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Existence of multiple solutions for a p(x)-biharmonic equation

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Abstract: The aim of this paper is to obtain at least three solutions for a Neumann problem involving the p(x)-biharmonic operator. The main tool used for obtaining our result is a three critical points theorem established by Ricceri.

Keywords: Neumann problem; p(x)-biharmonic operator; critical points.

1 Introduction

In recent years, increasing attention has been paid to the study of various mathematical problems. The interest in studying such problems was stimulated by their applications in nonlinear electrorheological fluids and elastic mechanics (such as [2, 3]). In addition, we point out that the fourth order equations can describe the satics from change of bean or the sport of tigid body, there are many authors have pointed out that type of nonlinearity furnishes a model to study traveling waves in suspension bridge(see [4, 5]). It is well known that, comparing with the p-biharmonic operator, the p(x)-biharmonic operator possesses more complicated nonlinear properties; for example, it is not homogeneous. This causes many problems, and some classical theories and methods, such as the Lagrange Multiplier Theorem, are not applicable.

In this paper, we consider the following p(x)-biharmonic problem with Neumann boundary condition,

$$(1.1)\begin{cases} \Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), & in\ \Omega, \\ \dfrac{\partial u}{\partial v} = 0, \quad \dfrac{\partial}{\partial v}(\Delta |u|^{p(x)-2}\Delta u) = 0, & on\ \partial\Omega. \end{cases}$$

where $\Omega \subset R^N (N > 1)$ is a nonempty bounded open domain with a sufficient smooth boundary $\partial\Omega$, and v is the outward unit normal to $\partial\Omega$. $\Delta^2_{p(x)}u = \Delta(\Delta|u|^{p(x)-2}\Delta u)$ is the so called p(x)-biharmonic operator of fourth order with $p \in C(\overline{\Omega}), 1 < p(x) < p_2^*(x)$, where

$$p_2^*(x) := \begin{cases} \dfrac{Np(x)}{N-2p(x)}, & if\ \ 2p(x) < N, \\ +\infty, & if\ \ 2p(x) \geq N. \end{cases}$$

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$\lambda,\mu\in[0,\infty)$, $a\in L^{\infty}(\Omega)$ such that $\inf\limits_{x\in\Omega} a(x)=a^{-}>0$.

Denote by $F(x,t)=\int_0^t f(x,s)ds, G(x,t)=\int_0^t g(x,s)ds$, $p^{-}=\inf\limits_{x\in\Omega} p(x), p^{+}=\sup\limits_{x\in\Omega} p(x)$ and $f,g:\Omega\times R\to R$ are *Carathe'odory* functions.

Currently, several variations of problem (1.1) have been studied in the literature. For instance Li, Feng and Pan [13] studied the problem

$$(1.2)\begin{cases}\Delta^2_{p(x)}u+|u|^{p(x)-2}u=\lambda f(x,u)+\mu g(x,u), & in\ \Omega,\\ u=0,\quad \Delta u=0, & on\ \partial\Omega.\end{cases}$$

In this article, using the three critical points theorem by Ricceri, the existence of at least three solutions was proved.

Abdel Rachid El Amrouse and Anass Ourraoui [11] studied the problem as follows

$$(1.3)\begin{cases}\Delta^2_{p(x)}u+a(x)|u|^{p(x)-2}u=f(x,u)+\lambda g(x,u), & in\ \Omega,\\ \dfrac{\partial u}{\partial v}=0,\quad \dfrac{\partial}{\partial v}(\Delta|u|^{p(x)-2}\Delta u)=0, & on\ \partial\Omega.\end{cases}$$

They proved that there exists $\lambda_*>0$ such that for any $\lambda\in]0,\lambda_*[$, the problem (1.3) has at least three weak solutions. Their technical approach is based on theorem obtain by B.Ricceri's variational principle and local mountain pass theorem without (PS) condition.

Motivated by these nice thoughts, we use Ricceri's three critical points theorem, which is a powful tool to study boundary problem of differential equation [10, 12, 13, 14], to study the problem (1.1). Moreover, we list an example, which meets the assumptions of the main theorem in our paper, cannot satisfy the result in [11,Theorem 1.2].

