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Local Property of a Factored Fourier Series using Absolute Norlund Indexed Summability

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Abstract

In this paper we have established a theorem on the local property of absolute Norlund indexed-summability of Factored Fourier series.

Keywords: $|N, p_n|_k$ - summability; $|N, p_n, \alpha_n|_k$ - summability; $|\overline{N}, p_n; \alpha_n, \delta|_k$ - summability and Fourier series.

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\{p_n\}$be a sequence of positive real constants such that

(1.1) $$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \; as \; n \to \infty \; (P_{-i} = p_{-i} = 0, i \geq 1)$$

The sequence-to-sequence transformation

(1.2) $$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu} s_\nu$$

defines (N, p_n) -mean of the sequence $\{s_n\}$generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if

(1.3) $$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty .$$

For k = 1, $|N, p_n|_k$ - summability is same as $|N, p_n|$ - summability.

When $p_n = 1$, for all n and $k = 1$, $|N, p_n|_k$ - summability is same as |C, 1| summability.

Let $\{\alpha_n\}$ be any sequence of positive numbers. The series $\sum a_n$ is said to be summable $|N, p_n, \alpha_n|_k$, $k \geq 1$, if

(1.4)
$$\sum_{n=1}^{\infty} \alpha_n^{k-1} |t_n - t_{n-1}|^k < \infty,$$

Where $\{t_n\}$ is as defined in (5.1.2).The series $\sum a_n$ is said to be summable $|N, p_n, \alpha_n; \delta|_k, k \geq 1, \delta \geq 0$, if

(1.5)
$$\sum_{n=1}^{\infty} \alpha_n^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty.$$

For $\delta = 0$, the summability method $|N, p_n, \alpha_n; \delta|_k$, $k \geq 1, \delta \geq 0$, reduces to the summabilty method $|N, p_n, \alpha_n|_k$, $k \geq 1$.

For any real number γ, the series $\sum a_n$ is said to be summable by the summabilty method $|N, p_n, \alpha_n; \delta, \gamma|_k, k \geq 1, \delta \geq 0$, if

(1.6)
$$\sum_{n=1}^{\infty} \alpha_n^{\gamma(\delta k + k - 1)} |t_n - t_{n-1}|^k < \infty .$$

For $\gamma = 1$, the summability method $|N, p_n, \alpha_n; \delta, \gamma|_k, k \geq 1, \delta \geq 0$, any real γ, reduces to the method $|N, p_n, \alpha_n; \delta|_k, k \geq 1, \delta \geq 0$.

A sequence $\{\lambda_n\}$ is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer n.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

(1.7)
$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t)$$

It is well known that the convergence of Fourier series at t = x is a local property of $f(t)$ (i.e., it depends only on the behavior of $f(t)$ in an arbitrarily small neighborhood of x) and hence the summability of the Fourier series at t = x by any regular linear method is also a local property of $f(t)$.

2. Known theorems

Dealing with the $|\overline{N}, p_n|_k$ - summability of an infinite series Bor [1] proved the following theorem:

Theorem-2.1

Let $k \geq 1$ and let the sequences $\{p_n\}$ and $\{\lambda_n\}$ be such that

(2.1.1)
$$\Delta X_n = O\left(\frac{1}{n}\right),$$

(2.1.2) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k + |\lambda_{n+1}|^k}{n} < \infty,$$

and

(2.1.3) $$\sum_{n=1}^{\infty} (X_n^k + 1)|\Delta\lambda_n| < \infty,$$

where $X_n = \dfrac{P_n}{np_n}$. Then the summability $\left|\overline{N}, p_n\right|_k$ of the factored Fourier series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property.

