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Differential Sandwich Theorems for p-valent Analytic Functions Defined by Cho–Kwon–Srivastava Operator

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Abstract.

By using of Cho–Kwon–Srivastava operator, we obtain some subordinations and superordinations results for certain normalized p-valent analytic functions.

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1. Introduction.

Let $H(U)$ be the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$, and let $H[a;p]$ be the subclass of $H(U)$ consisting of functions of the form :

$$f(z)=a+a_p z^p+a_{p+1}z^{p+1}+\dots \qquad (a\in\mathbb{C}),$$

For simplicity, $H[a]=H[a;1]$. Also, let $A(p)$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_n z^{n+p}. \quad (p\in\mathrm{N}=\{1,2,\dots\}), \tag{1.1}$$

Which are analytic and p-valent in U .

If $f,g\in H(U)$,we say that the function f is subordinate to g ,or the function g is superordinate to f , if there exists a Schwarz function ω, i.e., $\omega\in H(U)$ with $\omega(0)=0$ and $|\omega(z)|<1, z\in U$, such that $f(z)=g(\omega(z))$ for all $z\in U$. This subordination is usually denoted by $f(z)\prec g(z)$. It is well known that, if the function g is univalent in U, then $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$ (cf., e.g., [9], see also [5]).

Supposing that p, h are two analytic functions in U , let

$$\varphi(r,s,t;z): C^3 \times U \to C.$$

If $p(z)$ and $\varphi\left(p(z), zp^{'}(z), z^2 p^{"}(z); z\right)$ are univalent functions in U and $p(z)$ satisfies the second-order differential subordination

(1.2) $$h(z) \prec \varphi\left(p(z), zp^{'}(z), z^p p^{"}(z); z\right),$$

then $p(z)$ is called to be a solution of the differential superordination (1.2). An analytic function $q(z)$ is called a subordinant of the solution of the differential superordination (1.2) , if $q(z) \prec p(z)$ for all the functions p(z) satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all of the subordinants q of (1.2), is called the best subordinant (cf., e.g., [9], see also [5]).

Recently, Miller and Mocanu [10] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi\left(p(z), zp^{'}(z), z^p p^{"}(z); z\right) \Rightarrow q(z) \prec p(z).$$

For functions $f_j(z) \in A(p)$, given by

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{n+p} \quad (j = 1, 2),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{n+p} = (f_2 * f_1)(z) \quad (z \in U).$$

In terms of the Pochhammer symbol $(\theta)_n$ given by

$$(\theta)_n = \begin{cases} 1, & (n = 0) \\ \theta(\theta+1)....(\theta+n-1), & \left(n \in N = \{1, 2, ...\}\right), \end{cases}$$

we now define a function $\varphi_p(a, c; z)$ by

(1.3) $$\varphi_p(a, c; z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p}$$

$$\left(a \in R; c \in R \setminus Z_0^-; Z_0^- = \{0, -1, -2, ...\}; z \in U\right).$$

With the aid of the function $\varphi_p(a, c; z)$ defined by (1.3), we consider a function $\varphi_p^*(a, c; z)$ given by the following convolution

$$\varphi_p(a, c; z) * \varphi_p^*(a, c; z) = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p; z \in U)$$

which yields the following family of linear operators $I_p^{\lambda}(a,c)$:

(1.4) $$I_p^{\lambda}(a,c)f(z) = \varphi_p^{*}(a,c;z) * f(z) \quad \left(a,c \in R \setminus Z_0^{-}; \lambda > -p; z \in U\right).$$

For a function $f(z) \in A(p)$, given by (1.1), it is easily seen from (1.4) that

(1.5) $$I_p^{\lambda}(a,c)f(z) = z^p + \sum_{n=1}^{\infty} \frac{(c)_n (\lambda+p)_n}{(a)_n (1)_n} a_{p+n} z^{p+n} \quad (z \in U),$$

which readily yields the following properties of the operator $I_p^{\lambda}(a,c)$:

(1.6) $$z\left(I_p^{\lambda}(a,c) f(z)\right)' = (\lambda+p) I_p^{\lambda+1}(a,c)f(z) - \lambda I_p^{\lambda}(a,c)f(z)$$

and

(1.7) $$z\left(I_p^{\lambda}(a+1,c)f(z)\right)' = a I_p^{\lambda}(a,c)f(z) - (a-p) I_p^{\lambda}(a+1,c)f(z).$$

The operator $I_p^{\lambda}(a,c)$ was introduced and studied by Cho et al. [6].