This paper consists of four sections. In section 2, we start with some preliminary basic results on theory of Lesbegue-Sobolev spaces with variables exponent ([6, 7]), we recall Ricceri's three critical points theorem [8] and prove several lemmas which are needed later. In section 3, we give the proof of the main result. In section 4, we present one example to illustrate the main result.

2 Preliminaries

In this part, we introduce some definitions and results which will be used in the next section.

Firstly, we introduce some results on the space $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, and properties of p(x)-biharmonic operator.

For any $p(x)\in C_{+}(\overline{\Omega})$, and $C_{+}(\overline{\Omega})=\{p: p\in C(\overline{\Omega}), p(x)>1 \text{ for all } x\in(\overline{\Omega})\}$.

Denote $p^{+}=\sup\limits_{x\in\overline{\Omega}} p(x)$ and $p^{-}=\inf\limits_{x\in\overline{\Omega}} p(x)$, and for any $x\in\overline{\Omega}, k\geqslant 1$,

$$p^{*}(x):=\begin{cases}\dfrac{Np(x)}{N-p(x)}, & if\ \ p(x)<N,\\ +\infty, & if\ \ p(x)\geq N.\end{cases}$$

$$p_k^*(x) := \begin{cases} \dfrac{Np(x)}{N-kp(x)}, & if \;\; kp(x) < N, \\ +\infty, & if \;\; kp(x) \geq N. \end{cases}$$

We define the variable exponent Lebesgue space

$L^{p(x)}(\Omega) = \{$ u : u is a measurable real-valued function $\int_\Omega |u|^{p(x)}\, dx < \infty$ $\}$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\, dx \leq 1\}.$$

The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable and reflexive Banach space.

Define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $D^\alpha u = \dfrac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_N}^{\alpha_N}}$ with $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$.

The space $W^{k,p(x)}(\Omega)$ with the norm $\|u\|_{k,p(x)} := \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)}$ is a separable and reflexive Banach space.

Proposition 2.1 ([1]) For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding

$$W^{k,p(x)}(\Omega) \rightarrowtail L^{r(x)}(\Omega).$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Proposition 2.2 ([1]) For any $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$, we have

$$|\int_\Omega uv dx| \leq (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)},$$

where q(x) is the conjugate function of p(x); i.e. $\dfrac{1}{p(x)} + \dfrac{1}{q(x)} = 1$.

Note that the weak solutions of (1.1) are considered in the generalized Sobolev space $X := W^{2,p(x)}(\Omega)$, equipped with the norm

$$\|u\| = inf\{\lambda > 0 : \int_\Omega (|\frac{\Delta u(x)}{\lambda}|^{p(x)} + a(x)|\frac{u(x)}{\lambda}|^{p(x)})dx \leq 1\}.$$

Remark 2.1 *By [15], the norm* $\|u\|_{2,p(x)}$ *is equivalent to the norm* $|\Delta u|_{p(x)}$ *in the space X. Consequently, it is easy to see that* $\|u\|, \|u\|_{2,p(x)}$ *and* $|\Delta u|_{p(x)}$ *are equivalent. In this paper, for the convenience of discussion, we use the norm* $\|u\|$ *for X.*

Proposition 2.3 ([11]) Set $\rho_a(u) = \int_\Omega (|\Delta u|^{p(x)} + a(x)|u|^{p(x)})dx$. For $u, u_n \in W^{2,p(x)}(\Omega)$, we have,

(1) $\|u\| < (=;>)1 \Leftrightarrow \rho_a(u) < (=;>)1$.

(2) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le \rho_a(u) \le \|u\|^{p^-}$.

(3) $\|u\| \ge 1 \Rightarrow \|u\|^{p^+} \ge \rho_a(u) \ge \|u\|^{p^-}$.

(4) $\|u_n\| \to 0 \Leftrightarrow \rho_a(u_n) \to 0$.

(5) $\|u_n\| \to +\infty \Leftrightarrow \rho_a(u_n) \to +\infty$.

