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Pointwise Weighted Approximation of Functions with Endpoint Singularities by Combinations of Bernstein Operators

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Abstract. We give direct and inverse theorems for the weighted approximation of functions with endpoint singularities by combinations of Bernstein operators.

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Introduction

The set of all continuous functions, defined on the interval *I*, is denoted by *C(I)*. For any $f \in C([0,1])$, the corresponding *Bernstein operators* are defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x),$$

Where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ k = 0, 1, 2, \ldots, n,\ x \in [0,1].$$

Approximation properties of Bernstein operators have been studied very well (see [2], [3], [5]-[8], [12]-[14], for example). In order to approximate the functions with singularities, Della Vecchia et al. [3] and Yu-Zhao [12] introduced some kinds of *modified Bernstein operators*. Throughout the paper, *C* denotes a positive constant independent of *n* and *x*, which may be different in different cases.

Let

$$w(x) = x^{\alpha}(1-x)^{\beta},\ \alpha,\ \beta \geqslant 0,\ \alpha + \beta > 0,\ 0 \leqslant x \leqslant 1.$$

and

$$C_w := \{f \in C((0,1)) : \lim_{x \longrightarrow 1} (wf)(x) = \lim_{x \longrightarrow 0} (wf)(x) = 0\}.$$

The norm in C_w is defined by $\|wf\|_{C_w} := \|wf\| = \sup_{0 \leqslant x \leqslant 1} |(wf)(x)|$. Define

$$W_{w,\lambda}^r := \{f \in C_w : f^{(r-1)} \in A.C.((0,1)),\ \|w\varphi^{r\lambda} f^{(r)}\| < \infty\}.$$

For $f \in C_w$, define the *weighted modulus of smoothness* by

$$\omega^r_{\varphi^\lambda}(f,t)_w := \sup_{0<h\leqslant t}\{\|w\Delta^r_{h\varphi^\lambda} f\|_{[16h^2,1-16h^2]} + \|w\overrightarrow{\Delta}^r_h f\|_{[0,16h^2]} + \|w\overleftarrow{\Delta}^r_h f\|_{[1-16h^2,1]}\},$$

where

$$\Delta^r_{h\varphi} f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k} f(x+(\frac{r}{2}-k)h\varphi(x)),$$

$$\overrightarrow{\Delta}^r_h f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k} f(x+(r-k)h),$$

$$\overleftarrow{\Delta}^r_h f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k} f(x-kh),$$

and $\varphi(x) = \sqrt{x(1-x)}$. Della Vecchia *et al.* firstly introduced $B^*_n(f,x)$ and $\bar{B}_n(f,x)$ in [3], where the properties of $B^*_n(f,x)$ and $\bar{B}_n(f,x)$ are studied. Among others, they prove that

$$\|w(f - B^*_n(f))\| \leqslant C\omega^2_\varphi(f, n^{-1/2}),\ f \in C_w,$$

$$\|\bar{w}(f - \bar{B}_n(f))\| \leqslant \frac{C}{n^{3/2}}\sum_{k=1}^{[\sqrt{n}]} k^2\omega^2_\varphi(f,\frac{1}{k})^*_{\bar{w}},\ f \in C_{\bar{w}},$$

where $w(x) = x^\alpha(1-x)^\beta,\ \alpha,\ \beta \geqslant 0,\ \alpha+\beta>0,\ 0 \leqslant x \leqslant 1$. In [11], for any $\alpha,\ \beta > 0,\ n \geqslant 2r+\alpha+\beta$, there hold

$$\|wB^*_{n,r}(f)\| \leqslant C\|wf\|,\ f \in C_w,$$

$$\|w(B^*_{n,r}(f) - f)\| \leqslant \begin{cases} \frac{C}{n^r}(\|wf\| + \|w\varphi^{2r}f^{(2r)}\|), f \in W^{2r}_w, \\ C(\omega^{2r}_\varphi(f,n^{-1/2})_w + n^{-r}\|wf\|), f \in C_w. \end{cases}$$

$$\|w\varphi^{2r}B^{*(2r)}_{n,r}(f)\| \leqslant \begin{cases} Cn^r\|wf\|, f \in C_w, \\ C(\|wf\| + \|w\varphi^{2r}f^{(2r)}\|), f \in W^{2r}_w. \end{cases}$$

and for $0 < \gamma < 2r$,

$$\|w(B^*_{n,r}(f) - f)\| = O(n^{-\gamma/2}) \Longleftrightarrow \omega^{2r}_\varphi(f,t)_w = O(t^r).$$

Ditzian and Totik [5] extended this method of combinations and defined the following combinations of Bernstein operators:

$$B_{n,r}(f,x) := \sum_{i=0}^{r-1} C_i(n)B_{n_i}(f,x).$$

with the conditions

(a) $n = n_0 < n_1 < \cdots < n_{r-1} \leqslant Cn$,

(b) $\sum_{i=0}^{r-1}|C_i(n)| \leqslant C$,

(c) $\sum_{i=0}^{r-1} C_i(n) = 1$,

(d) $\sum_{i=0}^{r-1} C_i(n) n_i^{-k} = 0$, for $k = 1, \ldots, r-1$.