Subsequently, Misra et. al. [2] proved the following theorem on the local property of $\left|N, p_n, \alpha_n\right|_k$ summability of factored Fourier series:

Theorem-2.2

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $\{p_n\}$ be a sequence such that

(2.2.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(2.2.2) $$\frac{P_{n-r-1}}{P_n} = O\left(\frac{p_{n-r-1}}{P_{n-1}} \frac{P_r}{p_r}\right),$$

(2.2.3) $$\sum_{n=r+1}^{m+1} (\alpha_n)^{k-1} \frac{p_{n-r}}{P_n} = O\left(\frac{p_r}{P_r}\right),$$

(2.2.4) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k}{n} < \infty,$$

and

(2.2.5) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\Delta\lambda_n|^k}{n} < \infty,$$

Where $X_n = \dfrac{P_n}{np_n}$. Then the summability $\left|N, p_n, \alpha_n\right|_k$, $k \geq 1$ of the factored Fourier series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property, where $\{\alpha_n\}$ is a sequence of positive numbers.

Rrecently, Paikray et al. [3] proved the following theorem on the local property of $\left|N, p_n, \alpha_n, \delta\right|_k$ summability of factored Fourier series:

Theorem-2.3

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $\{p_n\}$ be a sequence such that

(2.3.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(2.3.2) $$\frac{P_{n-r-1}}{P_n} = O\left(\frac{p_{n-r-1}}{P_{n-1}}\frac{P_r}{p_r}\right),$$

(2.3.3) $$\sum_{n=r+1}^{m+1}(\alpha_n)^{\delta k+k-1}\frac{p_{n-r}}{P_n} = O\left(\frac{p_r}{P_r}\right),$$

(2.3.4) $$\sum_{n=1}^{\infty} X_n^{k-1}\frac{|\lambda_n|^k}{n} < \infty ,$$

and

(2.3.5) $$\sum_{n=1}^{\infty} X_n^{k-1}\frac{|\Delta\lambda_n|^k}{n} < \infty,$$

where $X_n = \frac{P_n}{np_n}$. Then the summability $|N, p_n, \alpha_n, \delta|_k$, $k \geq 1$ of the factored Fourier series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property, where $\{\alpha_n\}$ is a sequence of positive numbers.

In the present paper we have established the following theorem on $|N, p_n, \alpha_n, \delta, \gamma|_k$ - summabilty of a factored Fourier series through its local property.

3. Main theorem

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $\{p_n\}$ be a sequence such that

(3.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(3.2) $$\frac{P_{n-r-1}}{P_n} = O\left(\frac{p_{n-r-1}}{P_{n-1}}\frac{P_r}{p_r}\right)$$

(3.3) $$\sum_{n=r+1}^{m+1}(\alpha_n)^{\gamma(\delta k+k-1)}\frac{p_{n-r}}{P_n} = O\left(\frac{p_r}{P_r}\right) ,$$

(3.4) $$\sum_{n=1}^{\infty} X_n^{k-1}\frac{|\lambda_n|^k}{n} < \infty ,$$

and

(3.5) $$\sum_{n=1}^{\infty} X_n^{k-1}\frac{|\Delta\lambda_n|^k}{n} < \infty,$$

where $X_n = \frac{P_n}{np_n}$. Then the summability $\left|N, p_n, \alpha_n, \delta, \gamma\right|_k$, $k \geq 1$ of the factored Fourier series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property, where $\{\alpha_n\}$ is a sequence of positive numbers.

4. Required lemma

In order to prove the above theorem we require the following lemma:

Lemma

Let $k \geq 1$ and suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $\{p_n\}$ be a sequence such that the conditions (3.1)-(3.5) are satisfied. If $\{s_n\}$ is bounded, then for the sequence of positive numbers $\{\alpha_n\}$ the series $\sum_{n=1}^{\infty} a_n \lambda_n X_n$ is summable $\left|N, p_n, \alpha_n, \delta\right|_k^{\gamma}$, $k \geq 1, \delta \geq 0$.