Definition 2.1 *Let* $u \in X$ *, u is called a weak solution of problem (1.1) if for all* $v \in X$ *,*

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx + \int_\Omega a(x)|u|^{p(x)-2} uvdx = \lambda \int_\Omega f(x,u)vdx + \mu \int_\Omega g(x,u)vdx.$$

The energy functional corresponding to problem(1.1) is defined on X as

$$H(u) = \Phi(u) + \lambda\Psi(u) + \mu J(u).$$

where,

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + a(x)|u|^{p(x)})dx, \tag{2.1}$$

$$\Psi(u) = -\int_\Omega F(x,u)dx, \tag{2.2}$$

$$J(u) = -\int_\Omega G(x,u)dx. \tag{2.3}$$

Lemma 2.4 ([8]) Let X be a reflexive Banach space. $\Phi: X \to R$ is a continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X' and Φ is bounded on each bounded subset of X; $\Psi: X \to R$ is a continuously Gateaux differentiable functional whose Gateaux derivative is compact; I $\subseteq$ R an interval.

Assume that

$$\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty \tag{2.4}$$

for all $\lambda \in I$,and that there exists $h \in R$ such that

$$\sup_{\lambda\in I}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)+h))<\inf_{x\in X}\sup_{\lambda\in I}(\Phi(x)+\lambda(\Psi(x)+h)) \tag{2.5}$$

Then, there exists an open interval $\Lambda\subseteq I$ and a positive real number ρ with the following property: for every $\lambda\in\Lambda$ and every C^1 functional J: $X\mapsto R$ with compact derivative, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$ the equation

$$\Phi'(x)+\lambda\Psi'(x)+\mu J'(x)=0$$

has at least three solutions in X whose norms are less than ρ.

Proposition 2.5 ([9]) Let X be a non-empty set and Φ,Ψ two real functions on X. Assume that there are $r>0$ and $x_0,x_1\in X$ such that

$$\Phi(x_0)=-\Psi(x_0)=0,\quad \Phi(x_1)>r,\quad \sup_{x\in\Phi^{-1}(]-\infty,r])}-\Psi(x)<r\frac{-\Psi(x_1)}{\Phi(x_1)}$$

Then, for each h satisfying

$$\sup_{x\in\Phi^{-1}(]-\infty,r])}-\Psi(x)<h<r\frac{-\Psi(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)+h))<\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\Psi(x)+h)).$$

Lemma 2.6 *For any* $p(x)\in C_+(\overline{\Omega})$ *, we have*

(1) The functional $\Phi:X\to R$ $(X=W^{2,p(x)})$ is sequentially weakly lower semi continuous, $\Phi\in C^1(X,R)$.

(2) $(\Phi')^{-1}:X^*\to X$ exists and it is continuous.

Proof. (1) It is clear that Φ is well defined and $\Phi\in C^1(X,R)$. By the continuity and convexity of Φ, we deduce that Φ is sequentially weakly lower semi continuous.

(2) We only need to prove that Φ' is coercive, hemicontinuous, uniformly monotone.

Firstly, for any $u\in X$ with $\|u\|>1$, we have, $\dfrac{\langle\Phi'(u),u\rangle}{\|u\|}=\dfrac{\rho_a(u)}{\|u\|}\geq\|u\|^{p^- -1}$ and thus $\lim\limits_{\|u\|\to\infty}\dfrac{\langle\Phi'(u),u\rangle}{\|u\|}=\infty$, so Φ' is coercive.

Secondly, Φ' is hemicontinuous can be verified using standard arguments.

