for every $\sigma=(\sigma_1,\sigma_2)$.

(2.4)

Hence it follows that:

$\lim_{m,n\to\infty}\left|a_{m,n}^{(1)}\right| = 0$ and $\sum_{m,n=0}^{\infty}\left|a_{m,n}^{(1)}\right|\left|a_{m,n}^{(2)}\right|\exp(\sigma_1\lambda_m+\sigma_2\mu_n)<\infty$ (2.5)

And since $\left|a_{m,n}\right|\sim\left|a_{m,n}^{(1)}\right|\left|a_{m,n}^{(2)}\right|$, we have $\sum_{m,n=0}^{\infty}\left|a_{m,n}\right|\exp(\sigma_1\lambda_m+\sigma_2\mu_n)<\infty$ for every $\sigma=(\sigma_1,\sigma_2)$.hence f (s_1,s_2) is entire function .

Again using (1.6) for $f_1(s_1,s_2)$ and $f_2(s_1,s_2)$ we get

$\lim_{m,n\to\infty}\frac{-\log\left|a_{m,n}^{(1)}\right|}{\lambda_m\log\lambda_m+\mu_n\log\mu_n}=\frac{1}{\rho_1}$, since $\lambda_{1,m}\sim\lambda_{2,m}\sim\lambda_m$, $\mu_{1,n}\sim\mu_{2,n}\sim\mu_n$. And (2.6)

$\lim_{m,n\to\infty}\frac{-\log\left|a_{m,n}^{(2)}\right|}{\lambda_m\log\lambda_m+\mu_n\log\mu_n}=\frac{1}{\rho_2}$ (2.7)

Therefore, for every $\varepsilon>0$, we have for sufficiently large m, n .

$\frac{-\log\left|a_{m,n}^{(1)}\right|}{\lambda_m\log\lambda_m+\mu_n\log\mu_n}>\frac{1}{\rho_1}-\frac{\varepsilon}{2}$ (2.8)

$\frac{-\log\left|a_{m,n}^{(2)}\right|}{\lambda_m\log\lambda_m+\mu_n\log\mu_n}>\frac{1}{\rho_2}-\frac{\varepsilon}{2}$ (2.9)

Or $\frac{-\log\left|a_{m,n}^{(2)}\right|\left|a_{m,n}^{(1)}\right|}{\lambda_m\log\lambda_m+\mu_n\log\mu_n}>\frac{1}{\rho_1}+\frac{1}{\rho_2}+\varepsilon$ (2.10)

And, since $\left|a_{m,n}\right|\sim\left|a_{m,n}^{(1)}\right|\left|a_{m,n}^{(2)}\right|$, we get,

$\frac{-\log\left|a_{m,n}\right|}{\lambda_m\log\lambda_m+\mu_n\log\mu_n}>\frac{1}{\rho_1}+\frac{1}{\rho_2}$

Or $\frac{1}{\rho}\geq\frac{1}{\rho_1}+\frac{1}{\rho_2}$ (2.11)

Corollary:

Let $f_k(s_1,s_2)=\sum_{m,n=0}^{\infty}a_{m,n}^{(k)}\exp(s_1\lambda_{k,m}+s_2\mu_{k,n})$ **,** where k=1,2,….,p be entire functions of non-zero finite orders $\rho_1,\ldots\ldots,\rho_p$. respectively, then the function

$f(s_1,s_2)=\sum_{m,n=0}^{\infty}a_{m,n}\exp(s_1\lambda_m+s_2\mu_n)$, where

$\lambda_{k,m}\sim\lambda_m$,$\mu_{k,n}\sim\mu_n$

$\left|a_{m,n}\right|\sim\prod_{k=1}^{p}\left|a_{m,n}^{(k)}\right|$, is an entire function such that

$\frac{1}{\rho}\geq\sum_{k=1}^{p}\frac{1}{\rho_k}$, where ρ is the order of $f(s_1,s_2)$. (2.12)