2. The main results

Now, we can define our *new combinations of Bernstein operators* as follows:

$$B^{*}_{n,r}(f,x) := B_{n,r}(F_n,x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(F_n,x), \tag{2.1}$$

where $C_i(n)$ satisfy the conditions (a)-(d). For the details, it can be referred to [11]. Our main results are the following:

Theorem 2.1. *If $\alpha,\ \beta > 0$, for any $f \in C_w$, we have*

$$\|wB^{*(r)}_{n,r-1}(f)\| \leqslant Cn^r \|wf\|. \tag{2.2}$$

Theorem 2.2. *For any $\alpha,\ \beta > 0,\ 0 \leqslant \lambda \leqslant 1$, we have*

(2.3)

$$|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant \begin{cases} Cn^{r/2}\{max\{n^{r(1-\lambda)/2}, \varphi^{r(\lambda-1)}(x)\}\}\|wf\|, f \in C_w, \\ C\|w\varphi^{r\lambda} f^{(r)}\|, f \in W^r_{w,\lambda}. \end{cases}$$

Theorem 2.3. *For $f \in C_w,\ \alpha,\ \beta > 0,\ \alpha_0 \in (0,2),\ 0 \leqslant \lambda \leqslant 1$, we have*

(2.4)

$$w(x)|f(x) - B^{*}_{n,r-1}(f,x)| = O((n^{-\frac{1}{2}}\varphi^{-\lambda}(x)\delta_n(x))^{\alpha_0}) \Longleftrightarrow \omega^r_{\varphi^\lambda}(f,t)_w = O(t^{\alpha_0}).$$

3. Lemmas

Lemma 3.1. *([13]) For any non-negative real u and v, we have*

$$\sum_{k=1}^{n-1} (\frac{k}{n})^{-u}(1-\frac{k}{n})^{-v} p_{n,k}(x) \leqslant Cx^{-u}(1-x)^{-v}. \tag{3.1}$$

Lemma 3.2. *([3]) If $\gamma \in R$, then*

$$\sum_{k=0}^{n} |k-nx|^\gamma p_{n,k}(x) \leqslant Cn^{\frac{\gamma}{2}}\varphi^\gamma(x). \tag{3.2}$$

Lemma 3.3. *For any $f \in W^r_{w,\lambda}$, $0 \leqslant \lambda \leqslant 1$ and $\alpha,\ \beta > 0$, we have*

$$\|w\varphi^{r\lambda}F^{(r)}_n\| \leqslant C\|w\varphi^{r\lambda}f^{(r)}\|. \tag{3.3}$$

Proof. By symmetry, we only prove the above result when $x \in (0, 1/2]$, the others can be done similarly. Obviously, when $x \in (0, 1/n]$, by [5], we have

$$\begin{aligned} |L^{(r)}_r(f,x)| \leqslant C|\overrightarrow{\Delta}^r_{\frac{1}{r}} f(0)| \ &\leqslant\ Cn^{-\frac{r}{2}+1}\int_0^{\frac{r}{n}} u^{\frac{r}{2}}|f^{(r)}(u)|du \\ &\leqslant Cn^{-\frac{r}{2}+1}\|w\varphi^{r\lambda}f^{(r)}\|\int_0^{\frac{r}{n}} u^{\frac{r}{2}}w^{-1}(u)\varphi^{-r\lambda}(u)du. \end{aligned}$$

So

$$|w(x)\varphi^{r\lambda}(x)F_n^{(r)}(x)| \leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.$$

If $x \in [\frac{1}{n}, \frac{2}{n}]$, we have

$$|w(x)\varphi^{r\lambda}(x)F_n^{(r)}(x)| \leqslant |w(x)\varphi^{r\lambda}(x)f^{(r)}(x)| + |w(x)\varphi^{r\lambda}(x)(f(x) - F_n(x))^{(r)}| \\ := I_1 + I_2.$$