Thirdly, we have,

$$\begin{aligned}\langle \Phi'(u)-\Phi'(v),u-v\rangle &= \int_\Omega [(|\Delta u|^{p(x)-2}\Delta u-|\Delta v|^{p(x)-2}\Delta v)(\Delta u-\Delta v)\\ &\quad + a(x)(|u|^{p(x)-2}u-|v|^{p(x)-2}v)(u-v)]dx\\ &\geq \frac{1}{2^{p(x)}}\int_\Omega [|\Delta(u-v)|^{p(x)}+a(x)|u-v|^{p(x)}]dx\\ &\geq \frac{1}{2^{p+}}\int_\Omega [|\Delta(u-v)|^{p(x)}+a(x)|u-v|^{p(x)}]dx\\ &= \frac{1}{2^{p+}}\rho_a(u-v),\forall u,v\in X\end{aligned}$$

Next, we define the function $y:[0,\infty)\to[0,\infty)$ by

$$y(t):=\begin{cases}\frac{1}{2^{p^+}}\cdot t^{p^+-1}, & if\ t\leq 1,\\ \frac{1}{2^{p^+}}\cdot t^{p^--1}, & if\ t\geq 1.\end{cases}$$

It is easy to find that y is an increasing function with $y(0)=0$ and $\lim_{t\to\infty} y(t)=\infty$.

$$\langle \Phi'(u)-\Phi'(v),u-v\rangle \geq y(\|u-v\|)\cdot\|u-v\| \quad \forall u,v\in X.$$

So, Φ' is uniformly monotone.

Consequently, we conclude that $(\Phi')^{-1}$ exists and it is continuous.

Lemma 2.7 ([1]) If $f:\Omega\times R\to R$ is a $Carathe'odory$ function and $|f(x,s)|\leq a(x)+b|s|^{\frac{p_1(x)}{p_2(x)}},\forall x\in\overline{\Omega}$, $s\in R$, where $p_1(x),p_2(x)\in C_+(\overline{\Omega}),a(x)\in L^{p_2(x)},a(x)\geq 0$ and $b\geq 0$ is a constant, then the Nemytski operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$, defined by $(N_f(u))=f(x,u(x))$ is a continuous and bounded operator.

Lemma 2.8 *If* $f:\Omega\times R\to R$ *is a* $Carathe'odory$ function and $\sup_{(x,s)\in\Omega\times R}\frac{|f(x,s)|}{1+|s|^{t(x)-1}}<+\infty$, where $t(x)\in C_+(\overline{\Omega})$, $t(x)<p_2^*(x)$, Set $F(x,u)=\int_0^u f(x,t)dt$, $X=W^{2,p(x)}(\Omega)$, $\Psi(u)=-\int_\Omega F(x,u(x))dx$, then

$$\Psi(u)\in C^1(X,R),\ D\Psi(u,\varphi)=(\Psi'(u),\varphi)=-\int_\Omega f(x,u(x))\varphi dx.$$

And, the operator $\Psi':X\to X^*$ is compact.

Proof. From $\sup_{(x,s)\in\Omega\times R}\frac{|f(x,s)|}{1+|s|^{t(x)-1}}<+\infty$, we have $\exists C_1>0,C_2>0$, such that

$$|f(x,s)|\leq C_1+C_2|s|^{t(x)-1},\ \forall(x,s)\in\overline{\Omega}\times R \tag{2.6}$$

Also from the Mean-Value Theorem, we have (where $0\leq\theta(x,u(x),t\varphi(x))\leq 1$)

$$D\Psi(u,\varphi)=\lim_{t\to 0}\frac{\Psi(u+t\varphi)-\Psi(u)}{t}$$

$$=-\lim_{t\to 0}\int_\Omega \frac{F(x,u(x)+t\varphi(x))-F(x,u(x))}{t}dx$$

$$=-\lim_{t\to 0}\int_\Omega f(x,u(x)+t\theta\varphi(x))\varphi(x)dx$$

And using the inequality

$$(x+y)^p\le 2^{p-1}\cdot(|x|^p+|y|^p),\ p\ge 1 \tag{2.7}$$

By (2.6), (2.7) and Young's inequality, we obtain

$$|f(x,u(x)+t\theta\varphi(x)\varphi(x)|\le[C_1+C_2|u(x)+t\theta\varphi(x)|^{t(x)-1}]\cdot|\varphi(x)|$$

$$\le\frac{t(x)-1}{t(x)}[C_1+C_2|u(x)+t\theta\varphi(x)|^{t(x)-1}]^{\frac{t(x)}{t(x)-1}}+\frac{1}{t(x)}|\varphi(x)|^{t(x)}$$