Theorem 2.2: If$f_1(s_1,s_2)=\sum_{m,n=0}^{\infty} a_{m,n}^{(1)}\exp(s_1\lambda_{1,m}+s_2\mu_{1,n})$,$f_2(s_1,s_2)=\sum_{m,n=0}^{\infty} a_{m,n}^{(2)}\exp(s_1\lambda_{2,m}+s_2\mu_{2,n})$be entire functions of orders ρ_1 ($0\leq \rho_1 \leq \infty$) and ρ_2 ($0\leq \rho_2 \leq \infty$). Then the function $f(s_1,s_2)=\sum_{m,n=0}^{\infty} a_{m,n}$ $\exp(s_1\lambda_m+s_2\mu_n)$,where

(i) $\lambda_{1,m}\sim \lambda_{2,m}\sim\lambda_m$, $\mu_{1,n}\sim \mu_{2,n}\sim\mu_n$ and

(ii) $\log|a_{m,n}|^{-1}\sim \left|\{log\left|a_{m,n}^{(1)}\right|^{-1} log\left|a_{m,n}^{(2)}\right|^{-1}\}^{\frac{1}{2}}\right|$ (2.13)

Is an entire function suchthat:

$(\frac{1}{\rho})\geq(\frac{1}{\rho_1\rho_2})^{\frac{1}{2}}$, where ρ is the order of $f(s_1,s_2)$. (2.14)

Proof:-Since$f_1(s_1,s_2)$ and $f_2(s_1,s_2)$ are entire functions , therefore using (1.3) for these two functions , we have for an arbitrary

$\varepsilon>0$ and large ρ_1 , ρ_2 and since $\lambda_{1,m}\sim \lambda_{2,m}\sim\lambda_m$, $\mu_{1,n}\sim \mu_{2,n}\sim\mu_n$, we have :

$(L_1-\varepsilon)<\log(\left|a_{m,n}^{(1)}\right|^{-1})^{\frac{1}{\lambda_m+\mu_n}}<(L_2+\varepsilon)$

$(L_1-\varepsilon)<\log(\left|a_{m,n}^{(2)}\right|^{-1})^{\frac{1}{\lambda_m+\mu_n}}<(L_2+\varepsilon)$ (2.15)

Therefore for $(m+n)>k=\max(k_1,k_2)$ and let $L=\max(L_1,L_2)$ we have :

$(L-\varepsilon)<\log(\left|a_{m,n}^{(1)}\right|^{-1})^{\frac{1}{\lambda_m+\mu_n}}<(L+\varepsilon)$

$(L-\varepsilon)<\log(\left|a_{m,n}^{(2)}\right|^{-1})^{\frac{1}{\lambda_m+\mu_n}}<(L+\varepsilon)$ (2.16)

Multiply both sides we get:-

$(L-\varepsilon)^2<\{log\left|a_{m,n}^{(1)}\right|^{-1} log\left|a_{m,n}^{(2)}\right|^{-1}\}^{\frac{1}{\lambda_m+\mu_n}}<(L+\varepsilon)^2$

Or

$(L-\varepsilon)<\{(log\left|a_{m,n}^{(1)}\right|^{-1} log\left|a_{m,n}^{(2)}\right|^{-1})^{\frac{1}{2}}\}^{\frac{1}{\lambda_m+\mu_n}}<(L+\varepsilon)$ (2.17)

Since $\log|a_{m,n}|^{-1}\sim \left|\{log\left|a_{m,n}^{(1)}\right|^{-1} log\left|a_{m,n}^{(2)}\right|^{-1}\}^{\frac{1}{2}}\right|$, therefore for large m+n we have:

$(L-\varepsilon)<\{log|a_{m,n}|^{-1}\}^{\frac{1}{\lambda_m+\mu_n}}<(L+\varepsilon)$ (2.18)

