Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 5, Issue 5
Published online: November 26, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Optimal Control of Problems Governed by Obstacle Type for Infinite Order

S.A. El-Zahaby and Ibtissam El-Zoulati

Department of Mathematics, Faculty of Science,

Al-Azhar University Girls Branch, Nasr City, Cairo, Egypt.

Abstract.

In this paper we obtain optimal control problems governed by variational inequalities of obstacle type for infinite order with finite dimension. We obtain the optimality condition under classical assumption using a dual regularized functional, to interpret the variational inequality we use a penalty method to get first - order conditions.

Keywords: Variational inequalities, optimal control, infinite order, penalization method.

1 Introduction

The obstacle problem, in fact, is one of the main motivations for the development of the theory of variational inequalities, of which Signorini was one of the main architects. It is known that optimal control problems governed by variational inequality may be equivalently transformed into control problem with linear state equation and mixed type (state - control) constraints. This method was used by F. Mignot and J. P. Puel [12]. Gali et al [8] presented a set of inequalities defining a control of a system governed by elliptic operator of infinite order with finite dimension and in [9] Gali et al presented the distribution control problem for the infinite order. In [7] El-Zahaby presented the necessary conditions for control problems governed by elliptic variational inequalities of infinite order with finite dimension by using Mignot [11]. In the present paper, we use the theory of M. Bergounious et al [3] to obtain the necessary conditions of optimality for control problem governed by elliptic variational inequality of obstacle type of infinite order with finite dimension. By using a penalty method, we get the first order optimality conditions with additional assumptions.

2 Some Function Spaces:

We will define the infinite order sobolev space $w^{\infty}\{a_{\alpha},2\}$

$w^{\infty}\{a_\alpha,2\}=\{u(x)\in C^{\infty}(R^n), \sum_{1\alpha 1=0}^{\infty} a_\alpha \| D^\alpha u \|_2^2<\infty\}$ this space is sobolev of infinite order of periodic functions

Defined on all R^n such that:

$$|\alpha|=\alpha_1+\ldots+\alpha_n, D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} \qquad a_\alpha\geq 0$$

$\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multi-index of differentiation

$\| \quad \|_2$ is the norm of $L_2(R^2)$

In this case: $w^{\infty}\{a_\alpha,2\}\subseteq L_2(R^n)\subseteq w^{-\infty}\{a_\alpha,2\}$

$w^{-\infty}\{a_\alpha,2\}$ is the dual space of $w^{\infty}\{a_\alpha,2\}$ with respect of $L_2(R^n)$

Now let us consider an elliptic operator of infinite order with finite dimension.

$$Au=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2\alpha}u$$

3 Setting the problem:

Consider the following problem:

(P): min $J(y,v), \quad <T(y,v),z-y>\geq 0, \forall z\in k, y\in k, v\in U_{ad}$

$$V=w^{\infty}\{a_\alpha,2\}$$

$$U=L^2(R^n)$$

U_{ad} is non empty closed convex of U.

$$T(y,v)=Ay-V-f \;,\; f\in L^2(R^n)$$

We define a non empty, closed, convex subset of v such that:

$k=\{y\in v, y\geq\varphi\geq 0 \;\text{ a.e in } R^n\}$

And the cost function:

$$J(y,v)=\frac{1}{2}\left[\| y-y_d \|^2+\| v-v_d \|^2_{L^2(R^n)}\right]$$

This functional is convex and lower semicontinous.

The compactness of the injection of $w^{\infty}\{a_\alpha,2\}$ in $L^2(R^n)$ implies that T is weakly-strongly continuous with respect to both variables y and v i.e.

$$y_n \xrightarrow{w} y \text{ and } v_n \xrightarrow{w} v \Rightarrow T(y_n,v_n)\xrightarrow{str} T(y,v)$$

The linearity of A gives that T is differentiable in the frèchet sense.

We introduce a continuous bilinear form on $w^{\infty}\{a_{\alpha},2\}$

$$a(u,v):V\times V\to R$$

$$a(u,v)=\sum\left((-1)^{|\alpha|}a_{\alpha}D^{2\alpha}u(x),v(x)\right)_{L^{2}(R^{n})}+\left(a(x)u(x),v(x)\right)_{L^{2}(R^{n})}$$

$x\in R^{n},q(x)$ is a real valued function in $L_{2}(R^{n})$

Such that $q(x)\geq\beta,0\leq\beta\leq1$

The ellipticity of A is sufficient for the coerciveness of $a(u,v)$ on V .

i.e. $a(u,u)\geq C\|u\|^{2},\exists c>0,\forall u\in V$

and we can write $a(u,v)=<Au,V>,u.v\in V$

$\Rightarrow a(u,u)=<Au,u>\geq C\|u\|^{2},c>0,\forall u\in V$

According Barbu [1,2] the problem (P) has at least one optimal solution.