$$\le\frac{t(x)-1}{t(x)}\cdot 2^{\frac{1}{t(x)-1}}\{C_1^{\frac{t(x)}{t(x)-1}}+2^{t(x)-1}\cdot C_2^{\frac{t(x)}{t(x)-1}}[|u(x)|^{t(x)}+|\varphi(x)|^{t(x)}]\}$$

$$+\frac{1}{t(x)}|\varphi(x)|^{t(x)}\quad for|t|\le 1.$$

From Proposition 2.1, we have

$$|v|_{t(x)}\le C\|v\|_X,\ \forall v\in X \tag{2.8}$$

So, by the Lebesgue dominated convergence theorem and the continuity of f, we get

$$D\Psi(u,\varphi)=-\int_\Omega\lim_{t\to 0}f(x,u(x)+t\theta\varphi(x))\varphi(x)dx=-\int_\Omega f(x,u(x))\varphi(x)dx \tag{2.9}$$

From Lemma 2.7, we know that the Nemytski operator N: $u(x)\to f(x,u(x))$ is a continuous bounded operator from $L^{t(x)}$ to $L^{\frac{t(x)}{t(x)-1}}$, also in view of (2.8) and (2.9), we get

$$|D\Psi(u,\varphi)|=|\int_\Omega f(x,u(x))\varphi(x)dx|$$

$$\le 2|f(x,u(x))|_{\frac{t(x)}{t(x)-1}}|\varphi(x)|_{t(x)}$$

$$\le 2C|f(x,u(x))|_{\frac{t(x)}{t(x)-1}}\|\phi(x)\|_X,$$

which shows that $D\Psi(u,\varphi)$ is a linear bounded functional. Therefore the Gateaux derivative of the linear bounded functional $\Psi(u)$ exists and

$$D\Psi(u,\varphi)=(D\Psi(u),\varphi)=-\int_\Omega f(x,u(x))\varphi(x)dx,\ \forall u,\varphi\in X. \tag{2.10}$$

In the following, we need to prove that $\Psi^{'}:X\to X^*$ is completely continuous. For any $u,v,\varphi\in X$, from (2.8) and (2.10), we get

$$|(D\Psi(u)-D\Psi(v),\varphi)|=|\int_\Omega [f(x,v(x))-f(x,u(x))]\varphi(x)dx|$$

$$\leq 2|f(x,u(x))-f(x,v(x))|_{\frac{t(x)}{t(x)-1}}|\varphi(x)|_{t(x)}$$

$$\leq 2C|f(x,u(x))-f(x,v(x))|_{\frac{t(x)}{t(x)-1}}\|\varphi(x)\|_X$$

which implies that $\|D\Psi(u)-D\Psi(v)\|_{X^*}\leq 2C|f(x,u(x))-f(x,v(x))|_{\frac{t(x)}{t(x)-1}}$

So, the operator $Y:L^{\frac{t(x)}{t(x)-1}}\to X^*$ defined by $Y(f(x,u(x)))=D\Psi(u(x))$ is continuous. Moreover, the identity operator I from X to $L^{t(x)}$ is continuous, so the composite operator $D\Psi=Y\circ N\circ I:u\to D\Psi(u)$ from X to X^* is continuous. Therefore, this shows that $\Psi(u)\in C^1(X,R)$, and $D\Psi(u,\varphi)=(\Psi'(u),\varphi)=-\int_\Omega f(x,u(x))\varphi dx$. What's more, the identity operator I is compact, so the operator $\Psi':X\to X^*$ is compact.