For I_2, we have

$$f(x) - F_n(x) = (\psi(nx-1)+1)(f(x) - L_r(f,x)).$$

$$w(x)\varphi^{r\lambda}(x)|(f(x) - F_n(x))^{(r)}| = w(x)\varphi^{r\lambda}(x)\sum_{i=0}^{r} n^i|(f(x) - L_r(f,x))^{(r-i)}|.$$

By [5], then

$$|(f(x) - L_r(f,x))^{(r-i)}|_{[\frac{1}{n},\frac{2}{n}]} \leqslant C(n^{r-i}\|f - L_r\|_{[\frac{1}{n},\frac{2}{n}]} + n^{-i}\|f^{(r)}\|_{[\frac{1}{n},\frac{2}{n}]}),\ 0 < j < r.$$

Now, we estimate

$$I := w(x)\varphi^{r\lambda}(x)|f(x) - L_r(x)|. \tag{3.4}$$

By Taylor expansion, we have

$$f(\frac{i}{n}) = \sum_{u=0}^{r-1} \frac{(\frac{i}{n} - x)^u}{u!} f^{(u)}(x) + \frac{1}{(r-1)!}\int_x^{\frac{i}{n}} (\frac{i}{n} - s)^{r-1} f^{(r)}(s)ds, \tag{3.5}$$

It follows from (3.5) and the identities

$$\sum_{i=1}^{r} (\frac{i}{n})^v l_i(x) = Cx^v,\ v = 0, 1, \cdots, r.$$

We have

$$\begin{aligned} L_r(f,x) &= \sum_{i=1}^{r}\sum_{u=0}^{r-1} \frac{(\frac{i}{n} - x)^u}{u!} f^{(u)}(x)l_i(x) + \frac{1}{(r-1)!}\sum_{i=1}^{r} l_i(x)\int_x^{\frac{i}{n}} (\frac{i}{n} - s)^{r-1} f^{(r)}(s)ds \\ &= f(x) + \sum_{u=1}^{r-1} f^{(u)}(x)(\sum_{v=0}^{u} C_u^v(-x)^{u-v}\sum_{i=1}^{r}(\frac{i}{n})^v l_i(x)) \\ &\quad + \frac{1}{(r-1)!}\sum_{i=1}^{r} l_i(x)\int_x^{\frac{i}{n}} (\frac{i}{n} - s)^{r-1} f^{(r)}(s)ds, \end{aligned}$$

which implies that

$$w(x)\varphi^{r\lambda}(x)|f(x) - L_r(f,x)| = \frac{1}{r!}w(x)\varphi^{r\lambda}(x)\sum_{i=1}^{r} l_i(x)\int_x^{\frac{i}{n}} (\frac{i}{n} - s)^{r-1} f^{(r)}(s)ds,$$

since $|l_i(x)| \leqslant C$ for $x \in [0, \frac{2}{n}],\ i = 1, 2, \cdots, r.$

It follows from $\frac{|\frac{i}{n}-s|^{r-1}}{w(s)} \leqslant \frac{|\frac{i}{n}-x|^{r-1}}{w(x)}$, s between $\frac{i}{n}$ and x, then

$$\begin{aligned} w(x)\varphi^{r\lambda}(x)|f(x)-L_r(f,x)| &\leqslant Cw(x)\varphi^{r\lambda}(x)\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}|f^{(r)}(s)|ds \\ &\leqslant C\varphi^{r\lambda}(x)\|w\varphi^{r\lambda}f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\varphi^{-r\lambda}(s)ds \\ &\leqslant \frac{C}{n^r}\|w\varphi^{r\lambda}f^{(r)}\|. \end{aligned}$$

Thus

$$I \leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.$$

So, we get

$$I_2 \leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.$$

Above all, we have

$$|w(x)\varphi^{r\lambda}(x)F_n^{(r)}(x)| \leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.$$

□

Lemma 3.4. *If $f \in W^r_{w,\lambda}$, $0 \leqslant \lambda \leqslant 1$ and $\alpha,\ \beta > 0$, then*

$$|w(x)(f(x)-L_r(f,x))|_{[0,\frac{2}{n}]} \leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}f^{(r)}\|. \tag{3.6}$$

$$|w(x)(f(x)-R_r(f,x))|_{[1-\frac{2}{n},1]} \leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}f^{(r)}\|. \tag{3.7}$$

Proof. By Taylor expansion, we have

$$f(\frac{i}{n}) = \sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x) + \frac{1}{r!}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds, \tag{3.8}$$