Also we can use the dual methods of Bergounioux and Dietrich [3,4,5] which it is based on a functional.

$h_{\alpha}:V\times U\to R$ as following:

$$h_{\alpha}(y,v)=\sup_{z\in k}\left[<-\alpha T(y,v),z-y>-\frac{1}{2}\|z-y\|^{2}\right]$$

Theorem 3.1: For any $\alpha>0$ these two problems are equivalent:

$$(P_{\alpha})\min J(y,v),\quad h_{\alpha}(y,v)=0,y\in k,v\in U_{ad}$$

$$(P):\min J(y,v),\left\langle T(y,v),z-y\right\rangle\geq0\,\forall z\in k,y\in k,v\in U_{ad}$$

4 The Penalty Method

The classical mathematical programming problem is hopeless to obtain optimality systems so we use the dual penalty method:

Let $(y^{*},v^{*})\in k\times U_{ad}$ be a solution of (P).

We consider the following problem.

(P_{α}^{n}) which is a penalization of problem (P_{2})

$$(P_{\alpha}^{n}):\inf J_{n}(y,v),\forall y\in k,\forall v\in U_{ad}$$

With $Jn(y,v)=J(y,v)+nh_{\alpha}(y,v)+\frac{1}{2}\|y-y^{*}\|_{v}^{2}+\frac{1}{2}\|v-v^{*}\|_{4}^{2}$

For a $\varepsilon_{n}\to+0$ it is clear what there exists an element $(y_{n},v_{n})\in k\times U_{ad}$ such that:

$$J(J_n,V_n)+nh_\alpha(y_n,v_n)+\frac{1}{2}\|y_n-y^*\|_v^2+\frac{1}{2}\|v_n-v^*\|_4^2\leq \inf(P_\alpha^n)+\varepsilon_n$$

$$\Rightarrow J(y_n,v_n)+nh_\alpha(y_n,v_n)+\frac{1}{2}\|y_n-y^*\|_v^2+\frac{1}{2}\|v_n-v^*\|^2-\varepsilon_n\left\|\begin{pmatrix}y-y_n\\ v-v_n\end{pmatrix}\right\|$$

$$\leq J(y,v)+nh_\alpha(y,v)+\frac{1}{2}\|y-y^*\|_v^2+\frac{1}{2}\|v-v^*\|_v^2,\forall(y,v)\in k\times U_{ad}$$

Theorem 4.1

Assume that T is strongly monotone uniformly with respect to $v\in U$, i.e.

$$\exists\upsilon>0,\langle T(y,v)-T(z,v),y-z\rangle\geq\frac{\upsilon}{2}\|z-y\|_V^2\ \forall y,z\in k,\forall v\in U_{ad}$$

Then the sequence (y_n,v_n) converges strongly to $(y^*,v^*)\,\text{in}\,V\times U$

Proof:

The cost function $J:v\times u\to R$ is convex, continuous and Gateaux - differentiable we obtain:

From (1);

$$0\leq nh_\alpha(y_n,v_n)+\frac{1}{2}\|y_n-y^*\|_v^2+\frac{1}{2}\|v_n-v^*\|_u^2$$

$$<J(y^*,v^*)-J(y_n.v_n)+\varepsilon_n\left\|\begin{pmatrix}y_n-y^*\\ v_n-v^*\end{pmatrix}\right\|v\times U$$

With $(y,v)=(y^*,v^*)$

$$0\leq nh_\alpha(y_n,v_n)+\frac{1}{2}\|y_n-y^*\|_v^2+\frac{1}{2}\|v_n-v^*\|_u^2$$

$$\leq\|J`(y^*,v^*)\|\left\|\begin{pmatrix}y_n-y^*\\ v_n-v^*\end{pmatrix}\right\|_{V\times U}+\varepsilon_n\left\|\begin{pmatrix}y_n-y^*\\ v_n-v^*\end{pmatrix}\right\|_{V\times U}\ (2)$$

It follows that the sequences $\{V_n\}$ and $\{y_n\}$ are bounded there exist a weakly convergent subsequences of $\{v_n\}$ and $\{y_n\}$ still denoted $\{v_n\}$ and $\{y_n\}$ such that:

$$v_n\xrightarrow{w}\bar{v}\in U_{ad}\ \text{and}\ y_n\xrightarrow{w}\bar{y}\in K$$