We observe that:

$$I_p^{0}(p,1)f(z) = I_p^{1}(p+1,1)f(z) = f(z), I_p^{1}(p,1)f(z) = \frac{zf'(z)}{p},$$

$$I_p^{2}(p,1)f(z) = \frac{2zf'(z) + z^2 f''(z)}{p(p+1)};$$

$$I_p^{0}(a+1,1)f(z) = p\int_0^z \frac{f(t)}{p} dt,$$

$$I_p^{n}(a,a)f(z) = D^{n+p-1}f(z) \qquad \left(n \in \mathbb{N}, n > -p\right),$$

Where $D^{n+p-1}f(z)$ is the Ruscheweyh derivative of $(n+p-1)th$ order, see[8].Many interesting result of multivalent analytic functions associated with the linear operator $I_p^{\lambda}(a,c)$ have been studied in [6].

Also we observe that:

(i) $I^0(1,1) f(z) = I^1(2,1) f(z) = f(z), I^1(1,1) f(z) = zf'(z),$

$I^2(1,1) f(z) = \frac{1}{2}(2zf'(z) + z^2 f''(z));$

(ii) $I^{\mu}(\mu+2,1)f(z) = F_{\mu}(f)(z) (\mu > -1),$ where

$$F_{\mu}(f)(z) = \frac{\mu+1}{z^{\mu}} \int_0^z t^{\mu-1} f(t) dt \qquad (see\,[3]);$$

(iii) $I^0(n+1,1) f(z) = I_n f(z) (n \in N_0 = N \cup \{0\})$ $\left(Noor \text{ integral operator, see}[13]);\right)$

(iv) $I^{\lambda}(\mu+2,1)f(z) = I_{\lambda,\mu} f(z) (\lambda > -1; \mu > -2)$ $(Choi-Saigo-Srivastava\ operator, see\,[7]).$

Recently many authors ([1], [11], [12] and [14]) have used the results of Bulboacˇa [4] and shown some sufficient conditions applying first order differential subordinations and superordinations.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions f(z),g(z)in U such that $I_p^{\lambda}(a,c)g(z) \neq 0 \ for\ 0<|z|<1$ and satisfy

$$q_1(z) \prec \frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in U. Also, we obtain the number of known results as their special cases.

2. Definitions and preliminaries.

In order to prove our results, we shall make use of the following known results.

Definition 1 ([9]). Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} \setminus$ E(f), where

$$E(f)=\left\{\varsigma \in \partial U : \lim_{z\to\varsigma} f(z)=\infty\right\}$$

and are such that

$$f'(\varsigma)\neq 0 \ for\ \varsigma \in \partial U \setminus E(f).$$

Lemma 1 ([10]). Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing q(U) with $\varphi(\omega)\neq 0$ when $\omega \in$ q(U). Set

$$\psi(z)=zq'(z)\varphi(q(z)) \ and\ h(z)=(q(z))+\psi(z).$$

Suppose that

(i) $\psi(z)$ *is starlike univalent in* U,

(ii) $\operatorname{Re}\left\{\frac{zh'(z)}{\psi(z)}\right\}>0,\ z\in U.$

If p is analytic in U with $p(0)=q(0), p(U)\subseteq D$ and

(2.1) $$\theta(p(z))+zp'(z)\varphi(p(z))<\theta(q(z))+zq'(z)\varphi(q(z)),$$

then

$$p(z)\prec q(z)$$

and q is the best dominant.

Lemma 2 ([4]). Let q be convex univalent in the unit disk U and let θ and φ be analytic in a domain D containing q(U). Suppose that

$$(i)\ \mathrm{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0, \quad z \in U,$$

$(ii)\ \psi(z) = zq'(z)\varphi(q(z))$ *is univalent in* U.

If $p \in H[q(0),1] \cap Q$ *with* $p(U) \subseteq D$ *and* $\theta(P(z)) + zp'(z)\varphi(p(z))$ is univalent in U *and*

(2.2) $$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)),$$

then

$$q(z) \prec p(z),$$

and q is the best subordinant of (2.2).

3. Subordination results.

Using Lemma 1, we first prove the following theorem.