Or

$\lim_{m,n\to\infty} sup\{log|a_{m,n}|^{-1}\}^{\frac{1}{\lambda_m+\mu_n}}=\infty$ (2.19)

Hence $f(s_1,s_2)$ is an entire function . Now from (2.6) and (2.7), we have for sufficiently large (m+n)

$$\frac{-\log\left|a_{m,n}^{(1)}\right|}{\lambda_m \log\lambda_m + \mu_n \log\mu_n} > \frac{1}{\rho_1} - \frac{\varepsilon}{2}$$

$$\frac{-\log\left|a_{m,n}^{(2)}\right|}{\lambda_m \log\lambda_m + \mu_n \log\mu_n} > \frac{1}{\rho_2} - \frac{\varepsilon}{2} \qquad (2.20)$$

Or

$$-\log\left|a_{m,n}^{(1)}\right| > (\lambda_m \log\lambda_m + \mu_n \log\mu_n)\left(\frac{1}{\rho_1} - \frac{\varepsilon}{2}\right)$$

$$-\log\left|a_{m,n}^{(2)}\right| > (\lambda_m \log\lambda_m + \mu_n \log\mu_n)\left(\frac{1}{\rho_2} - \frac{\varepsilon}{2}\right)$$

$$\log\left|a_{m,n}^{(1)}\right| \log\left|a_{m,n}^{(2)}\right| > (\lambda_m \log\lambda_m + \mu_n \log\mu_n)^2\left(\frac{1}{\rho_1} - \frac{\varepsilon}{2}\right)\left(\frac{1}{\rho_2} - \frac{\varepsilon}{2}\right) \qquad (2.21)$$

Or

$$\left\{\log\left|a_{m,n}^{(1)}\right| \log\left|a_{m,n}^{(2)}\right|\right\}^{\frac{1}{2}} > (\lambda_m \log\lambda_m + \mu_n \log\mu_n)\left(\frac{1}{\rho_1\rho_2}\right)^{\frac{1}{2}}$$

Or

$$\lim_{m,n\to\infty} sup \frac{\left\{\log\left|a_{m,n}^{(1)}\right| \log\left|a_{m,n}^{(2)}\right|\right\}^{\frac{1}{2}}}{\lambda_m \log\lambda_m + \mu_n \log\mu_n} \geq \left(\frac{1}{\rho_1\rho_2}\right)^{\frac{1}{2}} \qquad (2.22)$$

Now if $\log\left|a_{m,n}\right|^{-1} \sim \left|\{\log\left|a_{m,n}^{(1)}\right|^{-1} \log\left|a_{m,n}^{(2)}\right|^{-1}\}^{\frac{1}{2}}\right|$, then we have

$$\lim_{m,n\to\infty} sup \frac{\log\left|a_{m,n}\right|^{-1}}{\lambda_m \log\lambda_m + \mu_n \log\mu_n} \geq \left(\frac{1}{\rho_1\rho_2}\right)^{\frac{1}{2}} \qquad (2.23)$$

Or

$$\frac{1}{\rho} \geq \left(\frac{1}{\rho_1\rho_2}\right)^{\frac{1}{2}}.$$

Corollary:

Let $f_k(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n}^{(k)} \exp(s_1\lambda_{k,m} + s_2\mu_{k,n})$, where k=1,2,….,pbe entire functions of non-zero finite orders $\rho_1, \ldots\ldots, \rho_p$. respectively, then the function

$f(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(s_1\lambda_m + s_2\mu_n)$, where

$\lambda_{k,m} \sim \lambda_m$,$\mu_{k,n} \sim \mu_n$,$\log\left|a_{m,n}\right|^{-1} \sim \left|\{\prod_{k=1}^{p} \log\left|a_{m,n}^{(k)}\right|^{-1}\}^{\frac{1}{p}}\right|$,is an entire function such that :

$\frac{1}{\rho} \geq \left(\frac{1}{\prod_{k=1}^{p}\rho_k}\right)^{\frac{1}{p}}$, where ρ is the order of $f(s_1,s_2)$.