Proof of the lemma

Let $\{T_n\}$ denote the (N, p_n) - mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n X_n$. Then by definition we have

$$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} \sum_{r=0}^{\nu} a_r \lambda_r X_r$$

$$= \frac{1}{P_n} \sum_{r=0}^{n} a_r \lambda_r X_r \sum_{\nu=r}^{n} p_{n-\nu}$$

$$= \frac{1}{P_n} \sum_{r=0}^{n} a_r P_{n-r} \lambda_r X_r$$

Hence

$$T_n - T_{n-1} = \frac{1}{P_n} \sum_{r=1}^{n} P_{n-r} a_r \lambda_r X_r - \frac{1}{P_{n-1}} \sum_{r=1}^{n-1} P_{n-r-1} a_r \lambda_r X_r$$

$$= \sum_{r=1}^{n} \left(\frac{P_{n-r}}{P_n} - \frac{P_{n-r-1}}{P_{n-1}} \right) a_r \lambda_r X_r$$

$$= \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n} \left(P_{n-r} P_{n-1} - P_{n-r-1} P_n\right) a_r \lambda_r X_r$$

$$= \frac{1}{P_n P_{n-1}} \left[\sum_{r=1}^{n-1} \Delta\left\{\left(P_{n-r} P_{n-1} - P_{n-r-1} P_n\right) \lambda_r X_r \right\} \right] \sum_{\nu=1}^{r} a_\nu$$

$$= \frac{1}{P_n P_{n-1}} \left[\sum_{r=1}^{n-1} \left(p_{n-r} P_{n-1} - p_{n-r-1} P_n\right) \lambda_r X_r s_r \right.$$

$$+ \sum_{r=1}^{n-1} \left(P_{n-r-1} P_{n-1} - P_{n-r-2} P_n\right) \Delta\lambda_r X_r s_r$$

$$\left. + \sum_{r=1}^{n-1} \left(P_{n-r-1} P_{n-1} - P_{n-r-2} P_n\right) \lambda_{r+1} \Delta X_r s_r \right]$$

(by Abel's transformation)

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} + T_{n,5} + T_{n,6}\ , \quad \text{(say).}$$

In order to complete the proof of the theorem by using Minokowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \alpha_n^{\gamma(\delta k+k-1)} \left| T_{n,i} \right|^k < \infty \quad for\ i = 1,2,3,4,5,6.$$

Now, we have

$$\sum_{n=2}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \left| T_{n,1} \right|^k = \sum_{n=2}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \left| \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n-1} p_{n-r} P_{n-1} \lambda_r X_r s_r \right|^k$$

$$\leq \sum_{n=2}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \frac{1}{P_n} \left(\sum_{r=1}^{n-1} p_{n-r} \left| \lambda_r \right|^k \left| s_r \right|^k X_r^k \right) \left(\frac{1}{P_n} \sum_{r=1}^{n-1} p_{n-r} \right)^{k-1}$$

$$= O(1) \sum_{r=1}^{m} \left| \lambda_r \right|^k X_r^k \sum_{n=r+1}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \left(\frac{p_{n-r}}{P_n} \right)$$

$$= O(1) \sum_{r=1}^{m} \left| \lambda_r \right|^k X_r^k \frac{p_r}{P_r} \quad \text{, by (3.3)}$$

$$= O(1) \sum_{r=1}^{m} \left| \lambda_r \right|^k X_r^{k-1} \frac{p_r}{P_r} \frac{P_r}{r p_r}, \text{ as } X_n = \frac{P_n}{n p_n}$$

$$= O(1) \sum_{r=1}^{m} X_r^{k-1} \frac{\left| \lambda_r \right|^k}{r}$$

$$= O(1) \quad as\ m \to \infty \text{, by (5.3.4).}$$

Next, $$\sum_{n=2}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \left| T_{n,2} \right|^k = \sum_{n=2}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \left| \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n-1} p_{n-r-1} P_n \lambda_r X_r s_r \right|^k$$

$$\leq \sum_{n=2}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \frac{1}{P_{n-1}} \left(\sum_{r=1}^{n-1} p_{n-r-1} \left| \lambda_r \right|^k \left| s_r \right|^k X_r^k \right) \left(\frac{1}{P_{n-1}} \sum_{r=1}^{n-1} p_{n-r-1} \right)^{k-1}$$

$$= O(1) \sum_{r=1}^{m} \left| \lambda_r \right|^k X_r^k \sum_{n=r+1}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \left(\frac{p_{n-r-1}}{P_{n-1}} \right)$$