It follows from (3.8) and the identities

$$\sum_{i=1}^{r-1}(\frac{i}{n})^v l_i(x) = Cx^v,\ v=0,1,\ldots,r.$$

We have

$$\begin{aligned} L_r(f,x) &= \sum_{i=1}^{r}\sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x)l_i(x) + \frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds \\ &= f(x) + \sum_{u=1}^{r-1}f^{(u)}(x)(\sum_{v=0}^{u}C_u^v(-x)^{u-v}\sum_{i=1}^{r}(\frac{i}{n})^v l_i(x)) \\ &\quad + \frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds, \end{aligned}$$

which implies that

$$w(x)|f(x)-L_r(f,x)| = \frac{1}{(r-1)!}w(x)\sum_{i=1}^{r} l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds,$$

since $|l_i(x)| \leqslant C$ for $x \in [0, \frac{2}{n}],\ i = 1, 2, \cdots, r$.

It follows from $\frac{|\frac{i}{n}-s|^{r-1}}{w(s)} \leqslant \frac{|\frac{i}{n}-x|^{r-1}}{w(x)}$, s between $\frac{i}{n}$ and x, then

$$\begin{aligned} w(x)|f(x)-L_r(f,x)| &\leqslant Cw(x)\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}|f^{(r)}(s)|ds \\ &\leqslant C\frac{\varphi^r(x)}{\varphi^{r\lambda}(x)}\|w\varphi^{r\lambda}f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\varphi^{-r}(s)ds \\ &\leqslant C\frac{\delta_n^r(x)}{\varphi^{r\lambda}(x)}\|w\varphi^{r\lambda}f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\varphi^{-r}(s)ds \\ &\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}f^{(r)}\|. \end{aligned}$$

The proof of (3.7) can be done similarly. □

Lemma 3.5. *([11]) For every $\alpha,\ \beta > 0$, we have*

$$\|wB^{*}_{n,r-1}(f)\| \leqslant C\|wf\|. \tag{3.9}$$

Lemma 3.6. *([15]) If $\varphi(x) = \sqrt{x(1-x)},\ 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \beta \leqslant 1$, then*

$$\int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}}\cdots\int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}}\varphi^{-r\beta}(x+\sum_{k=1}^{r}u_k)du_1\cdots du_r \leqslant Ch^r\varphi^{r(\lambda-\beta)}(x). \tag{3.10}$$

4. Proof of Theorems

4.1. **Proof of Theorem 2.1.** By symmetry, in what follows we will always assume that $x \in (0, \frac{1}{2}]$. It is sufficient to prove the validity for $B_{n,r-1}(F_n, x)$

instead of $B^{*}_{n,r-1}(f,x)$. When $x \in (0, \frac{1}{n})$, we have

$$\begin{aligned}
|w(x)B^{*(r)}_{n,r-1}(f,x)| &\leqslant w(x)\sum_{i=0}^{r-2}\frac{n_i!}{(n_i-r)!}\sum_{k=0}^{n_i-r}|\overrightarrow{\Delta}^{r}_{\frac{1}{n_i}}F_n(\frac{k}{n_i})|p_{n_i-r,k}(x) \\
&\leqslant Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=0}^{n_i-r}|\overrightarrow{\Delta}^{r}_{\frac{1}{n_i}}F_n(\frac{k}{n_i})|p_{n_i-r,k}(x) \\
&\leqslant Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=0}^{n_i-r}\sum_{j=0}^{r}C_r^j|F_n(\frac{k+r-j}{n_i})|p_{n_i-r,k}(x) \\
&\leqslant Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{j=0}^{r}C_r^j|F_n(\frac{r-j}{n_i})|p_{n_i-r,0}(x) \\
&+ Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{j=0}^{r}C_r^j|F_n(\frac{n_i-j}{n_i})|p_{n_i-r,n_i-r}(x) \\
&+ Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=1}^{n_i-r-1}\sum_{j=0}^{r}C_r^j|F_n(\frac{k+r-j}{n_i})|p_{n_i-r,k}(x) \\
&:= H_1+H_2+H_3.
\end{aligned}$$

We have

$$\begin{aligned}
H_1 &\leqslant Cw(x)\|wf\|\sum_{i=0}^{r-2}n_i^r w^{-1}(\frac{1}{n_i})p_{n_i-r,0}(x) \\
&\leqslant C\|wf\|\sum_{i=0}^{r-2}n_i^r(n_ix)^{\alpha}(1-x)^{n_i-r} \\
&\leqslant Cn^r\|wf\|.
\end{aligned}$$

When $1 \leqslant k \leqslant n_i-r-1$, we have $1 \leqslant k+2r-j \leqslant n_i-1$, and thus

$$\frac{w(\frac{k}{n_i-r})}{w(\frac{k+r-j}{n_i})} = (\frac{n_i}{n_i-r})^{\alpha+\beta}(\frac{k}{k+r-j})^{\alpha}(\frac{n_i-r-k}{n_i-r-k+j})^{\beta} \leqslant C.$$