We see that: $\lim\limits_{n\to+\infty}h_\alpha(y_n,v_n)=0$

The model of the system is strongly monotone so:

$$0\leq\frac{\upsilon}{2}\|y_n-\bar{y}\|^2\leq\langle T(y_n,v_n-T(\bar{y},v_n),y_n-\bar{y}\rangle\ \text{such that:}\ \alpha>\frac{1}{\upsilon}$$

Since $h_\alpha(y_n,v_n)=\sup_{\bar{y}\in k}\left[\left\langle -\alpha T(y_n,v_n),\bar{y}-y_n\right\rangle-\frac{1}{2}\|\bar{y}-y_n\|^2\right]$

$$h_\alpha(y_n,v_n)+\frac{1}{2}\|\bar{y}-y_n\|^2=\left\langle -\alpha T(y_n,v_n),\bar{y}-y_n\right\rangle$$

We find that:

$$0\le\frac{\alpha\upsilon}{2}\|y_n-\bar{y}\|^2\le-\alpha\left\langle T(y_n,v_n),\bar{y}-y_n\right\rangle-\alpha\left\langle T(\bar{y},v_n),y_n-\bar{y}\right\rangle$$

$$0\le\frac{\alpha\upsilon}{2}\|y_n-\bar{y}\|^2\le h_\alpha(y_n,v_n)+\frac{1}{2}\|\bar{y}-y_n\|^2-\alpha\left\langle T(\bar{y},v_n),y_n-\bar{y}\right\rangle$$

$$\Rightarrow\left(\frac{\alpha\upsilon-1}{2}\right)\|y_n-y\|^2\le h_\alpha(y_n,v_n)-\alpha\left\langle T(\bar{y},v_n),y_n-\bar{y}\right\rangle$$

The mapping $v\to T(y,v)$ is strongly continuous for any $y\in k$ so: $T(\bar{y},v_n)\xrightarrow{\text{strongly}}T(\bar{y},\bar{v})$

$$\Rightarrow\frac{\alpha\upsilon-1}{2}\|y_n-\bar{y}\|\le 0$$

$$\Rightarrow\lim_{n\to+\infty}y_n=\bar{y}$$

Now we want to prove that: $h_\alpha(\bar{y},\bar{v})=0$

In fact: $h_\alpha(\bar{y},\bar{v})=\lim_{n\to+\infty}h_\alpha(\bar{y},\bar{v})$

$$=\lim_{n\to+\infty}\left[\sup_{y_n\in k}\left[\ \left\langle -\alpha T(\bar{y},\bar{v}),y_n-\bar{y}\right\rangle-\frac{1}{2}\|y_n-\bar{y}\|^2\right.\right.$$

as $n\to+\infty$ $y_n\to\bar{y}$ so $h_\alpha(\bar{y},\bar{v})=0$

$\Rightarrow(\bar{y},\bar{v})$ is a solution for $(P_\alpha)\Rightarrow(y^*,v^*)=(\bar{y},\bar{v})$

from: (2)

$$0\le nh_\alpha(y_n,v_n)+\frac{1}{2}\|y_n-y^*\|_v^2+\frac{1}{2}\|v_n-v^*\|_u^2\le J(y^*,v^*)-J(y_n,v_n)\le 0$$

$$\lim_{n\to\infty}nh_\alpha(y_n,v_n)+\frac{1}{2}\|y_n-y^*\|+\frac{1}{2}\|v_n-v^*\|^2=0$$

$$y_n\to y^*\text{ and }v_n\to v^*\Rightarrow\lim_{n\to+\infty}nh_\alpha(y_n,v_n)=0$$

To get the optimality system we define a sequence $p_n=n(z_n-y_n)$ such that:

$Z_n=P_k\left(y_n-\alpha\wedge^{-1}T(y_n,v_n)\right)$ note that $\exists k>0;\forall n\in N;\|p_n\|_V\le K$

z_n is a bounded sequence, so we can extract a subsequence still denoted $\{p_n\}$ weakly convergent towards P^* as $n \to +\infty$

P^* is the adjoint state.

We can prove that:

$$\forall u \in U_{ad}, \left\langle J`_v(y_n, v_n) - \alpha T_v^*(y_n, v_n)P_n, u - v_n \right\rangle$$
$$+\left\langle \wedge_u (v_n - v^*), u - v_n \right\rangle + \varepsilon_n \| u - v_n \| \geq 0$$

[one can refer to [Berg] for more details].