Theorem 1. Let $\alpha \neq 0, \beta > 0$ and q(z) be convex univalent in U with q(0) = 1. Further assume that

(3.1) $$\mathrm{Re}\left\{\frac{\beta - p\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \qquad (z \in U).$$

If $f, g \in A(p)$ satisfy

(3.2) $$\gamma(f, g, \alpha, \beta) \prec (\beta - p\alpha)q(z) + \alpha q^2(z) + \alpha z'(z),$$

Where

(3.3) $$\begin{aligned}\gamma(f,g,\alpha,\beta) &= (\beta-(p+1)\alpha)\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)} + \alpha\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)}\right)^2 \\ &+\alpha(\lambda+p+1)\frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)} \\ &-\alpha(\lambda+p)\frac{I_p^{\lambda+1}(a,c)g(z)}{I_p^{\lambda}(a,c)g(z)}\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)}\right),\end{aligned}$$

then

$$\frac{I_p^{\lambda+1}(a,c)\,\mathrm{f(z)}}{I_p^{\lambda}(a,c)g(z)} \prec q(z)$$

And q is the best dominant.

Proof. Define the function p(z) by

(3.4) $$p(z)=\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)} \qquad (z\in U).$$

Then the function p(z) is analytic in U and p(0) = 1. Therefore, differentiating (3.4) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

(3.5) $$\begin{aligned}&\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)}\left[\beta-(p+1)\alpha+\alpha\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)}\right.\\ &\quad+\alpha(\lambda+p+1)\frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)\,\mathrm{f(z)}}-\alpha(\lambda+p)\frac{I_p^{\lambda+1}(a,c)g(z)}{I_p^{\lambda}(a,c)g(z)}\\ &\quad=(\beta-p\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z).\end{aligned}$$

By using (3.5) in (3.2), we have

(3.6) $$\left(\beta-p\alpha\right)p(z)+\alpha p^2(z)+\alpha zp'(z)\prec(\beta-p\alpha)q(z)+\alpha q^2(z)+\alpha z'(z).$$

By setting

$$\theta(\omega)=\alpha\omega^2+(\beta-p\alpha)\omega \quad and \quad \varphi(\omega)=\alpha$$

we can easily observe that θ(w) and $\varphi(\omega)$ are analytic in C\{0} and that $\varphi(\omega)\neq 0$. Hence the result now follows by using Lemma 1.

Remark 1. Putting λ =0, a = c =1 and taking $f(z)\equiv g(z)(z\in U)$ in Theorem 1, we obtain the result obtained by Murugusundaramoorthy and Magesh [9, Corollary 2.9].

Putting $f(z)\equiv g(z)(z\in U)$ in Theorem 1, we obtain the following corollary.

Corollary 1. Let $\alpha\neq 0,\beta>0$ and q be convex univalent in U with q(0) = 1 and (3.1) holds true. If $f\in A(p)$ satisfies

$$(\beta-(p+1)\alpha)\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}+\alpha(\lambda+p+1)\frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}-\alpha(\lambda+p-1)\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}\right)^2 \text{ then}$$

$$\prec(\beta-p\alpha)(q(z)+\alpha q^2(z)+\alpha\,\mathrm{z}'(z),$$

$$\frac{I_p^{\lambda+1}(a,\mathrm{c})\,\mathrm{f(z)}}{I_p^{\lambda}(a,c)f(z)}\prec q(z)$$

and q is the best dominant.

Putting $a=\mu+p+1(\mu>-(p+1))\,and\ c=1$ in Theorem 1, we obtain the following corollary.

Corollary 2. Let $\alpha\neq 0,\beta>0$ and q be convex univalent in U with q(0) = 1 and (3.1) holds true. If $f,g\in A(p)$ satisfy

$$\gamma_1(f,g,\alpha,\beta)\prec(\beta-(p+1)\alpha)q(z)+\alpha q^2(z)+\alpha zq'(z)$$

where

(3.7)
$$\gamma_1(f,g,\alpha,\beta)=(\beta-(p+1)\alpha)\frac{I_p^{\lambda+1,\mu}f(z)}{I_p^{\lambda,\mu}g(z)}+\alpha\left(\frac{I_p^{\lambda+1,\mu}f(z)}{I_p^{\lambda,\mu}g(z)}\right)^2$$
$$+\alpha(\lambda+p+1)\frac{I_p^{\lambda+2,\mu}f(z)}{I_p^{\lambda,\mu}g(z)}-\alpha(\lambda+p)\frac{I_p^{\lambda+1,\mu}f(z)}{I_p^{\lambda,\mu}g(z)}\left(\frac{I_p^{\lambda+1,\mu}f(z)}{I_p^{\lambda,\mu}g(z)}\right),$$

then

$$\frac{I_p^{\lambda+1,\mu}f(z)}{I_p^{\lambda,\mu}g(z)}\prec q(z),$$

and q is the best dominant.