$$= O(1) \sum_{r=1}^{m} \left| \lambda_r \right|^k X_r^k \frac{p_r}{P_r} \quad \text{, by(5.3.3)}$$

$$= O(1) \sum_{r=1}^{m} \left| \lambda_r \right|^k X_r^{k-1} \frac{p_r}{P_r} \frac{P_r}{r p_r}, \text{ as } X_n = \frac{P_n}{n p_n}$$

$$= O(1)\sum_{r=1}^{m} X_r^{k-1}\frac{|\lambda_r|^k}{r}$$

$$= O(1) \quad as\ m \to \infty, \text{ by (3.4).}$$

Further,

$$\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}|T_{n,3}|^k = \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\frac{1}{P_nP_{n-1}}\sum_{r=1}^{n-1}P_{n-r-1}P_{n-1}\,\Delta\lambda_r X_r s_r\right|^k$$

$$\leq \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\frac{1}{P_n}\left(\sum_{r=1}^{n-1}P_{n-r-1}|\Delta\lambda_r|^k|s_r|^k X_r^k\right)\left(\frac{1}{P_n}\sum_{r=1}^{n-1}P_{n-r-1}|\Delta\lambda_r|\right)^{k-1}$$

$$= O(1)\sum_{r=1}^{m}|\Delta\lambda_r|^k X_r^k\sum_{n=r+1}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left(\frac{P_{n-r-1}}{P_n}\right)$$

$$\left(Since\ \frac{1}{P_n}\sum_{r=1}^{n-1}P_{n-r-1}|\Delta\lambda_r| \leq \sum_{r=1}^{n-1}|\Delta\lambda_r| = O(1)\right)$$

$$= O(1)\sum_{r=1}^{m}|\Delta\lambda_r|^k X_r^k\frac{p_r}{P_r} \text{ , by (3.3)}$$

$$= O(1)\sum_{r=1}^{m}|\Delta\lambda_r|^k X_r^{k-1}\frac{p_r}{P_r}\frac{P_r}{rp_r}, \text{ as } X_n = \frac{P_n}{np_n}$$

$$= O(1)\sum_{r=1}^{m} X_r^{k-1}\frac{|\Delta\lambda_r|^k}{r}$$

$$= O(1) \quad as\ m \to \infty, \text{ by (3.5).}$$

Now,

$$\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}|T_{n,4}|^k = \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\frac{1}{P_nP_{n-1}}\sum_{r=1}^{n-1}P_{n-r-2}P_n\,\Delta\lambda_r X_r s_r\right|^k$$

$$\leq \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\frac{1}{P_{n-1}}\left(\sum_{r=1}^{n-1}P_{n-r-2}|\Delta\lambda_r|^k|s_r|^k X_r^k\right)\left(\frac{1}{P_{n-1}}\sum_{r=1}^{n-1}P_{n-r-2}|\Delta\lambda_r|\right)^{k-1}$$

$$= O(1)\sum_{r=1}^{m}|\Delta\lambda_r|^k X_r^k\sum_{n=r+1}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left(\frac{P_{n-r-2}}{P_{n-1}}\right) \text{ (as above)}$$

$$= O(1)\sum_{r=1}^{m}|\Delta\lambda_r|^k X_r^k\frac{p_r}{P_r} \text{ , by (5.3.3)}$$

$$= O(1)\sum_{r=1}^{m}|\Delta\lambda_r|^k X_r^{k-1}\frac{p_r}{P_r}\frac{P_r}{rp_r}, \text{ as } X_n = \frac{P_n}{np_n}$$

$$= O(1)\sum_{r=1}^{m} X_r^{k-1}\frac{|\Delta\lambda_r|^k}{r}$$

$$= O(1) \quad as\ m \to \infty, \text{ by (3.5).}$$

Again

$$\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}|T_{n,5}|^k = \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\frac{1}{P_nP_{n-1}}\sum_{r=1}^{n-1}P_{n-r-1}P_{n-1}\lambda_{r+1}\Delta X_r s_r\right|^k$$