Thereof, by (3.1), we have

$$\begin{aligned}
H_3 &\leqslant Cw(x)\|wF_n\|\sum_{i=0}^{r-2}n_i^r\sum_{k=1}^{n_i-r-1}\sum_{j=0}^{r}\frac{1}{w(\frac{k+r-j}{n_i})}p_{n_i-r,k}(x) \\
&\leqslant Cw(x)\|wF_n\|\sum_{i=0}^{r-2}n_i^r\sum_{k=1}^{n_i-r-1}\frac{1}{w(\frac{k}{n_i-r})}p_{n_i-r,k}(x) \\
&\leqslant Cn^r\|wF_n\| \ \leqslant \ Cn^r\|wf\|.
\end{aligned}$$

Similarly, we can get $H_2 \leqslant Cn^r\|wf\|$. So

$$|w(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant Cn^r\|wf\|,\ x \in (0,\frac{1}{n}). \tag{4.1}$$

When $x \in [\frac{1}{n}, \frac{1}{2}]$, according to [5], we have

$$\begin{aligned} &|w(x)B^{*(r)}_{n,r-1}(f,x)| \\ &= |w(x)B^{(r)}_{n,r-1}(F_n,x)| \\ &\leqslant w(x)(\varphi^2(x))^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}|Q_j(x,n_i)|n_i^j\sum_{k=0}^{n}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x). \end{aligned}$$

Then
$Q_j(x,n_i) = (n_ix(1-x))^{[\frac{r-j}{2}]}$, and $(\varphi^2(x))^{-r}Q_j(x,n_i)n_i^j \leqslant C(n_i/\varphi^2(x))^{\frac{r+j}{2}}$, we have

$$\begin{aligned} |w(x)B^{*(r)}_{n,r-1}(f,x)| &\leqslant Cw(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}(\frac{n_i}{\varphi^2(x)})^{\frac{r+j}{2}}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ &\leqslant C\|wF_n\|w(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}(\frac{n_i}{\varphi^2(x)})^{\frac{r+j}{2}}\sum_{k=0}^{n_i}\frac{|x-\frac{k}{n_i}|^j}{w(\frac{k^*}{n_i})}p_{n_i,k}(x), \end{aligned} \tag{4.2}$$

where $k^* = 1$ for $k = 0$, $k^* = n_i - 1$ for $k = n_i$ and $k^* = k$ for $1 < k < n_i$. Note that

$$w^2(x)\frac{p_{n_i,0}(x)}{w^2(\frac{1}{n_i})} \leqslant C(n_ix)^{2\alpha}(1-x)^{n_i} \leqslant C,$$

and

$$w^2(x)\frac{p_{n_i,n_i}(x)}{w^2(1-\frac{1}{n_i})} \leqslant Cn_i^\beta x^{n_i} \leqslant C\frac{n_i^\beta}{2^{n_i}} \leqslant C.$$

By (3.1), we have

$$\sum_{k=0}^{n_i}\frac{1}{w^2(\frac{k^*}{n_i})}p_{n_i,k}(x) \leqslant Cw^{-2}(x). \tag{4.3}$$

Now, applying Cauchy's inequality, by (3.2) and (4.3), we have

$$\begin{aligned} \sum_{k=0}^{n_i}\frac{|x-\frac{k}{n_i}|^j}{w(\frac{k^*}{n_i})}p_{n_i,k}(x) &\leqslant (\sum_{k=0}^{n_i}|x-\frac{k}{n_i}|^{2j}p_{n_i,k}(x))^{1/2}(\sum_{k=0}^{n_i}\frac{1}{w^2(\frac{k^*}{n_i})}p_{n_i,k}(x))^{1/2} \\ &\leqslant Cn_i^{-j/2}\varphi^j(x)w^{-1}(x). \end{aligned}$$

Substituting this to (4.2), we have

$$|w(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant Cn^r\|wf\|,\ x \in [\frac{1}{n},\frac{1}{2}]. \tag{4.4}$$

We get Theorem 2.1 by (4.1) and (4.4). $\square$

4.2. **Proof of Theorem 2.2.** (1) When $f \in C_w$, we proceed it as follows:

Case 1. If $0 \leqslant \varphi(x) \leqslant \frac{1}{\sqrt{n}}$, by (2.2), we have

(4.5)
$$|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant Cn^{-r\lambda/2}|w(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant Cn^{r(1-\lambda/2)}\|wf\|.$$