Putting a = μ + p+1 (μ>-p), c =1 and λ = μ in Theorem 1, we obtain the following corollary.

Corollary 3. Let $\alpha\neq 0,\beta>0$ and q be convex univalent in U with q(0) = 1 and (3.1) holds true. If $f,g\in A(p)$ satisfy

$$\gamma_2(f,g,\alpha,\beta)\prec(\beta-\alpha+\alpha\mu)q(z)+\alpha q^2(z)+\alpha z'(z),$$

where

(3.8)
$$\gamma_2(f,g,\alpha,\beta)=(\beta-\alpha+\alpha\mu)\frac{f(z)}{F_{\mu,p}(g)(z)}+\alpha\left(\frac{f(z)}{F_{\mu,p}(g)(z)}\right)^2$$
$$+\alpha\frac{zf'(z)}{F_{\mu,p}(g)(z)}-\alpha(\mu+p)\frac{\mathrm{g}(z)}{F_{\mu,p}(g)(z)}\frac{f(z)}{F_{\mu,p}(g)(z)}.$$

then

$$\frac{f(z)}{F_{\mu,p}g(z)}\prec q(z),$$

and q is the best dominant.

Corollary 4. Let $\alpha\neq 0,\beta>0$ and q be convex univalent in U with q(0) = 1 and (3.1) holds true. If $f\in A(p)$ satisfies

$$\gamma_3(f,\alpha,\beta)\prec(\beta-p\alpha)q(z)+\alpha q^2(z)+\alpha zq'(z),$$

where

$$\gamma_3(f,\alpha,\beta)=(\beta-\alpha+\alpha\mu)\frac{f(z)}{F_{\mu,p}(\mathrm{f})(z)}+\alpha\frac{zf'(z)}{F_{\mu,p}(\mathrm{f})(z)}$$

(3.9)
$$-\alpha(\mu+p-1)\left(\frac{f(z)}{F_{\mu,p}(\mathrm{f})(z)}\right)^2$$

then

$$\frac{f(z)}{F_{\mu,p}(f)(z)}\prec q(z),\quad (\mu>-p),$$

and q is the best dominant.

4. Superordination and sandwich results.

Theorem 2. Let $\alpha\neq 0, \beta>0$. Let q be convex univalent in U with q(0) = 1. Assume that

(4.1)
$$\mathrm{Re}\{q(z)\}\geq \mathrm{Re}\left\{\frac{p\alpha-\beta}{(1+p)\alpha}\right\}.$$

Let $f,g\in A(p), \frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)\mathrm{g}(z)}\in H[q(0),1]\cap Q$,Let $\gamma(f,g,\alpha,\beta)$ be univalent in U and

(4.2)
$$(\beta-p\alpha)q(z)+\alpha q^2(z)+\alpha zq'(z)\prec\gamma(f,g,\alpha,\beta),$$

where γ(f, g, α, β) is given by (3.3), then

(4.3)
$$q(z)\prec\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)\mathrm{g}(z)}.$$

and q is the best subordinant.

Proof. Let p(z) be defined by (3.4). Therefore, differentiating (3.4) with respect to z and using the identity (1.6) in the resulting equation, we have

$$\gamma(f,g,\alpha,\beta)=(\beta-p\alpha)\mathrm{p}(z)+\alpha p^2(z)+\alpha zp'(z),$$

then

$$(\beta-p\alpha)\mathrm{q}(z)+\alpha q^2(z)+\alpha zq'(z)\prec(\beta-p\alpha)\mathrm{p(z)}+\alpha\,\mathrm{p}^2\mathrm{(z)}+\alpha z\ \mathrm{p'(z)}.$$

By setting $\theta(\omega)=\alpha\omega^2+(\beta-p\alpha)\omega$ *and* $\varphi(\omega)=\alpha,$ it is easily observed that $\theta(\omega)$ is analytic in C. Also, $\varphi(\omega)$ is analytic in C\{0} and that $\varphi(\omega)\neq 0.$. Since q(z) is convex univalent, it follows that

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\}=\operatorname{Re}\left\{\frac{\beta-p\alpha}{\alpha}+2q(z)\right\}>0 \quad (z\in U).$$

Now Theorem 2 follows by applying Lemma 2. D

Putting f(z) ≡ g(z) (z ∈ U) in Theorem 2, we obtain the following corollary.