3 Main result

We need the following assumptions

(f_0) $\sup_{(x,s\in\Omega\times R)}\frac{|f(x,s)|}{1+|s|^{t(x)-1}}<+\infty$, where $t(x)\in C_+(\overline{\Omega})$, $t(x)<p_2^*(x)$,

(f_1) $\exists\alpha>0$, such that $F(x,s)>0$ for a.e. $x\in\Omega$ and all $s\in[0,\alpha]$,

(f_2) $\exists h>0$ and a function $q(x)\in C(\overline{\Omega})$, $1<q^-\leq q^+<p^-$, such that,

$$|F(x,s)|\leq h(1+|s|^{q(x)}) \text{ for a.e. } x\in\Omega \text{ and all } s\in R,$$

(f_3) $\exists r(x)\in C(\overline{\Omega})$, and $p^+<r^-<r(x)<p_2^*(x)$, such that

$$\lim_{s\to 0}\sup_{x\in\Omega}\frac{F(x,s)}{|s|^{r(x)}}<+\infty.$$

Theorem 3.1 *Assume that (f_0), (f_1), (f_2) and (f_3) hold, then, there exist an open interval $\Lambda\subseteq(0,+\infty)$ and $l>0$ with the following property: for each $\lambda\in\Lambda$ and each function*

$g(x,s):\Omega\times R\to R$ satisfying $\sup_{(x,s\in\Omega\times R)}\frac{|g(x,s)|}{1+|s|^{p_2(x)-1}}<+\infty$ $(p_2(x)\in C_+(\overline{\Omega}), p_2(x)<p_2^*(x))$.

There exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, problem (1.1) has at least three weak solutions whose norms in X are less than l.

Proof. There, $\Phi(u)$, $\Psi(u)$ and $J(u)$ as (2.1) (2.2) (2.3). So, for each $u,v\in X$, one has

$$\langle \Phi'(u), v \rangle = \int_\Omega (|\Delta u|^{p(x)-2} \Delta u \Delta v + |u|^{p(x)-2} uv) dx,$$

$$\langle \Psi'(u), v \rangle = -\int_\Omega f(x,u) v dx,$$

$$\langle J'(u), v \rangle = -\int_\Omega g(x,u) v dx.$$

From Lemma 2.6, it is easy to see that Φ is a continuous Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X', and under our assumptions Φ is bounded on each bounded subset of X. From Lemma 2.8, Ψ and J are continuously Gateaux differentiable functionals whose Gateaux derivative is compact.

From Proposition 2.3, we can conclude: if $\|u\| \geq 1$, then

$$\frac{1}{p^+}\|u\|^{p^-} \leq \frac{1}{p^+}\int_\Omega (|\Delta u|^{p(x)} + a(x)|u|^{p(x)}) dx \leq \Phi(u)$$

$$\leq \frac{1}{p^-}\int_\Omega (|\Delta u|^{p(x)} + a(x)|u|^{p(x)}) dx \leq \frac{1}{p^-}\|u\|^{p^+} \tag{3.1}$$

Moreover, using condition (f_2), we can obtain, for each $\lambda \in \Lambda$,

$$\begin{aligned}\lambda\Psi(u) &= -\lambda\int_\Omega F(x,u) dx \\ &\geq -\lambda\int_\Omega h(1+|u|^{q(x)}) dx \\ &= -\lambda h(|\Omega| + |u|_{q(x)}^{q(x)}) \\ &\geq -\lambda h[|\Omega| + (C\|u\|)^{q(x)}] \\ &\geq -\lambda h[|\Omega| + C^{q(x)}\|u\|^{q(x)}].\end{aligned}$$

From $\|u\| \geq 1$, we get $\|u\|^{q(x)} \geq \|u\|^{q^+}$, also we have $C^{q(x)} \geq max\{C^{q^+}, C^{q^-}\}$.

So

$$\lambda\Psi(u) \geq -\lambda h[|\Omega| + max\{C^{q^+}, C^{q^-}\} \cdot \|u\|^{q^+}]$$

$$\geq -C_1(1+\|u\|^{q^+}). \tag{3.2}$$

for any $u \in X$, where C_1 is positive constant. Combining (3.1) and (3.2), we obtain

$$\Phi(u)+\lambda\Psi(u)\geq\frac{1}{p^{+}}\|u\|^{p^{-}}-C_{1}(1+\|u\|^{q^{+}}).$$

Since $q^{+}<p^{-}$, we have

$$\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u))=+\infty\ \ \forall u\in X\ ,\ \lambda\in[0,+\infty).$$

So, assumption (2.4) in Lemma 2.4 is satisfied.