$\wedge : w^{\infty}\{a_\alpha, 2\} \to w^{-\infty}\{a_\alpha, 2\}$

Is the canonical isomorphism we recall that:

$$\forall (y,z) \in V \times V \qquad (y,z)_v = \langle y, z \rangle$$

Theorem 4.2

Assume that $U_{ad} \subseteq L^2(R^n)$ and let (y^*, v^*) be an optional solution of (P). then there exists $P^* \in L^2(R^n)$ such that:

$(P(v^* - vd) + \alpha P^*) \geq 0$, $\forall v \in U_{ad}$

$\forall z \in ky^* . (y^* - y_d - A^* q^*, z)_{L^*(R^n)} \geq 0$

Such that $ky^* = \{z \in k_{y^*}(f^*, z)_{L^2(R^n)} = 0\}$

$f^* = Ay^* - f - v^*$

Proof:

(y^*, v^*) is a solution of our problem, from (3)

$$\forall\, v \in U_{ad} \quad (v_n, y_n) \in U_{ad} \times k$$

$$\left\langle J`_v(j_n, v_n, v\right\rangle - \alpha \left\langle T`v^*(y_n, v_n)P_n, v \right\rangle + (v_n - v^*, v) + \varepsilon_n \| V \| \geq 0$$

$$\Rightarrow (f(v^* - vd), v) - \alpha(P_n, T`_v v) + (v_n - v^*, v) + \varepsilon_n \| v \| \geq 0$$

$$\Rightarrow (P(v^* - v_d) + v_n - v^*, v) + (\alpha P_n, v) + \varepsilon_n \| v \| \geq 0$$

Passage to the limit:

Use $n \to +\infty$

$$\left(P(v^* - v_d), v\right) + \alpha(p^*, v) \geq 0$$

$$\Rightarrow (P(v^* - v_d) + \alpha P^*, v) \geq 0$$

We can easily prove that:

$$\left\langle J`y(y^*,v^*)-T`y^*(y^*,v^*)q^*,2-y^*\right\rangle \geq 0, q^*=\alpha p^*$$

See Bergounioux and Dietrich [3] th. 4.3.

$$\Rightarrow \left\langle y^*-y_d, z-y^*\right\rangle - \left\langle q^*, T`y(y^*,v^*).(z-y^*)\right\rangle \geq 0$$

$$\Rightarrow (y^*-y_d, z-y^*)-(A^*q^*, z-y^*)\geq 0$$

Because A`=A

$$\Rightarrow (y^*-y_d-A^*q^*, z-y^*)\geq 0\ ,\ \forall z\in k \quad (5)$$

Let $z`=z-y^*, z`\in ky^*$

$$\Rightarrow (y^*-y_d-A^*q^*, z`)\geq 0\ ,\ \forall z`\in ky^*$$

References

[1] V. Barbu, Optimal control of variational inequalities, Research Notes in mathematics, 100 pitman, Boston, 1984.

[2] V. Barbu, Analysis and control of non linear infinte dimensional systems ,math in science and engineering ,vol.190,Academic Press 1993.

[3] M. Bergounioux, H. Dietrich, Optimal control of problems governed by obstacle type variational inequalities: a dual regularization penalization approach, Vol. 5, no. 2, (1997), pp. 32-351.

[4] M. Bergounioux, Optimal control of an obstacle problem, Applied mathematics and optimazation ,36,pp.147-172,1997.

[5] M. Bergounioux, Optimal control of problems governed by abstract elliptic variational inequalities with state constraints siam journal on control and optimization,vol.36,n 1,January 1998.

[6] Ju. A. Dubinskii, Some imbedding theorems for Sobolev spaces of infinite order, Soviet Math. Dokl., Vol. 19, (1978), pp. 1271-1274.

[7] S.A. El.zahaby and GH.H. Mostafa. Optimal control of variational inequalities of infinite order, journal of mathematical egyptian society 13,2005, pp.99-106.

[8] I. M. Gali, Optimal control of system governed by elliptic operators of infinite order, ordinary and partial differential equations, Proceedings, Dundee, Lecture Notes in Mathematics, (1984), pp. 263-272.

[9] I.M. Gali, and H.A. El-Saify, and S.A. El-Zahaby, Distributed Control of a System Governed by Dirichlet and Neumann Problems for Elliptic Equations of Infinite Order Proc. Of the International Conference of Functional Differential System and Related topics III, Pland, 22-29, (1983).

[10] J. L. Lions and G. Stampacchia, Variational inequalities, Comm. Pure Applied Math., Vol. 20, (1967), pp. 493-519.

[11] F. Mignot, Control dans les inequations variationnelles elliptiques, J. Funct. Anal., Vol. 22, (1976), pp. 130-185.

[12] F. Mignot and J. P. Puel, Optimal control in some vriational inequalities, SIAM J. Control and Optimization, Vol. 22, (1984), pp. 466-476.

[13] E. Zeidlar, Nonlinear functional analysis and its applications, Vol. IIB: Nonlinear Monotone Operators, Springs, Berlin 1992.