Theorem 2.3:-If $f_1(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n}^{(1)} \exp(s_1\lambda_{1,m} + s_2\mu_{1,n})$,$f_2(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n}^{(2)} \exp(s_1\lambda_{2,m} + s_2\mu_{2,n})$ be entire functions of orders ρ_1 $(0 \leq \rho_1 \leq \infty)$,ρ_2 $(0 \leq \rho_2 \leq \infty)$ and types $T_1(0 \leq T_1 \leq \infty)$,$T_2(0 \leq T_2 \leq \infty)$ respectively , Then

The function f $(s_1,s_2)=\sum_{m,n=0}^{\infty} a_{m,n}$ exp$(s_1\lambda_m+s_2\mu_n)$, Where

(i)$|a_{m,n}| \sim \left|\{|a_{m,n}^{(1)}|\,|a_{m,n}^{(2)}|\}^{\frac{1}{2}}\right|$, (ii)$\lambda_{1,m} \sim \lambda_{2,m} \sim \lambda_m$, $\mu_{1,n} \sim \mu_{2,n} \sim \mu_n$. Is an entire function such that

$(\rho T)^{\frac{2}{\rho}} \leq (\rho_1 T_1)^{\frac{1}{\rho_1}} (\rho_2 T_2)^{\frac{1}{\rho_2}}$, Where ρ and Tare the order and type off (s_1,s_2)respectively and $\frac{2}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$

Proof

We prove as in the proof of theorem 2.1 that f (s_1,s_2) is an entire function when $|a_{m,n}| \sim \left|\{|a_{m,n}^{(1)}|\,|a_{m,n}^{(2)}|\}^{\frac{1}{2}}\right|$,Further using (1.5)and condition (ii) we have :

$$\mathrm{e}\rho_1 T_1 = \lim_{m,n\to\infty} sup\{\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n}\,\left|a_{m,n}^{(1)}\right|^{\rho_1}\}^{\frac{1}{\lambda_m+\mu_n}} \tag{2.24}$$

$$\mathrm{e}\rho_2 T_2 = \lim_{m,n\to\infty} sup\{\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n}\,\left|a_{m,n}^{(2)}\right|^{\rho_2}\}^{\frac{1}{\lambda_m+\mu_n}} \tag{2.25}$$

Or

$$\mathrm{e}\rho_1 T_1 = \lim_{m,n\to\infty} sup\{(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n})^{\frac{1}{\rho_1}}\left|a_{m,n}^{(1)}\right|\}^{\frac{\rho_1}{\lambda_m+\mu_n}} \tag{2.26}$$

$$\mathrm{e}\rho_2 T_2 = \lim_{m,n\to\infty} sup\{(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n})^{\frac{1}{\rho_2}}\left|a_{m,n}^{(2)}\right|\}^{\frac{\rho_2}{\lambda_m+\mu_n}} \tag{2.27}$$

From (2.26) and (2.27), we get for arbitrary ε>0, we have:

$$\{(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n})^{\frac{1}{\rho_1}}\left|a_{m,n}^{(1)}\right|\}^{\frac{1}{\lambda_m+\mu_n}} < \{\mathrm{e}\rho_1(T_1+\varepsilon)\}^{\frac{1}{\rho_1}} \tag{2.28}$$

For $m+n>k_1$

$$\{(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n})^{\frac{1}{\rho_2}}\left|a_{m,n}^{(2)}\right|\}^{\frac{1}{\lambda_m+\mu_n}} < \{\mathrm{e}\rho_2(T_2+\varepsilon)\}^{\frac{1}{\rho_2}} \tag{2.29}$$