Case 2. If $\varphi(x) > \frac{1}{\sqrt{n}}$, we have

$$|B^{*(r)}_{n,r-1}(f,x)| = |B^{(r)}_{n,r-1}(F_n,x)|$$
$$\leqslant (\varphi^2(x))^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}|Q_j(x,n_i)C_i(n)|n_i^j\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x),$$

$Q_j(x,n_i) = (n_ix(1-x))^{[\frac{r-j}{2}]}$, and $(\varphi^2(x))^{-2r}Q_j(x,n_i)n_i^j \leqslant C(n_i/\varphi^2(x))^{\frac{r+j}{2}}$.
So

$$|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)|$$
$$\leqslant Cw(x)\varphi^{r\lambda}(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}(\frac{n_i}{\varphi^2(x)})^{\frac{r+j}{2}}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x)$$
$$\leqslant Cn^{\frac{r}{2}}\varphi^{r(\lambda-1)}(x). \tag{4.6}$$

It follows from combining with (4.5) and (4.6) that the first inequality is proved.
(2) When $f \in W^r_{w,\lambda}$, we have

$$B^{(r)}_{n,r-1}(F_n,x) = \sum_{i=0}^{r-2}C_i(n)n_i^r\sum_{k=0}^{n_i-r}\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{k}{n_i})p_{n_i-r,k}(x). \tag{4.7}$$

If $0 < k < n_i - r$, we have

$$|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{k}{n_i})| \leqslant Cn_i^{-r+1}\int_0^{\frac{r}{n_i}}|F_n^{(r)}(\frac{k}{n_i}+u)|du, \tag{4.8}$$

If $k = 0$, we have

$$|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(0)| \leqslant C\int_0^{\frac{r}{n_i}}u^{r-1}|F_n^{(r)}(u)|du, \tag{4.9}$$

Similarly

$$|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{n_i-r}{n_i})| \leqslant Cn_i^{-r+1}\int_{1-\frac{r}{n_i}}^1(1-u)^{\frac{r}{2}}|F_n^{(r)}(u)|du. \tag{4.10}$$

By (4.7)-(4.10), we have

$$|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)|$$

$$\leqslant Cw(x)\varphi^{r\lambda}(x)\|w\varphi^{r\lambda}F^{(r)}_n\|\sum_{i=0}^{r-2}\sum_{k=0}^{n_i-r}(w\varphi^{r\lambda})^{-1}(\frac{k^*}{n_i})p_{n_i-r,k}(x), \tag{4.11}$$

where $k^* = 1$ for $k = 0$, $k^* = n_i - r - 1$ for $k = n_i - r$ and $k^* = k$ for $1 < k < n_i - r$. By (3.1), we have

$$\sum_{k=0}^{n_i-r}(w\varphi^{r\lambda})^{-1}(\frac{k^*}{n_i})p_{n_i-r,k}(x) \leqslant C(w\varphi^{r\lambda})^{-1}(x). \tag{4.12}$$

which combining with (4.12) give

$$|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.\square$$

So we get the second inequality of the Theorem 2.2.

4.3. **Proof of Theorem 2.3.**

4.3.1. *The direct theorem.* We know

(4.13)

$$F_n(t) = F_n(x) + F'_n(t)(t-x) + \cdots + \frac{1}{(r-1)!}\int_x^t (t-u)^{r-1}f^{(r)}(u)du,$$

$$B_{n,r-1}((\cdot - x)^k, x) = 0,\ k = 1, 2, \cdots, r-1. \tag{4.14}$$

According to the definition of $W^r_{w,\lambda}$, for any $g \in W^r_{w,\lambda}$, we have $B^*_{n,r-1}(g,x) = B_{n,r-1}(G_n(g),x)$, and $w(x)|G_n(x)-B_{n,r-1}(G_n,x)| = w(x)|B_{n,r-1}(R_r(G_n,t,x),x)|$, thereof $R_r(G_n,t,x) = \int_x^t(t-u)^{r-1}G^{(r)}_n(u)du$. It follows from $\frac{|t-u|^{r-1}}{w(u)} \leqslant \frac{|t-x|^{r-1}}{w(x)}$, u between t and x, we have

$$\begin{aligned} w(x)|G_n(x) - B_{n,r-1}(G_n,x)| &\leqslant C\|w\varphi^{r\lambda}G^{(r)}_n\|w(x)B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{w(u)\varphi^{r\lambda}(u)}du, x) \\ &\leqslant C\|w\varphi^{r\lambda}G^{(r)}_n\|w(x)(B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{\varphi^{2r\lambda}(u)}du, x))^{\frac{1}{2}}\cdot \end{aligned}$$

$$(B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{w^2(u)}du, x))^{\frac{1}{2}}. \tag{4.15}$$

also

$$\int_x^t \frac{|t-u|^{r-1}}{\varphi^{2r\lambda}(u)}du \leqslant C\frac{|t-x|^r}{\varphi^{2r\lambda}(x)},\ \int_x^t \frac{|t-u|^{r-1}}{w^2(u)}du \leqslant \frac{|t-x|^r}{w^2(x)}. \tag{4.16}$$