Corollary 5. Let $\alpha\neq 0,\beta\geq 1$ and q be convex univalent in U with q(0) = 1 and (4.1) holds true.

Let $f\in A(p),\dfrac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}\in H[q(0),1]\cap Q.$

Let

$$\gamma(f,\alpha,\beta)=(\beta-(p+1)\alpha)\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}+\alpha(\lambda+p+1)\frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}$$
$$-\alpha(\lambda+p-1)\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}\right)^2,$$

be univalent in U and

$$(\beta-p\alpha)q(z)+aq^2(z)+\alpha zq'(z)\prec\gamma(f,\alpha,\beta),$$

then

$$q(z)\prec\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)},$$

and q is the best subordinant.

Putting a = μ +p+1 (μ > −(p+1)) and c =1 in Theorem 2, we obtain the following corollary.

Corollary 6. Let $\alpha\neq 0,\beta>0$ and q be convex univalent in U with q(0) = 1 and (4.1) holds true.

Let $f,g\in A(p),\dfrac{I_p^{\lambda+1,\mu}f(z)}{I_p^{\lambda,\mu}g(z)}\in H[q(0),1]\cap Q.$ Let $\gamma_1(f,g,\alpha,\beta)$ be univalent in U and

$$(\beta-p\alpha)q(z)+\alpha q^2(z)+\alpha zq'(z)\prec(f,g,\alpha,\beta),$$

Where $\gamma_1(f,g,\alpha,\beta)$ is given by (3.7) , then

$$q(z)\prec\frac{I_p^{\lambda+1,\mu}f(z)}{I_p^{\lambda,\mu}g(z)},$$

and q is the best subordinant .

Putting $a=\mu+p+1(\mu>-(p+1)),c=1$ and $\lambda=\mu$ in Theorem 2, we obtain the following corollary.

Corollary 7. Let $\alpha\neq 0,\beta\geq 1$ and q be convex univalent in U with $q(0)=1$ and (4.1) holds true.

Let $f,g\in A(p)$, $\dfrac{f(z)}{F_{\mu,p}(g)(z)}\in H[q(0),1]\cap Q.$ Let $\gamma_2(f,g,\alpha,\beta)$ be univalent in U and

$$(\beta - p\alpha)q(z) + \alpha q^2(z) + \alpha z q'(z) \prec \gamma_2(f, g, \alpha, \beta),$$

Where $\gamma_2(f, g, \alpha, \beta)$ *is given by* (3.8), then

$$q(z) \prec \frac{f(z)}{F_{\mu,p}(g)(z)},$$

and q is the best subordinant .

Putting f(z) ≡ g(z) (z ∈ U)in Corollary 7,we obtain the following corollary .

Corollary 8. Let $\alpha \neq 0, \beta \geq 1$ and q be convex univalent in U with q(0) = 1 and(4.1) holds true. Let $f \in A(p), \frac{f(z)}{F_{\mu,p}(f)(z)} \in H[q(0),1] \cap Q$. *Let* $\gamma_3(f, \alpha, \beta)$ be univalent in U and

$$(\beta - p\alpha)q(z) + \alpha q^2(z) + \alpha z'(z) \prec \gamma_3(f, \alpha, \beta),$$

where $\gamma_3(f, \alpha, \beta)$ *is given by* (3.9),

$$q(z) \prec \frac{f(z)}{F_{\mu,p}(f)(z)} \quad (\mu > -p).$$

and q is the best subordinant .

We conclude this section by stating the following sandwich result.

Theorem 3. Let q1 and q2 be convex univalent in U, $\alpha \neq 0$ and β ≥ 1. Suppose q2 satisfies (3.1) and q1 satisfies (4.1). Moreover, suppose

$$\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)} \in H[1,1] \cap Q$$

and $\gamma(f, g, \alpha, \beta)$ is univalent in U. If $f, g \in A(p)$ satisfy

$$(\beta - p\alpha)q_1(z) + \alpha q_1^2(z) + \alpha z q_1'(z) \prec \gamma(f, g, \alpha, \beta) \prec (\beta - p\alpha)q_2(z) + \alpha q_2^2(z) + \alpha z q'(z),$$

where γ(f, g, α, β) is given by (3.3),then

$$q_1(z) \prec \frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)g(z)} \prec q_2(z)$$

and q1, q2 are, respectively, the best subordinant and the best dominant.