Next,we will prove that assumption (2.5) in Lemma 2.4 is also satified. Let $u_0=0$, we can easily have

$$\Phi(u_0)=-\Psi(u_0)=0.$$

We'll apply proposition 2.5 to verify (2.5).

From (f_3) there exist $\rho\in[0,1]$, $C_2>0$, such that

$$F(x,s)<C_2|s|^{r(x)}<C_2|s|^{r^{-}}\quad \forall s\in[-\rho,\rho],\text{ a.e. } x\in\Omega.$$

Then, from (f_2), we can find a positive constant M such that

$$F(x,s)<M|s|^{r^{-}}$$

for all $s\in R$ and a.e. $x\in\Omega$. Consequently, by the Sobolev embedding theorem ($X\rightarrowtail L^{r^{-}}(\Omega)$ is continuous), we have

$$-\Psi(u)=\int_{\Omega}F(x,u)dx<M\int_{\Omega}|u|^{r(x)}\,dx\leq C_3\|u\|^{r^{-}}\leq C_4 l^{r^{-}/p^{+}},$$

when $\frac{\|u\|^{p^{+}}}{p^{+}}\leq l$. It follows from $r>p^{+}$ that

$$\lim_{l\to0^{+}}\frac{\sup_{\|u\|^{p^{+}}/p^{+}\leq l}-\Psi(u)}{l}=0. \tag{3.3}$$

Let $u_1\in C^2(\Omega)$ be a function positive in Ω, with $u_1|_{\partial\Omega}=0$ and $\max_{\Omega}u_1\leq\alpha$. Consequently, we have, $u_1\in X$ and $\Phi(u_1)>0$. Under the assumption (f_1), we also have

$$-\Psi(u_1)=\int_{\Omega}F(x,u_1(x))dx>0.$$

Therefore, from (3.3), we can find $l\in(0,min\{\Phi(u_1),\frac{1}{p^{+}}\})$ such that

$$\sup_{\|u\|^{p^{+}}/p^{+}\leq l}(-\Psi(u))<l\frac{-\Psi(u_1)}{\Phi(u_1)}.$$

Now, we'll show $\Phi^{-1}((-\infty,l])\subset\{u\in X:\frac{1}{p^{+}}\|u\|^{p^{+}}\leq l\}$. Let $u\in\Phi^{-1}((-\infty,l])$.

Then, $\rho_a(u) = \int_\Omega (|\Delta u|^{p(x)} + a(x)|u|^{p(x)})dx \le lp^+ < 1$. From Proposition 2.3, $\|u\| < 1$. Consequently,

$$\frac{1}{p^+}\|u\|^{p^+} \le \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + a(x)|u|^{p(x)})dx \le l .$$

Therefore,

$$\sup_{u\in\Phi^{-1}(]-\infty,l])} -\Psi(u) < l\frac{-\Psi(u_1)}{\Phi(u_1)}.$$

From Proposition 2.5, for each h satisfying $\sup_{x\in\Phi^{-1}(]-\infty,r])} -\Psi(x) < h < r\frac{-\Psi(x_1)}{\Phi(x_1)}$, (2.5) holds. By Lemma 2.4, there exists an open interval $\Lambda \subseteq I$ and a positive real number ρ with the following property: for every $\lambda \in \Lambda$ and every C^1 functional J: $X \mapsto R$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$ the equation

$$\Phi'(x) + \lambda\Psi'(x) + \mu J'(x) = 0$$

has at least three solutions in X whose norms are less than ρ. That is, the problem (1.1) has at least three weak solutions whose norms in X are less than ρ.

4 Example

Let

$$f(x,s) = |s|^{p(x)-2}s - s,\ a(x) = 1,$$

where $p(x) \in C(\overline{\Omega})$, $1 < p(x) < min\{2, p_2^*(x)\}$, we can easily verify that $f(x,u)$ satisfies condition (f_0), (f_1), (f_2) and (f_3), and then the conclusion of Theorem 3.1 holds true. But we cannot use the main result in [11,Theorem 1.2] to solve this problem, since the function $f(x,u)$ does not satisfy condition (F_2') in [11].

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