By (3.2), (4.15) and (4.16), we have

$$\begin{aligned}
w(x)|G_n(x)-B_{n,r-1}(G_n,x)| &\leqslant C\|w\varphi^{r\lambda}G_n^{(r)}\|\varphi^{-r\lambda}(x)B_{n,r-1}(|t-x|^r,x)\\
&\leqslant Cn^{-\frac{r}{2}}\frac{\varphi^r(x)}{\varphi^{r\lambda}(x)}\|w\varphi^{r\lambda}G_n^{(r)}\|\\
&\leqslant Cn^{-\frac{r}{2}}\frac{\delta_n^r(x)}{\varphi^{r\lambda}(x)}\|w\varphi^{r\lambda}G_n^{(r)}\|\\
&= C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}G_n^{(r)}\|. \qquad (4.17)
\end{aligned}$$

By (3.3), (3.6), (3.7) and (4.17), when $g\in W^r_{w,\lambda}$, then

$$\begin{aligned}
w(x)|g(x)-B^*_{n,r-1}(g,x)| &\leqslant w(x)|g(x)-G_n(g,x)|+w(x)|G_n(g,x)-B^*_{n,r-1}(g,x)|\\
&\leqslant |w(x)(g(x)-L_r(g,x))|_{[0,\frac{2}{n}]}+|w(x)(g(x)-R_r(g,x))|_{[1-\frac{2}{n},1]}\\
&\quad + \ C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}G_n^{(r)}\|\\
&\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}g^{(r)}\|. \qquad (4.18)
\end{aligned}$$

For $f\in C_w$, we choose proper $g\in W^r_{w,\lambda}$, by (3.9) and (4.18), then

$$\begin{aligned}
w(x)|f(x)-B^*_{n,r-1}(f,x)| &\leqslant w(x)|f(x)-g(x)|+w(x)|B^*_{n,r-1}(f-g,x)|\\
&\quad +w(x)|g(x)-B^*_{n,r-1}(g,x)|\\
&\leqslant C(\|w(f-g)\|+(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}g^{(r)}\|)\\
&\leqslant C\omega^r_{\varphi^\lambda}(f,\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})_w.\square
\end{aligned}$$

4.3.2. *The inverse theorem.* We define the weighted main-part modulus for $D=R_+$ by

$$\Omega^r_{\varphi^\lambda}(C,f,t)_w=\sup_{0<h\leqslant t}\|w\Delta^r_{h\varphi^\lambda}f\|_{[Ch^*,\infty]},$$
$$\Omega^r_{\varphi^\lambda}(1,f,t)_w=\Omega^r_{\varphi^\lambda}(f,t)_w.$$

where $C>2^{1/\beta(0)-1}$, $\beta(0)>0$ and h^* is given by

$$h^*=\begin{cases}(Ar)^{1/1-\beta(0)}h^{1/1-\beta(0)}, 0\leqslant\beta(0)<1,\\ 0,\beta(0)\geqslant 1.\end{cases}$$

The main-part K-functional is given by

$$K_{r,\varphi^\lambda}(f,t^r)_w=\sup_{0<h\leqslant t}\inf_g\{\|w(f-g)\|_{[Ch^*,\infty]}+t^r\|w\varphi^{r\lambda}g^{(r)}\|_{[Ch^*,\infty]},$$

where $g^{(r-1)} \in A.C.((Ch^*, \infty))\}$. By ([5]), we have

$$C^{-1}\Omega^r_{\varphi^\lambda}(f,t)_w \leqslant \omega^r_{\varphi^\lambda}(f,t)_w \leqslant C\int_0^t \frac{\Omega^r_{\varphi^\lambda}(f,\tau)_w}{\tau}d\tau, \tag{4.19}$$

$$C^{-1}K_{r,\varphi^\lambda}(f,t^r)_w \leqslant \Omega^r_{\varphi^\lambda}(f,t)_w \leqslant CK_{r,\varphi^\lambda}(f,t^r)_w. \tag{4.20}$$