By making use of Corollaries 2 and 6, we obtain the following corollary.

Corollary 9. *Let q1 and q2 be convex univalent in U* $\alpha \neq 0$ *and* $\beta \geq 1$. *Suppose q2 satisfies (3.1) and q1 satisfies (4.1). Moreover, suppose*

$$\frac{I_p^{\lambda+1,\mu}(a,c)\,\mathrm{f(z)}}{I_p^{\lambda,\mu}(a,\mathrm{c})\,\mathrm{g(z)}} \in H\left[1,1\right]\cap Q$$

and γ1(f, g, α, β) is univalent in U. If $f, g \in A(p)$ satisfy

$$(\beta - p\alpha)q_1(z) + \alpha q_1^{\,2}(\mathrm{z}) + \alpha\, \mathrm{z}q_1^{'}(z) \prec \gamma_1(f, g, \alpha, \beta) \prec (\beta - p\alpha)q_2(z) + \alpha q_2^{\,2}(\mathrm{z}) + \alpha\, \mathrm{z}q_2^{'}(z),$$

where $\gamma_1(f, g, \alpha, \beta)$ *is given by* (3.7),*then*

$$q_1(z) \prec \frac{I_p^{\lambda+1,\mu} f(z)}{I_p^{\lambda,\mu} g(z)} \prec q_2(z) \ \ (\mu > -(p+1))$$

and q_1, q_2 are, respectively, the best subordinant and the best dominant.

By making use of Corollaries 3 and 7, we obtain the following corollary.

Corollary 10. *Let* q_1 *and* q_2 *be convex univalent in U,* $\alpha \neq 0,$ *and* $\beta \geq 1.$ *. Suppose* q_2 *satisfies (3.1) and* q_1 *satisfies (4.1). Moreover, suppose*

$$\frac{f(z)}{F_\mu(g)(\mathrm{z})} \in H\left[1,1\right]\cap Q$$

$\gamma_2(f, g, \alpha, \beta)$ is univalent in U. If $f, g \in A(p)$ satisfy

$$(\beta - p\alpha)q_1(z) + \alpha q_1^{\,2}(z)\alpha z q_1^{'}(z) \prec \gamma_2(f, g, \alpha, \beta) \prec (\beta - p\alpha)q_2(z) + \alpha q_2^{\,2}(z)\alpha z q_2^{'}(z),$$

where $\gamma_2(\mathrm{f,g},\alpha,\beta)$ is given by (3.8), then

$$q_1(z) \prec \frac{f(z)}{F_{\mu,p}(g)(z)} \prec q_2(z) \ \ (\mu > -p)$$

and q_1, q_2 are, respectively, the best subordinant and the best dominant.

By making use of Corollaries 4 and 8, we obtain the following corollary.

Corollary 11. *Let* q_1 *and* q_2 *be convex univalent in U,* $\alpha \neq 0$ *and* $\beta \geq 1.$ *Suppose* q_2 *satisfies (3.1) and* q_1 *satisfies (4.1). Moreover, suppose*

$$\frac{f(z)}{F_{\mu,p}(f)(\mathrm{z})} \in H\left[1,1\right]\cap Q$$

And $\gamma_3(f, \alpha, \beta)$ is univalent in U , $f \in A(p)$ satisfies

$$(\beta - \alpha)q_1(z) + \alpha q_1^{\,2}(z) + \alpha z_1^{'}(z) \prec \gamma_3(f, \alpha, \beta) \prec (\beta - \alpha)q_2(z) + \alpha q_2^{\,2}(z) + \alpha z_2^{'}(z)$$

Where $\gamma_3(f, \alpha, \beta)$ is given by (3.9), then

$$q_1(z) \prec \frac{f(z)}{F_{\mu,p}(f)(z)} \prec q_2(z) \qquad (\mu > -p)$$

and q_1, q_2 ,are respectively, the best subordinant and the best dominant.

Remark 2. Putting p=1, we obtain the results obtained by] Aouf and El-ashwah [2].

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