$$= \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\sum_{r=1}^{n-1}\frac{P_{n-r-1}}{P_n}\lambda_{r+1}\Delta X_r s_r\right|^k$$

$$= \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-1}}{P_{n-1}}\frac{P_r}{p_r}\lambda_{r+1}\Delta X_r s_r\right|^k \quad \text{by (.3.2)}$$

$$= \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-1}}{P_{n-1}}\frac{P_r}{p_r}\lambda_{r+1}s_r\frac{1}{r}\right|^k \quad \text{by (3.1)}$$

$$= \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-1}}{P_{n-1}}\frac{P_r}{p_r}\lambda_{r+1}s_r X_r\frac{p_r}{P_r}\right|^k \quad \text{, as } X_n = \frac{P_n}{np_n}$$

$$= \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left\{\sum_{r=1}^{n-1}\frac{p_{n-r-1}}{P_{n-1}}|\lambda_{r+1}|^k|s_r|^k X_r^k\right\}\left\{\sum_{r=1}^{n-1}\frac{p_{n-r-1}}{P_{n-1}}\right\}^{k-1}$$

$$= O(1)\sum_{r=1}^{m}|\lambda_{r+1}|^k X_r^k\sum_{n=r+1}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left(\frac{p_{n-r-1}}{P_{n-1}}\right)$$

$$= O(1)\sum_{r=1}^{m}|\lambda_{r+1}|^k X_r^{k-1}\frac{p_r}{P_r}\frac{P_r}{rp_r}, \text{ as } X_n = \frac{P_n}{np_n} \text{ and by (3.3)}$$

$$= O(1)\sum_{r=1}^{m}\frac{|\lambda_{r+1}|^k}{r}X_r^{k-1},$$

$$= O(1) \quad as\ m \to \infty, \text{ by (3.4).}$$

Finally,

$$\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}|T_{n,6}|^k = \sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\frac{1}{P_nP_{n-1}}\sum_{r=1}^{n-1}P_{n-r-2}P_n\lambda_{r+1}\Delta X_r s_r\right|^k$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\sum_{r=1}^{n-1}\frac{P_{n-r-2}}{P_{n-1}}\lambda_{r+1}\Delta X_r s_r\right|^k$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\frac{P_r}{p_r}\lambda_{r+1}\Delta X_r s_r\right|^k \quad \text{, by (3.2)}$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\frac{P_r}{p_r}\lambda_{r+1}s_r\frac{1}{r}\right|^k \quad \text{, by (3.1)}$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\frac{P_r}{p_r}\lambda_{r+1}s_r X_r\frac{p_r}{P_r}\right|^k \quad \text{, as } X_n=\frac{P_n}{np_n}$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left\{\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}|\lambda_{r+1}|^k|s_r|^k X_r^k\right\}\left\{\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\right\}^{k-1}$$

$$=O(1)\sum_{r=1}^{m}|\lambda_{r+1}|^k X_r^k\sum_{n=r+1}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left(\frac{p_{n-r-2}}{P_{n-2}}\right)$$

$$=O(1)\sum_{r=1}^{m}|\lambda_{r+1}|^k X_r^{k-1}\frac{p_r}{P_r}\frac{P_r}{rp_r}\text{, as } X_n=\frac{P_n}{np_n} \text{ and by(3.3)}$$

$$=O(1)\sum_{r=1}^{m}\frac{|\lambda_{r+1}|^k}{r}X_r^{k-1},$$

$$=O(1) \quad as\ m\to\infty\text{, by (3.4).}$$

This completes the proof of the Lemma.

5. Proof of the theorem

Since the behavior of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behavior of the function in the immediate neighborhood of this point only, the truth of the theorem is necessarily the consequence of the Lemma.

6. Conclusion

Putting δ = 0 and $\alpha=\frac{P_n}{p_n}$ with δ=0, the result of Misra et al. [2] and the result of H.Bor [1] can be achieved respectively from the result established in the present chapter under a few varying condition. Further there is a reach scope to work in this area for different indexed summability methods with additional parameter.

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