Proof. Let $\delta > 0$, by (4.20), we choose proper g so that

$$\|w(f-g)\| \leqslant C\Omega^r_{\varphi^\lambda}(f,\delta)_w,\ \|w\varphi^{r\lambda}g^{(r)}\| \leqslant C\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w. \tag{4.21}$$

then

$$\begin{aligned}
|w(x)\Delta^r_{h\varphi^\lambda}f(x)| &\leqslant |w(x)\Delta^r_{h\varphi^\lambda}(f(x)-B^*_{n,r-1}(f,x))| + |w(x)\Delta^r_{h\varphi^\lambda}B^*_{n,r-1}(f-g,x)| \\
&\quad + |w(x)\Delta^r_{h\varphi^\lambda}B^*_{n,r-1}(g,x)| \\
&\leqslant \sum_{j=0}^r C_r^j(n^{-\frac{1}{2}}\delta_n(x+(\frac{r}{2}-j)h\varphi^\lambda(x)))^{\alpha_0} \\
&\quad + \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\cdots\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} w(x)B^{*(r)}_{n,r-1}(f-g,x+\sum_{k=1}^r u_k)du_1\cdots du_r \\
&\quad + \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\cdots\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} w(x)B^{*(r)}_{n,r-1}(g,x+\sum_{k=1}^r u_k)du_1\cdots du_r \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{4.22}$$

Obviously

$$J_1 \leqslant C(n^{-\frac{1}{2}}\delta_n(x))^{\alpha_0}. \tag{4.23}$$

By (2.2) and (4.21), we have

$$\begin{aligned}
J_2 &\leqslant Cn^r\|w(f-g)\|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\cdots\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} du_1\cdots du_r \\
&\leqslant Cn^rh^r\varphi^{r\lambda}(x)\|w(f-g)\| \\
&\leqslant Cn^rh^r\varphi^{r\lambda}(x)\Omega^r_{\varphi^\lambda}(f,\delta)_w.
\end{aligned} \tag{4.24}$$

By the first inequality of (2.3), we let $\lambda = 1$, and (3.10) as well as (4.21), we have

$$\begin{aligned}
J_2 &\leqslant Cn^{\frac{r}{2}}\|w(f-g)\|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\cdots\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \varphi^{-r}(x+\sum_{k=1}^r u_k)du_1\cdots du_r \\
&\leqslant Cn^{\frac{r}{2}}h^r\varphi^{r(\lambda-1)}(x)\|w(f-g)\| \\
&\leqslant Cn^{\frac{r}{2}}h^r\varphi^{r(\lambda-1)}(x)\Omega^r_{\varphi^\lambda}(f,\delta)_w.
\end{aligned} \tag{4.25}$$

By the second inequality of (2.3) and (4.21), we have

$$J_3 \leqslant C\|w\varphi^{r\lambda}g^{(r)}\|w(x)\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\cdots\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} w^{-1}(x+\sum_{k=1}^{r}u_k)\varphi^{-r\lambda}(x+\sum_{k=1}^{r}u_k)du_1\cdots du_r$$

$$\leqslant Ch^r\|w\varphi^{r\lambda}g^{(r)}\|$$

$$\leqslant Ch^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w. \tag{4.26}$$

Now, by (4.22)-(4.26), we get

$$|w(x)\Delta^r_{h\varphi^\lambda}f(x)| \leqslant C\{(n^{-\frac{1}{2}}\delta_n(x))^{\alpha_0}+h^r(n^{-\frac{1}{2}}\delta_n(x))^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w+h^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w\}.$$

When $n \geqslant 2$, we have

$$n^{-\frac{1}{2}}\delta_n(x) < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x) \leqslant \sqrt{2}n^{-\frac{1}{2}}\delta_n(x),$$

Choosing proper $x, n \in N$, so that

$$n^{-\frac{1}{2}}\delta_n(x) \leqslant \delta < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x),$$

Therefore

$$|w(x)\Delta^r_{h\varphi^\lambda}f(x)| \leqslant C\{\delta^{\alpha_0}+h^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w\}.$$

By Borens-Lorentz lemma, we get

$$\Omega^r_{\varphi^\lambda}(f,t)_w \leqslant Ct^{\alpha_0}. \tag{4.27}$$

So, by (4.27), we get

$$\omega^r_{\varphi^\lambda}(f,t)_w \leqslant C\int_0^t \frac{\Omega^r_{\varphi^\lambda}(f,\tau)_w}{\tau}d\tau = C\int_0^t \tau^{\alpha_0-1}d\tau = Ct^{\alpha_0}.$$

□

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