For $m+n>k_2$

Thus for $m+n>k=\max(k_1,k_2)$, and $(\frac{2}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2})$, we have:

$$\{(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n})^{\frac{2}{\rho}}\left|a_{m,n}^{(1)}\right|\left|a_{m,n}^{(2)}\right|\}^{\frac{1}{\lambda_m+\mu_n}} < \{\mathrm{e}\rho_1(T_1+\varepsilon)\}^{\frac{1}{\rho_1}}\{\mathrm{e}\rho_2(T_2+\varepsilon)\}^{\frac{1}{\rho_2}}$$

Or

$$\{[(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n})^{\frac{1}{\rho}}(\left|a_{m,n}^{(1)}\right|\left|a_{m,n}^{(2)}\right|)^{\frac{1}{2}}]^2\}^{\frac{1}{\lambda_m+\mu_n}} < \{\mathrm{e}\rho_1(T_1+\varepsilon)\}^{\frac{1}{\rho_1}}\{\mathrm{e}\rho_2(T_2+\varepsilon)\}^{\frac{1}{\rho_2}} \tag{2.30}$$

Since$|a_{m,n}| \sim \left|\{|a_{m,n}^{(1)}|\,|a_{m,n}^{(2)}|\}^{\frac{1}{2}}\right|$, we get

$$\lim_{m,n\to\infty} sup\{(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n})^{\frac{1}{\rho}}|a_{m,n}|\}^{\frac{1}{\lambda_m+\mu_n}} < \{\mathrm{e}\rho_1(T_1+\varepsilon)\}^{\frac{1}{2\rho_1}}\{\mathrm{e}\rho_2(T_2+\varepsilon)\}^{\frac{1}{2\rho_2}}$$

Or

$$\lim_{m,n\to\infty} sup\{[(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n}\,)|a_{m,n}|^{\rho}]^{\frac{1}{\rho}}\}^{\frac{1}{\lambda_m+\mu_n}} < \{e\rho_1(T_1+\varepsilon)\}^{\frac{1}{2\rho_1}}\{e\rho_2(T_2+\varepsilon)\}^{\frac{1}{2\rho_2}}$$

Or

$$\lim_{m,n\to\infty} sup\{[(\lambda_m{}^{\lambda_m}\,\mu_n{}^{\mu_n}\,)|a_{m,n}|^{\rho}]^{\frac{1}{\lambda_m+\mu_n}}\}^{\frac{1}{\rho}} < \{e\rho_1(T_1+\varepsilon)\}^{\frac{1}{2\rho_1}}\{e\rho_2(T_2+\varepsilon)\}^{\frac{1}{2\rho_2}}$$

Or

$(\rho T)^{\frac{2}{\rho}} \leq (\rho_1 T_1)^{\frac{1}{\rho_1}}(\rho_2 T_2)^{\frac{1}{\rho_2}}$(2.31)

Where ρ and T are the order and type off (s_1, s_2) respectively , hence $(\rho T)^{\frac{2}{\rho}} \leq (\rho_1 T_1)^{\frac{1}{\rho_1}}(\rho_2 T_2)^{\frac{1}{\rho_2}}$.

Corollary:

Let $f_k(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n}^{(k)}$ exp($s_1\lambda_{k,m}$ +$s_2\mu_{k,n}$) **,** where k=1,2,….,p be entire functions of non-zero finite orders $\rho_1, \dots \dots, \rho_p$. And types $T_1 T_2,$..............T_p, then the function

f (s_1,s_2) $=\sum_{m,n=0}^{\infty} a_{m,n}$ exp($s_1\lambda_m$+$s_2\mu_n$) , where

$\lambda_{k,m} \sim \lambda_m$,$\mu_{k,n} \sim \mu_n$

$|a_{m,n}| \sim \left|(\prod_{k=1}^{p}|a_{m,n}^{(k)}|)^{\frac{1}{\rho}}\right|$, is an entire function such that:

$(\rho T)^{\frac{p}{\rho}} \leq \prod_{k=1}^{p}(\rho_k T_k)^{\frac{1}{\rho_k}}$, Where ρ and T are the order and type off (s_1,s_2)respectively.

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